## **Contents**

Pı	reface		xiii
С	HAP	TER 1 COMPUTER NUMBERS	1
1	Th	e Algebra of Complex Numbers	1
		Square Roots	1 3 4 6 9
2	Th	e Geometric Representation of Complex Numbers	12
	$\frac{2.2}{2.3}$	Geometric Addition and Multiplication The Binomial Equation Analytic Geometry The Spherical Representation	12 15 17 18
C	HAP	TER 2 COMPLEX FUNCTIONS	21
1	Int	roduction to the Concept of Analytic Function	21
	1.1 1.2 1.3 1.4	Analytic Functions	22 24 28 30
2	Ele	mentary Theory of Power Series	33
	$\frac{2.1}{2.2}$	Sequences Series	33 35
			vii

## viii CONTENTS

	2.3	Uniform Convergence	35
	2.4	Power Series	38
	2.5	Abel's Limit Theorem	41
3	Th	e Exponential and Trigonometric Functions	42
	3.1	The Exponential	42
	3.2	The Trigonometric Functions	43
	3.3	The Periodicity	44
	3.4	The Logarithm	46
CI	HAP	TER 3 ANALYTIC FUNCTIONS AS MAPPINGS	49
1	Ele	mentary Point Set Topology	50
	1.1	Sets and Elements	50
	1.2	Metric Spaces	51
	1.3	Connectedness	54
	1.4	Compactness	59
	1.5	Continuous Functions	63
	1.6	Topological Spaces	66
2	Co	nformality	67
	2.1	Arcs and Closed Curves	67
	2.2	Analytic Functions in Regions	69
	2.3	Conformal Mapping	73
	2.4	Length and Area	75
3	Lin	ear Transformations	76
	3.1	The Linear Group	76
	3.2	The Cross Ratio	78
	3.3	Symmetry	80
	3.4	Oriented Circles	83
	3.5	Families of Circles	84
4	Ele	mentary Conformal Mappings	89
	4.1	The Use of Level Curves	89
	4.2	A Survey of Elementary Mappings	93
	4.3	Elementary Riemann Surfaces	97
CI	łΑΡ	TER 4 COMPLEX INTEGRATION	101
1	Fu	ndamental Theorems	101
	1.1	Line Integrals	101
	1.2	Rectifiable Arcs	101
	1.3	Line Integrals as Functions of Arcs	105
	1.4	Cauchy's Theorem for a Rectangle	109
	1.5	Cauchy's Theorem in a Disk	112

	CONTEN	ITS	ix
2	Cauchy's Integral Formula		114
	2.1 The Index of a Point with Respect to a Closed Curve		114
	2.2 The Integral Formula		118
	2.3 Higher Derivatives		120
3	Local Properties of Analytical Functions		124
	3.1 Removable Singularities. Taylor's Theorem		124
	3.2 Zeros and Poles		126
	3.3 The Local Mapping		130
	3.4 The Maximum Principle		133
4	The General Form of Cauchy's Theorem		137
	4.1 Chains and Cycles		137
	4.2 Simple Connectivity		138
	4.3 Homology		141
	<ul><li>4.4 The General Statement of Cauchy's Theorem</li><li>4.5 Proof of Cauchy's Theorem</li></ul>		141 142
	4.6 Locally Exact Differentials		144
	4.7 Multiply Connected Regions		146
5	The Calculus of Residues		148
	5.1 The Residue Theorem		148
	5.2 The Argument Principle		152
	5.3 Evaluation of Definite Integrals		154
6	Harmonic Functions		162
	6.1 Definition and Basic Properties		162
	6.2 The Mean-value Property		165
	6.3 Poisson's Formula		166
	6.4 Schwarz's Theorem		168
	6.5 The Reflection Principle		172
CI	HAPTER 5 SERIES AND PRODUCT DEVELOPMENTS		175
1	Power Series Expansions		175
	1.1 Weierstrass's Theorem		175
	1.2 The Taylor Series		179
	1.3 The Laurent Series		184
2	Partial Fractions and Factorization		187
	2.1 Partial Fractions		187
	2.2 Infinite Products		191
	2.3 Canonical Products		193
	2.4 The Gamma Function		198
	2.5 Stirling's Formula		201

## x CONTENTS

3	Entire Functions	206		
	<ul><li>3.1 Jensen's Formula</li><li>3.2 Hadamard's Theorem</li></ul>	207 208		
4	The Riemann Zeta Function	212		
	<ul> <li>4.1 The Product Development</li> <li>4.2 Extension of ζ(s) to the Whole Plane</li> <li>4.3 The Functional Equation</li> <li>4.4 The Zeros of the Zeta Function</li> </ul>	213 214 216 218		
5	Normal Families			
	<ul> <li>5.1 Equicontinuity</li> <li>5.2 Normality and Compactness</li> <li>5.3 Arzela's Theorem</li> <li>5.4 Families of Analytic Functions</li> <li>5.5 The Classical Definition</li> </ul>	219 220 222 223 225		
C	CHAPTER 6 CONFORMAL MAPPING. DIRICHL PROBLEM	ET'S 229		
1	The Riemann Mapping Theorem	229		
	<ul> <li>1.1 Statement and Proof</li> <li>1.2 Boundary Behavior</li> <li>1.3 Use of the Reflection Principle</li> <li>1.4 Analytic Arcs</li> </ul>	229 232 233 234		
2	Conformal Mapping of Polygons			
	<ul> <li>2.1 The Behavior at an Angle</li> <li>2.2 The Schwarz-Christoffel Formula</li> <li>2.3 Mapping on a Rectangle</li> <li>2.4 The Triangle Functions of Schwarz</li> </ul>	235 236 238 241		
3	A Closer Look at Harmonic Functions			
	<ul><li>3.1 Functions with the Mean-value Property</li><li>3.2 Harnack's Principle</li></ul>	242 243		
4	The Dirichlet Problem			
	<ul><li>4.1 Subharmonic Functions</li><li>4.2 Solution of Dirichlet's Problem</li></ul>	245 248		
5	Canonical Mappings of Multiply Connected Regions			
	<ul><li>5.1 Harmonic Measures</li><li>5.2 Green's Function</li><li>5.3 Parallel Slit Regions</li></ul>	252 257 259		

		CONTENTS	xi	
Cł	HAPTER 7 ELLIPTIC FUNCTIONS		263	
1	Simply Periodic Functions		263	
	<ul> <li>1.1 Representation by Exponentials</li> <li>1.2 The Fourier Development</li> <li>1.3 Functions of Finite Order</li> </ul>		263 264 264	
2	Doubly Periodic Functions		265	
	<ul> <li>2.1 The Period Module</li> <li>2.2 Unimodular Transformations</li> <li>2.3 The Canonical Basis</li> <li>2.4 General Properties of Elliptic Functions</li> </ul>		265 266 268 270	
3	The Weierstrass Theory			
	<ul> <li>3.1 The Weierstrass ρ-function</li> <li>3.2 The Functions ζ(z) and σ(z)</li> <li>3.3 The Differential Equation</li> <li>3.4 The Modular Function λ(τ)</li> <li>3.5 The Conformal Mapping by λ(τ)</li> </ul>		272 273 275 277 279	
CI	HAPTER 8 GLOBAL ANALYTIC FUNCTION	ONS	283	
1	Analytic Continuation		283	
	<ul> <li>1.1 The Weierstrass Theory</li> <li>1.2 Germs and Sheaves</li> <li>1.3 Sections and Riemann Surfaces</li> <li>1.4 Analytic Continuations along Arcs</li> <li>1.5 Homotopic Curves</li> <li>1.6 The Monodromy Theorem</li> <li>1.7 Branch Points</li> </ul>		283 284 287 289 291 295 297	
2	Algebraic Functions		300	
	<ul> <li>2.1 The Resultant of Two Polynomials</li> <li>2.2 Definition and Properties of Algebraic Functions</li> <li>2.3 Behavior at the Critical Points</li> </ul>	ions	300 301 304	
3	Picard's Theorem		306	
	3.1 Lacunary Values		307	
4	Linear Differential Equations			
	<ul> <li>4.1 Ordinary Points</li> <li>4.2 Regular Singular Points</li> <li>4.3 Solutions at Infinity</li> <li>4.4 The Hypergeometric Differential Equation</li> <li>4.5 Riemann's Point of View</li> </ul>		309 311 313 315 318	
Ir	ndex		323	