

## Foreword

This book is the fourth volume of a series of monographs on functional analysis appearing under the title "Generalized Functions." It should not, however, be considered a direct sequel to the preceding volumes. In writing this volume the authors have striven for the maximum independence from the preceding volumes. Only that material which is discussed in the first two chapters of Volume I must be considered as the indispensable minimum which the reader is required to know. In view of this, certain topics which were discussed in the preceding volumes are briefly repeated here.

This book is devoted to two general topics: recent developments in the theory of linear topological spaces and the construction of harmonic analysis in  $n$ -dimensional Euclidean and infinite-dimensional spaces.

After the appearance of a theory of topological spaces, the question arose of distinguishing a class of topological spaces, defined by rather simple axioms and including all (or nearly all) spaces which arise in applications. In the same way, after a theory of linear topological spaces was created, it became necessary to ascertain which class of spaces is most suitable for use in mathematical analysis. Such a class of linear topological spaces—nuclear spaces—was singled out by the French mathematician A. Grothendieck.

The class of nuclear spaces includes all or nearly all linear topological spaces which are presently used in analysis, and has a number of extremely important properties: the kernel theorem of L. Schwartz is valid in nuclear spaces, as is also the theorem on the spectral resolution of a self-adjoint operator. Furthermore, any measure on the cylinder sets in the conjugate space of a nuclear space is countably additive. The first and fourth chapters of this book are devoted to the discussion of these questions. In connection with spectral analysis, the concept of a rigged Hilbert space is introduced, which turns out, apparently, to be very useful also in many other questions in mathematics.

The second question which we study in this volume is the harmonic analysis of functions in various spaces. Harmonic analysis in Euclidean space (the Fourier integral) has already been discussed to some extent in previous volumes. We have given up the idea of repeating here the material in the preceding volumes which was devoted to the Fourier integral (possibly, had all of the volumes been written at the same time, many questions in the theory of the Fourier integral, for example the Paley–Wiener theorem for generalized functions, would have found their natural setting in this volume). We discuss here only questions of harmonic analysis in Euclidean space which were left unclarified in the previous

volumes. Namely, we consider the Fourier transformation of measures having one or another order of growth (the theory of generalized positive definite functions) and its application in the theory of generalized random processes. The Fourier transformation of measures in linear topological spaces is considered at the same time.

In the following, fifth volume, we single out questions of harmonic analysis on homogeneous spaces (in particular, harmonic analysis on groups) and intimately related questions of integral geometry on certain spaces of constant curvature. This theory, which is very rich in the diversity of its results (connected, for example, with the theory of special functions, analytic functions of several complex variables, etc.) could not, of course, be discussed in its entirety within the confines of the fifth volume. We have restricted ourselves to discussing only questions of harmonic analysis on the Lorentz group. It should be remarked that harmonic analysis on the Lorentz group and the related homogeneous spaces is a considerably richer subject than harmonic analysis in the "degenerate" case of a Euclidean space. For example, in the case of a Euclidean space only the smoothness of the Fourier transform of a function is influenced by specifying one kind of behavior or another at infinity of the function itself. But in the case of the Lorentz group, specifying the behavior of the function at infinity leads to certain algebraic relations among the values of its Fourier transform at different points. However, at the present time these questions are only in the initial stages of investigation.

The material of this fourth volume represents a complete unit in itself, and, as we have said, the exposition is practically independent of the preceding volumes. In spite of the relation of one chapter to another, one can begin a reading of this book with the first chapter, which contains the general theory of nuclear and rigged Hilbert spaces, or with the second chapter, which discusses the more elementary theory of positive definite generalized functions.

We mention that certain chapters contain, together with general results, others of a more specialized nature; these can be passed over at the first reading.

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