Preface

Perhaps the most dramatic shift in our understanding of the physical world occurred in 1925, with the publication of Heisenberg's remarkable paper 'Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen'. Heisenberg demonstrated that one could deduce quantum phenomena from the equations of Newtonian physics provided one interpreted the time dependent variables as standing for infinite matrices rather than functions. In contrast to functions, matrices need not commute under multiplication. Heisenberg subsequently showed that this unusual feature of 'quantum variables' may be physically understood in terms of his famous Uncertainty Principle.

Heisenberg's 'matrix mechanics' quickly attracted the attention of a number of leading mathematicians, including Jordan, von Neumann, and Weyl. In particular, von Neumann pointed out that Heisenberg's matrices were more precisely modelled by self-adjoint Hilbert space operators. Nevertheless, despite these early contributions to the codification of quantum physics, mathematicians have, at times, been reluctant to consider the implications of Heisenberg's discovery for their own discipline.

There is now a consensus among scientists that the classical and relativistic notions of measurement and geometry that underlie so much of modern mathematics no longer correspond to our understanding of the real world. Von Neumann was the first to fully appreciate this fact, and he concluded that we should seek 'quantized' analogues of mathematics. He proposed that, as in physics, we should begin by replacing functions by operators.

Von Neumann took the first steps toward mathematical quantization in collaboration with Murray. In a remarkable series of articles, Murray and von Neumann (1936), (1937), and (1943), and von Neumann (1929), (1940), and (1949), succeeded in formulating an operator version of integration theory. They began by replacing the algebras of bounded complex functions that naturally arise in classical integration theory (or more precisely, the L_∞-*algebras*) by $*$-*algebras of bounded operators on Hilbert spaces.* During the past sixty years, such *operator algebras* have been shown to have a profound structure theory. As von Neumann had anticipated, they provide a natural framework for quantizing other areas of mathematics, including portions of topology, geometry, analysis, probability theory, and algebra.

In this monograph we are concerned with a more recent innovation, the *quantization of Banach space theory.* In retrospect, this development was both straightforward and unambiguous. We recall that a normed space E can always be realized as a *function space*; that is, a linear space of bounded functions on a set S, together with the uniform norm (see §2.1). By analogy, we define a (concrete) *operator space* V to be a linear space of bounded operators on a Hilbert space H. Although such a space V is normed by the operator norm, it actually inherits a more elaborate structure. Owing to the fact that an $n \times n$ matrix of operators on a Hilbert space H may again be realized as an operator on H^n, there is a *distinguished norm* on each of the *matrix spaces* $M_n(V)$. The appropriate morphisms for this structure are the linear mappings which are *completely bounded*; that is to say, which induce uniformly bounded mappings of the matrix spaces.

Long before operator space theory was axiomatized, operator algebraists had used completely bounded mappings to study the structure of C^*-algebras and von Neumann algebras. With this new framework, it has now become clear that some of the most important invariants of operator algebras, such as injectivity, exactness, and local reflexivity, are best understood as being attributes of their underlying operator spaces. In principle, however, operator spaces should have a much wider applicability than these algebraic results might suggest. There are natural operator space analogues for all the 'classical Banach spaces', and in particular, they provide a natural context for studying non-commutative integration theory. The latter subject is playing an increasingly important role in non-commutative analysis. It is now evident that various difficulties that arise in Fourier analysis of non-commutative groups may be overcome once we acknowledge the underlying operator space structures. There are also reasons to believe that this theory will be essential in the study of harmonic analysis on quantum groups. Finally, it seems inevitable that operator spaces and their Frechet generalizations will provide the correct functional analytic settings for other areas of quantized analysis, including the differential systems that naturally arise in non-commutative geometry.

Throughout this work we use the 'classical' theory of *functions* to motivate the 'quantized' theory of *operators*. As in the physical theory, this goal cannot be fully achieved since operator space theory involves phenomena that do not have classical analogues. Nevertheless, function theory has continued to provide our most fruitful guide for the development of the subject. Perhaps one of the most attractive aspects of operator space theory is the manner in which routine notions in Banach space theory re-emerge as deep and beautiful ideas in operator space theory. It is our hope that we shall succeed in communicating the excitement of this subject to our Banach space colleagues.

The monograph is divided into five parts, followed by an appendix in which we have summarized some of the elementary results we shall use from functional analysis.

In a preliminary chapter, we introduce the reader to spaces with 'matrix coefficients'. Although we might have used Banach modules for this purpose, we have adopted a less sophisticated approach, which is closer to that generally used in the literature.

Part I is devoted to the three fundamental results upon which operator space theory is based: the *representation theorem* of Ruan, the Arveson–Wittstock generalization of the *Hahn–Banach theorem*, and the Paulsen–Wittstock *decomposition theorem* for complete contractions. We have attempted to give completely accessible proofs for these important facts. Although there are now elegant tensor product approaches to all of these results, we feel that they may be too forbidding to the newcomer. As an application of this material, we characterize the injective operator spaces in §6.1.

As we explain at the beginning of Part II, tensor products have been crucial in the development of operator space theory. We consider the three most important tensor products. These are the *operator space projective* and *injective tensor products*, which are quite similar to their classical analogues, and the *Haagerup tensor product*, which is quite unlike anything that may be found in Banach space theory. The operator space injective tensor product reduces to the usual spatial tensor product for C^*-algebras, whereas the operator space projective tensor product may be used to construct the predual of the von Neumann algebraic spatial tensor product (see §7.2). The Haagerup tensor product has proved to be especially useful in more algebraic contexts, some of which we briefly consider in Part V.

In his pioneering work on Banach spaces, Grothendieck used a categorical approach that is particularly amenable to quantization. In particular, his study of the links between mapping spaces and tensor products carries over to this new context. In Part III we use the operator space projective and injective tensor products to generalize Grothendieck's theory of approximation properties. We then introduce analogues of the three most important Banach mapping spaces: the *nuclear*, the *integral*, and the *absolutely summing mappings*. It is here that the more subtle behaviour of operator spaces first becomes apparent. Grothendieck's characterization of the dual of the Banach space injective tensor product in terms of integral mappings does not carry over to operator spaces (see (12.1.8) and Proposition 14.2.2). On the other hand, his ingenious use of absolute summing mappings to study the Dvoretzky–Rogers theorem on unconditional summability applies as well to operator space theory (see §13.4).

Perhaps the deepest results in operator space theory are concerned with the unexpected new phenomena that occur in operator space theory. These largely centre around three notions that are key to C^*-algebra theory: *nuclearity*, *exactness*, and *local reflexivity*. These developments are explained in Part IV, which may be regarded as the central portion of this monograph. This is an area of great beauty and depth, and it represents one of the triumphs of the subject. Owing to the work of Kirchberg, Haagerup, Pisier and the recent developments in (Effros et al. 1998, 1999), we now have a very precise understanding of how these invariants are related. In particular, we have for any operator space V,

$$V \text{ is nuclear } \Rightarrow V \text{ is exact } \Rightarrow V \text{ is locally reflexive,}$$

and

$$V \text{ is nuclear } \Leftrightarrow V \text{ is locally reflexive and } V^{**} \text{ is semidiscrete.}$$

It has been suggested by Kirchberg that if G is an arbitrary discrete group, then its reduced group C^*-algebra $C^*_\lambda(G)$ is exact. On the other hand, he showed that extensions of exact C^*-algebras need not be exact. Turning to local reflexivity, C^*-algebras need not be locally reflexive, but, surprisingly, *all* C^*-algebraic duals are locally reflexive. This theorem is proved in §15.3.

In Part V we briefly consider some of the algebraic applications of operator space theory. We use the operator space projective tensor product to introduce the notion of a '*quantized Banach algebra*' and we reformulate amenability in this context. The most important examples of such objects are the Fourier algebras of locally compact groups, and more generally the 'L_1-*convolution algebras*' of quantum groups. In contrast to the classical theory of amenability, the Fourier algebra of a locally compact group is amenable in the completely bounded sense if and only if the group is amenable. We have included the proof of this for the simple case of compact groups. We have also included a proof of the strikingly elegant abstract characterization for the unital, not necessarily self-adjoint operator algebras. This 'non-self-adjoint Gelfand–Naimark' theorem illustrates some of the remarkable properties of the Haagerup tensor product.

Our goal in this monograph has been to explain the deep analogy between linear spaces of bounded functions and linear spaces of bounded operators. We have made every effort to provide 'elementary' proofs that will be accessible to readers with a rudimentary knowledge of functional analysis, even when that has restricted the scope of our treatment. This should not be regarded as an encylopaedic summary of the subject, and in particular, the references, notes, and bibliography primarily contain various items that are useful to the topics considered.

It is our conviction that the extraordinary array of techniques developed by Banach space theorists will have many applications in non-commutative analysis, and that conversely, operator space theory will provide Banach space theorists with exciting new vistas for research. More generally, it is our hope that this new formalism will help to unravel some of the difficulties associated with quantization.

Operator space theory has evolved rapidly, and this has made it difficult to write an up-to-date monograph on the subject. Numerous drafts of the book were overtaken by irresistible new approaches, prompting us to repeatedly scrap the text and begin over again. We are indebted to Garth Dales who suggested the project, offered encouragement, and never complained about unfulfilled schedules. We also wish to thank the editors of the Oxford University Press, who gave unflagging support for the project.

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