Contents

Translator's Note v

Foreword vii

Chapter I

Radon Transform of Test Functions and Generalized Functions on a Real Affine Space 1

1. The Radon Transform on a Real Affine Space 1

 1.1. Definition of the Radon Transform 1
 1.2. Relation between Radon and Fourier Transforms . . . 4
 1.3. Elementary Properties of the Radon Transform 5
 1.4. The Inverse Radon Transform 8
 1.5. Analog of Plancherel's Theorem for the Radon Transform 12
 1.6. Analog of the Paley-Wiener Theorem for the Radon Transform 15
 1.7. Asymptotic Behavior of Fourier Transforms of Characteristic Functions of Regions 19

2. The Radon Transform of Generalized Functions 21

 2.1. Definition of the Radon Transform for Generalized Functions 22
 2.2. Radon Transform of Generalized Functions Concentrated on Points and Line Segments 25
 2.3. Radon Transform of $(x_1)_+^\lambda\ \delta(x_2, ..., x_n)$ 26
 2.3a. Radon Transform of $(x_1)_+^k\ \delta(x_2, ..., x_n)$ for Nonnegative Integer k 27
 2.4. Integral of a Function over a Given Region in Terms of Integrals over Hyperplanes 31
 2.5. Radon Transform of the Characteristic Function of One Sheet of a Cone 35
 Appendix to Section 2.5 38
 2.6. Radon Transform of the Characteristic Function of One Sheet of a Two-Sheeted Hyperboloid 40

2.7. Radon Transform of Homogeneous Functions 43
2.8. Radon Transform of the Characteristic Function of an Octant 44
2.9. The Generalized Hypergeometric Function 51

3. Radon Transforms of Some Particular Generalized Functions . 55

3.1. Radon Transforms of the Generalized Functions $(P + i0)^\lambda$, $(P - i0)^\lambda$, and P_+^λ for Nondegenerate Quadratic Forms P 56
Appendix to Section 3.1 59
3.2. Radon Transforms of $(P + c + i0)^\lambda$, $(P + c - i0)^\lambda$, and $(P + c)_+^\lambda$ for Nondegenerate Quadratic Forms . . . 61
3.3. Radon Transforms of the Characteristic Functions of Hyperboloids and Cones 63
3.4. Radon Transform of a Delta Function Concentrated on a Quadratic Surface 66

4. Summary of Radon Transform Formulas 69

Chapter II

Integral Transforms in the Complex Domain 75

1. Line Complexes in a Space of Three Complex Dimensions and Related Integral Transforms 77

1.1. Plücker Coordinates of a Line 77
1.2. Line Complexes 78
1.3. A Special Class of Complexes 80
1.4. The Problem of Integral Geometry for a Line Complex . 82
1.5. The Inversion Formula. Proof of the Theorem of Section 1.4 86
1.6. Examples of Complexes. 89
1.7. Note on Translation Operators 92

2. Integral Geometry on a Quadratic Surface in a Space of Four Complex Dimensions 94

2.1. Statement of the Problem 94
2.2. Line Generators of Quadratic Surfaces 95
2.3. Integrals of $f(z)$ over Quadratic Surfaces and along Complex Lines 98

2.4. Expression for $f(z)$ on a Quadratic Surface in Terms of Its Integrals along Line Generators 100
2.5. Derivation of the Inversion Formula 103
2.6. Another Derivation of the Inversion Formula 107
2.7. Rapidly Decreasing Functions on Quadratic Surfaces. The Paley-Wiener Theorem 111

3. The Radon Transform in the Complex Domain 115

3.1. Definition of the Radon Transform 115
3.2. Representation of $f(z)$ in Terms of Its Radon Transform 117
3.3. Analog of Plancherel's Theorem for the Radon Transform 121
3.4. Analog of the Paley-Wiener Theorem for the Radon Transform 123
3.5. Radon Transform of Generalized Functions 124
3.6. Examples 125
3.7. The Generalized Hypergeometric Function in the Complex Domain 131

Chapter III

Representations of the Group of Complex Unimodular Matrices in Two Dimensions 133

1. The Group of Complex Unimodular Matrices in Two Dimensions and Some of Its Realizations 134

1.1. Connection with the Proper Lorentz Group 134
1.2. Connection with Lobachevskian and Other Motions . . 137

2. Representations of the Lorentz Group Acting on Homogeneous Functions of Two Complex Variables 139

2.1. Representations of Groups 139
2.2. The D_χ Spaces of Homogeneous Functions 141
2.3. Two Useful Realizations of the D_χ 142
2.4. Representation of G on D_χ 144
2.5. The $T_\chi(g)$ Operators in Other Realizations of D_χ . . . 145
2.6. The Dual Representations. 147

3. Summary of Basic Results concerning Representations on D_χ . 148

3.1. Irreducibility of Representations on the D_χ and the Role of Integer Points 148

3.2. Equivalence of Representations on the D_χ and the Role of Integer Points 151
3.3. The Problem of Equivalence at Integer Points 153
3.4. Unitary Representations 156

4. Invariant Bilinear Functionals 157

4.1. Statement of the Problem and the Basic Results 157
4.2. Necessary Condition for Invariance under Parallel Translation and Dilation 159
4.3. Conditions for Invariance under Inversion 163
4.4. Sufficiency of Conditions for the Existence of Invariant Bilinear Functionals (Nonsingular Case) . . . 165
4.5. Conditions for the Existence of Invariant Bilinear Functionals (Singular Case) 168
4.6. Degeneracy of Invariant Bilinear Functionals 174
4.7. Conditionally Invariant Bilinear Functionals. 175

5. Equivalence of Representations of G 178

5.1. Intertwining Operators 178
5.2. Equivalence of Two Representations 182
5.3. Partially Equivalent Representations 184

6. Unitary Representations of G 189

6.1. Invariant Hermitian Functionals on D_χ 189
6.2. Positive Definite Invariant Hermitian Functionals . . . 190
6.3. Invariant Hermitian Functionals for Noninteger ρ, $|\rho| \geqslant 1$ 193
6.4. Invariant Hermitian Functionals in the Special Case of Integer $n_1 = n_2$ 196
6.5. Unitary Representations of G by Operators on Hilbert Space 198
6.6. Subspace Irreducibility of the Unitary Representations . 200

Chapter IV

Harmonic Analysis on the Group of Complex Unimodular Matrices in Two Dimensions 202

1. Definition of the Fourier Transform on a Group. Statement of the Problems and Summary of the Results 202

1.1. Fourier Transform on the Line 202

1.2. Functions on G . . . 204
1.3. Fourier Transform on G . . . 205
1.4. Domain of Definition of $F(\chi)$ . . . 207
1.5. Summary of the Results of Chapter IV . . . 209
Appendix. Functions on G . . . 213

2. Properties of the Fourier Transform on G . . . 216
2.1. Simplest Properties . . . 216
2.2. Fourier Transform as Integral Operator . . . 218
2.3. Geometric Interpretation of $K(z_1, z_2; \chi)$. The Functions $\varphi(z_1, z_2; \lambda)$ and $\Phi(u, v; u', v')$ . . . 220
2.4. Properties of $K(z_1, z_2; \chi)$ . . . 222
2.5. Continuity of $K(z_1, z_2; \chi)$ . . . 223
2.6. Asymptotic Behavior of $K(z_1, z_2; \chi)$ . . . 225
2.7. Trace of the Fourier Transform . . . 226

3. Inverse Fourier Transform and Plancherel's Theorem for G . . . 227
3.1. Statement of the Problem . . . 227
3.2. Expression for $\varphi(z_1, z_2; \lambda)$ in Terms of $K(z_1, z_2; \chi)$ . . . 230
3.3. Expression for $f(g)$ in Terms of $\varphi(z_1, z_2; \lambda)$ . . . 232
3.4. Expression for $f(g)$ in Terms of Its Fourier Transform $F(\chi)$ 235
3.5. Analog of Plancherel's Theorem for G . . . 237
3.6. Symmetry Properties of $F(\chi)$ . . . 240
3.7. Fourier Integral and the Decomposition of the Regular Representation of the Lorentz Group into Irreducible Representations . . . 242

4. Differential Operators on G . . . 247
4.1. Tangent Space to G . . . 247
4.2. Lie Operators . . . 248
4.3. Relation between Left and Right Derivative Operators . 250
4.4. Commutation Relations for the Lie Operators . . . 252
4.5. Laplacian Operators . . . 253
4.6. Functions on G with Rapidly Decreasing Derivatives . . 254
4.7. Fourier Transforms of Lie Operators . . . 255

5. The Paley-Wiener Theorem for the Fourier Transform on G . . 256
5.1. Integrals of $f(g)$ along "Line Generators" . . . 257
5.2. Behavior of $\Phi(u, v; u', v')$ under Translation and Differentiation of $f(g)$ . . . 258

5.3. Differentiability and Asymptotic Behavior of $\Phi(u, v; u', v')$ 260
5.4. Conditions on $K(z_1, z_2; \chi)$ 262
5.5. Moments of $f(g)$ and Their Expression in Terms of the Kernel 265
5.6. The Paley-Wiener Theorem for the Fourier Transform on G 267

Chapter V

Integral Geometry in a Space of Constant Curvature 273

1. Spaces of Constant Curvature 274

1.1. Spherical and Lobachevskian Spaces 274
1.2. Some Models of Lobachevskian Spaces 276
1.3. Imaginary Lobachevskian Spaces 277
1.3a. Isotropic Lines of an Imaginary Lobachevskian Space . 278
1.4. Spheres and Horospheres in a Lobachevskian Space . . 280
1.5. Spheres and Horospheres in an Imaginary Lobachevskian Space 282
1.6. Invariant Integration in a Space of Constant Curvature . 285
1.7. Integration over a Horosphere 287
1.8. Measures on the Absolute. 288

2. Integral Transform Associated with Horospheres in a Lobachevskian Space 290

2.1. Integral Transform Associated with Horospheres . . . 291
2.2. Inversion Formula for $n = 3$ 293
2.3. Inversion Formula for Arbitrary Dimension 300
2.4. Functions Depending on the Distance from a Point to a Horosphere, and Their Averages 302

3. Integral Transform Associated with Horospheres in an Imaginary Lobachevskian Space 304

3.1. Statement of the Problem and Preliminary Remarks . . 304
3.2. Regularizing Integrals by Analytic Continuation in the Coordinates 308
3.3. Derivation of the Inversion Formula 314
3.4. Derivation of the Inversion Formula (Continued) . . . 319
3.4a. Parallel Isotropic Lines 324
3.5. Calculation of $\Phi(x, a; \mu)$ 326

Chapter VI

Harmonic Analysis on Spaces Homogeneous with Respect to the Lorentz Group 331

1. Homogeneous Spaces and the Associated Representations of the Lorentz Group 331

   1.1. Homogeneous Spaces 331
   1.2. Representations of the Lorentz Group Associated with Homogeneous Spaces 331
   1.3. The Relation between Representation Theory and Integral Geometry 332
   1.4. Homogeneous Spaces and Associated Subgroups of Stability 334
   1.5. Examples of Spaces Homogeneous with Respect to the Lorentz Group 335
   1.6. Group-Theoretical Definition of Horospheres 339
   1.7. Fourier Integral Expansions of Functions on Homogeneous Spaces 345

2. Representations of the Lorentz Group Associated with the Complex Affine Plane and with the Cone, and Their Irreducible Components 349

   2.1. Unitary Representations of the Lorentz Group Associated with the Complex Affine Plane 349
   2.2. Unitary Representation of the Lorentz Group Associated with the Cone 352

3. Decomposition of the Representation of the Lorentz Group Associated with Lobachevskian Space 356

   3.1. Representation of the Lorentz Group Associated with Lobachevskian Space 356
   3.2. Decomposition by the Horosphere Method 357
   3.3. The Analog of Plancherel's Theorem for Lobachevskian Space 362

4. Decomposition of the Representation of the Lorentz Group Associated with Imaginary Lobachevskian Space 364

   4.1. Representation of the Lorentz Group Associated with Imaginary Lobachevskian Space 364

4.2. Decomposition of the Representation Associated with Horospheres of the First Kind 365
4.3. Decomposition of the Representation Associated with Isotropic Lines 367
4.4. Decomposition of the Representation Associated with Imaginary Lobachevskian Space 373
4.5. The Analog of Plancherel's Theorem for Imaginary Lobachevskian Space 381
4.6. Integral Transform Associated with Planes in Lobachevskian Space 383

5. Integral Geometry and Harmonic Analysis on the Point Pairs on the Complex Projective Line 385

Chapter VII

Representations of the Group of Real Unimodular Matrices in Two Dimensions 390

1. Representations of the Real Unimodular Matrices in Two Dimensions Acting on Homogeneous Functions of Two Real Variables 390

1.1. The D_χ Spaces of Homogeneous Functions 390
1.2. Two Useful Realizations of D_χ. 392
1.3. Representation of G on D_χ 392
1.4. The $T_\chi(g)$ Operators in Other Realizations of D_χ 393
1.5. The Dual Representations. 394

2. Summary of the Basic Results concerning Representations on D_χ 395

2.1. Irreducibility of Representations on D_χ 395
2.2. Equivalence of Representations on D_χ and the Role of Integer Points 397
2.3. The Problem of Equivalence at Integer Points 398
2.4. Unitary Representations 399

3. Invariant Bilinear Functionals 400

3.1. Invariance under Translation and Dilation 401
3.2. Necessary and Sufficient Conditions for the Existence of an Invariant Bilinear Functional 404

3.3. Degenerate Invariant Bilinear Functionals for Analytic Representations 409
3.4. Conditionally Invariant Bilinear Functionals 411

4. Equivalence of Two Representations 413

4.1. Intertwining Operators 413
4.2. Equivalence of Two Representations 416
4.3. Partially Equivalent Representations 418
4.4. Other Models of F_s^+ and F_s^- 423

5. Unitary Representations of G 424

5.1. Existence of an Invariant Hermitian Functional 424
5.2. Positive Definite Invariant Hermitian Functionals (Nonanalytic Representations) 426
5.3. Invariant Hermitian Functionals for Analytic Representations 429
5.4. Invariant Positive Definite Hermitian Functionals on the Analytic Function Spaces F_s^+ and F_s^- 432
5.5. Unitary Representations of G by Operators on Hilbert Space 434
5.6. Inequivalence of the Representations of the Discrete Series 437
5.7. Subspace Irreducibility of the Unitary Representations . 438

Notes and References to the Literature 440

Bibliography 442

Index 445