TRANSLATOR'S PREFACE

THIS BOOK IS A translation of the last of Carathéodory's celebrated text books, *Funktionentheorie,* which he completed shortly before his recent death.

A number of misprints and minor errors have been corrected in this first American edition of the two-volume work.

It may be noted that in a first course in which considerations of time may necessitate the omission of some of the material, the chapter on Inversion Geometry (Part I, Chapter Three) may be bypassed without interruption of continuity, except for a short glance at § 86 where chordal (spherical) distance is defined. The spherical metric is used consistently, and to great advantage, throughout the rest of the book.

F. Steinhardt

EDITOR'S PREFACE

THE AUTHOR OF THIS book, shortly before he died, had the pleasure of being able to see it through the press. Like his *Vorlesungen über Reelle Funktionen* (Treatise on Real Functions), which was written during the First World War and which is today among the classics on the subject, the present work also was conceived and carried out in times of great external disorder and change. In spite of the extreme hardships imposed on him during these difficult years, Carathéodory lost none of his well-known creativity and endurance. Even the illness which after the end of the war confined him to bed again and again for weeks at a time scarcely slowed him down in the writing of this work. During this time, Carathéodory familiarized me with all the details of the book, and I undertook the task of going over the text from a critical point of view as well as from the student's point of view. With characteristic vitality, Carathéodory took great pleasure in acquainting friends and colleagues, who stayed for hours visiting, with the growing manuscript. Thus I always felt that he was especially devoted to the present work.

This book is primarily a textbook, even if its title does not make this explicit. However, the expert too will find in it much that is new and of interest as regards both subject matter and presentation. The division of the work into two volumes is not intended to imply a corresponding division of the material, but was done merely to make it easier for the student to purchase the book. For this reason, the usual practice of furnishing each of the two volumes with a subtitle has not been followed.

It is traditional for textbooks on the Theory of Functions written in German (and in English) to develop the theory of analytic functions as soon as possible after the introduction of complex numbers; at a later point there may be given, by way of application, a more or less brief account of the geometry of circles (theory of inversions and general Moebius transformations). The author of the present treatise has chosen the reverse order, placing the geometry of circles first. From it he evolves an effortless treatment of the Euclidean, Spherical, and Non-Euclidean Geometries and Trigonometries. When the student is then introduced to the Theory of Functions proper, he will have acquired a grasp of basic concepts and a certain familiarity with the way of thinking appropriate to the subject. Part Two of the book presents first some material from Point-Set Theory and from Topology that is needed for subsequent developments; this material is, in the main, compiled for purposes of reference rather than treated exhaustively. There follows an exposition of the basic concept of the complex contour integral and of the most important theorems attaching to this concept.

As is well known, there are various different ways of introducing the analytic functions. Carathéodory chose Riemann's definition, which is based on the property of differentiability of a complex function. This choice is significant for all of the further development of the theory of analytic functions, the main theorems of which are proved in Part Three. The fruitful concept of harmonic measure, which has assumed great importance in recent years, is introduced in its simplest form only, and its generalization is merely hinted at. A more detailed study would not lie within the scope of this work and will hardly be missed, especially since we have an unexcelled account of this subject in R. Nevanlinna's well-known book on single-valued analytic functions.

Part Four deals with the construction of analytic functions by means of various limiting processes. The approach followed here, offering great advantages and hardly explored in the textbook literature up to now, makes full use of the concept of normal families, due to P. Montel, and the concept of limiting oscillation, due to A. Ostrowski. This results in much greater elegance of proof and makes the theory very attractive.

The fifth and last part of the first volume consists of three chapters dealing with special functions. The author has taken particular care with the exposition of the trigonometric functions. In connection with the logarithm and the general power function, the student is introduced to the simplest types of Riemann surfaces. One of the points of interest in the study of these functions and of the gamma function, and not the least important one at that, is for the student to see that there is no fundamental difference between real and complex analytic functions and that the theorems of Function Theory can actually lead to numerical results.

It will be in keeping, I believe, with the wishes of my highly esteemed teacher if I express thanks in his name to all those who took an interest in this book and helped to further its progress. Particular thanks are due to Professor R. Fueter, who not only read all of the galley proofs but also conducted all of the negotiations with the publishers and was responsible for seeing it through the press. Professor E. Schmidt read large parts of the manuscript and deserves thanks for many a valuable suggestion and improvement. We also thank Professors R. Nevanlinna and A. Ostrowski for their active interest and help, the latter especially for his detailed critical appraisal. The publishers, Verlag Birkhäuser, are to be thanked for the excellence of workmanship of the book and for their untiring cooperation in carrying out our wishes.

May the spirit of this great teacher and master of the Theory of Functions continue through this work to inspire the hearts of students with love for this beautiful and important branch of our science.

Munich, May 1950 **L. Weigand**

AUTHOR'S PREFACE

THE THEORY OF Analytic Functions has its roots in eighteenth-century mathematics. It was L. Euler (1707-1783) who first compiled an enormous amount of material bearing on our subject, the elaboration of which kept the mathematicians of many generations fully occupied and which even today is not exhausted. The first mathematician who essayed to build a systematic theory of functions was J.-L. Lagrange (1736-1813), whose bold idea it was to develop the entire theory on the basis of power series. The state of the science at that time can best be gleaned from the comprehensive treatise by P. Lacroix (2nd ed., 1810-1819).

However, most of the results known at that time were not too well secured. For many individual facts, C. F. Gauss (1777-1855) supplied the first proofs that meet our present-day standards. But Gauss never published the most important of the ideas that he had conceived in this connection. Thus it remained for A.-L. Cauchy (1789-1857), who invented (in 1813) and systematically developed the concept of the complex contour integral, to create the first coherent structure for Function Theory. After the discovery of the elliptic functions in the years 1828-1830 by N. H. Abel (1802-1829) and C. G. J. Jacobi (1804-1851), Cauchy's theory was carried forward, especially by J. Liouville (1809-1882), and was set down in the treatise by Briot and Bouquet (1859) which even today is still worth reading. The genius of B. Riemann (1826-1865) intervened not only to bring the Cauchy theory to a certain completion, but also to create the foundations for the geometric theory of functions. At almost the same time, K. Weierstrass (1815-1897) took up again the above-mentioned idea of Lagrange's, on the basis of which he was able to arithmetize Function Theory and to develop a system that in point of rigor and beauty cannot be excelled. The Weierstrass tradition was carried on in an especially pure form by A. Pringsheim (1850-1941), whose book (1925-1932) is extremely instructive.

During the last third of the 19th Century the followers of Riemann and those of Weierstrass formed two sharply separated schools of thought. However, in the 1870's Georg Cantor (1845-1918) created the Theory of Sets. This discipline is one of the most original creations that mathematics has brought forth, comparable perhaps only to the achievements of the ancient mathematicians of the Fifth and Fourth Centuries B.C., who came up, so to speak, out of nowhere with strict geometric proofs. With the aid of Set Theory it was possible for the concepts and results of Cauchy's and Riemann's theories to be put on just as firm a basis as that on which Weierstrass' theory rests, and this led to the discovery of great new results in the Theory of Functions as well as of many simplifications in the exposition.

In yet another plane lie the developments that have in time led to the somewhat changed aspects of Function Theory today. For in the course of the present century, various processes were introduced that had a profound influence on the directions in which the theory grew. In the first place, we mention in this connection *Schwarz's Lemma,* which in combination with elementary mappings from the geometry of circles allows for new types of arguments such as were unknown prior to its discovery. The new methods thus created are equivalent to a general principle due to E. Lindelöf (1870-1946), discovered a little later than Schwarz's Lemma and used by some writers in place of this lemma. At almost the same time, the concept of *normal families* was introduced into Function Theory, and this concept has gradually taken over a central role in a large class of function-theoretic proofs. Although the beginnings of these developments can be traced back somewhat further, say to T. J. Stieltjes (1856-1894), it may be said that P. Montel first gave an exact definition of normal families and illustrated the usefulness of this concept by one new application after another. Likewise of great help is an idea of A. Ostrowski's, which is, to make use of the spherical distance on the Riemann sphere in order to avoid the exceptional role that is otherwise played by the number ∞ in various limiting processes. I have not hesitated to exploit systematically all of the advantages which these various methods have to offer.

The greatest difficulty in planning a textbook on Function Theory lies in the selection of material. Since a book that is too voluminous is impractical for various reasons, one must decide beforehand to omit all those problems the presentation of which requires very lengthy preparatory material. For this reason I have not even mentioned such things as the theory of algebraic functions, or the definition of the most general analytic function that can be obtained from one of its functional elements—things that have always been regarded as forming an indispensable part of a textbook of Function Theory. For the same reason, the general definition of a Riemann surface has not been given, nor is there an account of the theory of uniformization.

The book begins with a treatment of Inversion Geometry (geometry of circles). This subject, of such great importance for Function Theory, is taught in great detail in France in the "Classes de mathématiques spéciales," whereas in German-language (and English-language) universities, it is usually dealt with in much too cursory a fashion. It seems to me, however, that this branch of geometry forms the best avenue of approach to the Theory of Functions; it was, after all, his knowledge of Inversion Geometry that enabled H. A. Schwarz (1843-1921) to achieve all of his celebrated successes.