

# Preface

## First Edition

Several years ago, while reading Weil's *Number Theory: An Approach Through History*, I noticed a conjecture of Euler concerning primes of the form  $x^2 + 14y^2$ . That same week I picked up Cohn's *A Classical Invitation to Algebraic Numbers and Class Fields* and saw the same example treated from the point of view of the Hilbert class field. The coincidence made it clear that something interesting was going on, and this book is my attempt to tell the story of this wonderful part of mathematics.

I am an algebraic geometer by training, and number theory has always been more of an avocation than a profession for me. This will help explain some of the curious omissions in the book. There may also be errors of history or attribution (for which I take full responsibility), and doubtless some of the proofs can be improved. Corrections and comments are welcome!

I would like to thank my colleagues in the number theory seminars of Oklahoma State University and the Five Colleges (Amherst College, Hampshire College, Mount Holyoke College, Smith College and the University of Massachusetts) for the opportunity to present material from this book in preliminary form. Special thanks go to Dan Flath and Peter Norman for their comments on earlier versions of the manuscript. I also thank the reference librarians at Amherst College and Oklahoma State University for their help in obtaining books through interlibrary loan.

*Amherst, Massachusetts*  
*August 1989*

## Second Edition

The philosophy of the second edition is to preserve as much of the original text as possible. The major changes are:

- A new §15 on Shimura Reciprocity has been added, based on work of Peter Stevenhagen and Alice Gee [53, 54, 126] and Bumkyo Cho [22].
- The fifteen sections are now organized into four chapters:
  - The original §§1–13, which present a complete solution of  $p = x^2 + ny^2$ , now constitute Chapters 1, 2 and 3.
  - The new Chapter 4 consists of the original §14 (on elliptic curves) and the new §15 (on Shimura Reciprocity).
- An “Additional References” section has been added to supplement the original references [1]–[112]. This section is divided into five parts:
  - The first part consists of references [A1]–[A24] that are cited in the text. These references (by no means complete) provide updates to the book.

- The remaining four parts give some references (also not complete) for further reading that are relevant to the topics covered in Chapters 1, 2, 3 and 4.
- The expanded Notation section now includes all notation used in the book. Specialized notation is listed according to the page where it first appears.

The other changes to the text are very minor, mostly to enhance clarity, improve formatting, and simplify some of the proofs. One exception is the addition of new exercises: at the end of §12, Exercise 12.31 shows how Ramanujan could have derived Weber’s formula for  $f_1(\sqrt{-14})^2$  (thanks to Heng Huat Chan), and at the end of §14, Exercise 14.24 gives an elliptic curve primality test for Mersenne numbers due to Dick Gross [59] (thanks to Alice Silverberg).

I would like to thank the following people for the errors they found in the first edition and for the suggestions they made: Michael Baake, Dominique Bernardi, Jeff Beyerl, Reinier Bröker, Tony Feng, Nicholas Gavrielides, Lee Goswik, Christian Guenther, Shiv Gupta, Kazuo Hata, Yves Hellegouarach, Norm Hurt, Tim Hutchinson, Trevor Hyde, Maurice Kostas, Susumu Kuninaga, Franz Lemmermeyer, Joseph Lipman, Mario Magioladitis, David May, Stephen Mildenhall, Takashi Ono, Frans Oort, Alf van der Poorten, Jerry Shurman, Alice Silverberg, Neil Sloane, Steve Swanson, Cihangir Tezcan, Satoshi Tomabechei, Fan Xingyuan and Noriko Yui.

My hope is that the second edition of *Primes of the Form  $x^2 + ny^2$*  will help bring this wonderful part of number theory to a new audience of students and researchers.

*Amherst, Massachusetts*  
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### Third Edition with Solutions

The goal of the new edition of *Primes of the Form  $x^2 + ny^2$*  is to make this wonderful part of number theory available to readers in a form especially suited to self-study, mainly because complete solutions to all exercises are included. The inspiration was an email correspondence with Roger Lipsett. The questions Roger asked about the exercises and the solutions he wrote for most of them led to the two major changes in this edition:

- All exercises in the book have now been carefully checked. Small errors have been fixed and many hints have been clarified and/or expanded. For some exercises, the changes are more substantial, including complete replacement in a few cases. There are also a small number of new exercises.
- Roger and I completed and revised the solutions he wrote. These now appear at the end of the book, along with some suggestions for how to use the solutions. (Briefly, our advice is that rather than just passively reading the solutions, you should actively engage with the exercises and use our solutions as extended hints for what to do.)

This explains why the new edition is labeled “Third Edition with Solutions.”

Apart from these changes and typographical corrections, the text is largely the same. Here are two exceptions:

- The second edition added §15 on Shimura Reciprocity, with many details left to the exercises. The process of writing solutions for these exercises revealed some problems, including an error in the statement of Theorem 15.17 in the second edition. The corrected statement is now Theorem 15.22. Revisions to §15 include

fixing this theorem and its proof, together with a thorough rewrite of the entire section. This resulted in significant changes to the exercises.

- There is now a unified bibliography. The 112 references of the first edition were supplemented in the second edition with references A1–A24 that were listed separately. These have now been combined into a single bibliography with the addition of a few new references.

The web site for the book is

<https://dacox.people.amherst.edu/primes.html>

This website includes typographical errors for all editions and a link to supplementary exercises for §§1–3 written by Jeffrey Stopple.

I am especially grateful to the AMS for making this edition of *Primes of the Form  $x^2 + ny^2$*  possible. I thank Ina Mette for helpful suggestions and the staff of AMS author support for their help with numerous questions about L<sup>A</sup>T<sub>E</sub>X. And, of course, huge thanks to Roger.

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