CHAPTER 1

Harmonic Functions

1.1. Guide

In this chapter we will use various methods to study harmonic functions. These include mean value properties, fundamental solutions, maximum principles, and energy methods. The four sections in this chapter are relatively independent of each other.

The materials in this chapter are rather elementary, but they contain several important ideas on the whole subject, and thus should be covered thoroughly. While doing Sections 1.2 and 1.3, the classic book by Protter and Weinberger [**13**] may be a very good reference. Also, when one reads Section 1.4, some statements concerning the Hopf maximal principle in Section 2.2 can be selected as exercises. The interior gradient estimates of Section 2.4 follow from the same arguments as those in the proof of Proposition 1.31 in Section 1.4.

1.2. Mean Value Properties

We begin this section with the definition of mean value properties. We assume that Ω is a connected domain in $\mathbb{R}^n$.

DEFINITION 1.1 For $u \in C(\Omega)$ we define

(i) u satisfies the first mean value property if

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int\limits_{\partial B_r(x)} u(y) dS_y \quad \text{for any } B_r(x) \subset \Omega;$$

(ii) u satisfies the second mean value property if

$$u(x) = \frac{n}{\omega_n r^n} \int\limits_{B_r(x)} u(y) dy \quad \text{for any } B_r(x) \subset \Omega$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$.

REMARK 1.2. These two definitions are equivalent. In fact, if we write (i) as

$$u(x) r^{n-1} = \frac{1}{\omega_n} \int\limits_{\partial B_r(x)} u(y) dS_y,$$

we may integrate to get (ii). If we write (ii) as

$$u(x)r^n = \frac{n}{\omega_n} \int\limits_{B_r(x)} u(y)dy,$$

we may differentiate to get (i).

REMARK 1.3. We may write the mean value properties in the following equivalent ways:

(i) u satisfies the first mean value property if

$$u(x) = \frac{1}{\omega_n} \int\limits_{|w|=1} u(x+rw)dS_w \quad \text{for any } B_r(x) \subset \Omega;$$

(ii) u satisfies the second mean value property if

$$u(x) = \frac{n}{\omega_n} \int\limits_{|z|\le 1} u(x+rz)dz \quad \text{for any } B_r(x) \subset \Omega.$$

Now we prove the maximum principle for the functions satisfying mean value properties.

PROPOSITION 1.4 *If $u \in C(\bar{\Omega})$ satisfies the mean value property in Ω, then u assumes its maximum and minimum only on $\partial\Omega$ unless u is constant.*

PROOF: We only prove for the maximum. Set

$$\Sigma = \{x \in \Omega : u(x) = M \equiv \max_{\bar{\Omega}} u\} \subset \Omega.$$

It is obvious that Σ is relatively closed. Next we show that Σ is open. For any $x_0 \in \Sigma$, take $\bar{B}_r(x_0) \subset \Omega$ for some $r > 0$. By the mean value property we have

$$M = u(x_0) = \frac{n}{\omega_n r^n} \int\limits_{B_r(x_0)} u(y)dy \le M \frac{n}{\omega_n r^n} \int\limits_{B_r(x_0)} dy = M.$$

This implies $u = M$ in $B_r(x_0)$. Hence Σ is both closed and open in Ω. Therefore either $\Sigma = \phi$ or $\Sigma = \Omega$. □

DEFINITION 1.5 A function $u \in C^2(\Omega)$ is harmonic if $\triangle u = 0$ in Ω.

THEOREM 1.6 *Let $u \in C^2(\Omega)$ be harmonic in Ω. Then u satisfies the mean value property in Ω.*

PROOF: Take any ball $B_r(x) \subset \Omega$. For $\rho \in (0, r)$, we apply the divergence theorem in $B_\rho(x)$ and get

$$(*) \qquad \begin{aligned} \int\limits_{B_\rho(x)} \triangle u(y)dy &= \int\limits_{\partial B_\rho} \frac{\partial u}{\partial \nu} dS = \rho^{n-1} \int\limits_{|w|=1} \frac{\partial u}{\partial \rho}(x+\rho w)dS_w \\ &= \rho^{n-1} \frac{\partial}{\partial \rho} \int\limits_{|w|=1} u(x+\rho w)dS_w. \end{aligned}$$

Hence for harmonic function u we have for any $\rho \in (0, r)$

$$\frac{\partial}{\partial \rho} \int\limits_{|w|=1} u(x + \rho w) dS_w = 0.$$

Integrating from 0 to r we obtain

$$\int\limits_{|w|=1} u(x + rw) dS_w = \int\limits_{|w|=1} u(x) dS_w = u(x)\omega_n$$

or

$$u(x) = \frac{1}{\omega_n} \int\limits_{|w|=1} u(x + rw) dS_w = \frac{1}{\omega_n r^{n-1}} \int\limits_{\partial B_r(x)} u(y) dS_y.$$

□

REMARK 1.7. For a function u satisfying the mean value property, u is not required to be smooth. However a harmonic function is required to be C^2. We prove these two are equivalent.

THEOREM 1.8 *If $u \in C(\Omega)$ has mean value property in Ω, then u is smooth and harmonic in Ω.*

PROOF: Choose $\varphi \in C_0^\infty(B_1(0))$ with $\int_{B_1(0)} \varphi = 1$ and $\varphi(x) = \psi(|x|)$; i.e.,

$$\omega_n \int_0^1 r^{n-1} \psi(r) dr = 1.$$

We define $\varphi_\varepsilon(z) = \frac{1}{\varepsilon^n} \varphi(\frac{z}{\varepsilon})$ for $\varepsilon > 0$. Now for any $x \in \Omega$ consider $\varepsilon <$ dist $(x, \partial\Omega)$. Then we have

$$\begin{aligned}
\int\limits_{\Omega} u(y)\varphi_\varepsilon(y - x) dy &= \int u(x + y)\varphi_\varepsilon(y) dy \\
&= \frac{1}{\varepsilon^n} \int\limits_{|y|<\varepsilon} u(x + y)\varphi\left(\frac{y}{\varepsilon}\right) dy \\
&= \int\limits_{|y|<1} u(x + \varepsilon y)\varphi(y) dy \\
&= \int_0^1 r^{n-1} dr \int\limits_{\partial B_1(0)} u(x + \varepsilon rw)\varphi(rw) dS_w \\
&= \int_0^1 \psi(r) r^{n-1} dr \int\limits_{|w|=1} u(x + \varepsilon rw) dS_w \\
&= u(x)\omega_n \int_0^1 \psi(r) r^{n-1}\, dr = u(x)
\end{aligned}$$

where in the last equality we used the mean value property. Hence we get

$$u(x) = (\varphi_\varepsilon * u)(x) \quad \text{for any } x \in \Omega_\varepsilon = \{y \in \Omega;\ d(y, \partial\Omega) > \varepsilon\}.$$

Therefore u is smooth. Moreover, by formula $(*)$ in the proof of Theorem 1.2 and the mean value property we have

$$\int_{B_r(x)} \Delta u = r^{n-1} \frac{\partial}{\partial r} \int_{|w|=1} u(x + rw) dS_w$$
$$= r^{n-1} \frac{\partial}{\partial r} (\omega_n u(x)) = 0 \quad \text{for any } B_r(x) \subset \Omega.$$

This implies $\Delta u = 0$ in Ω. □

REMARK 1.9. By combining Theorem 1.6 and Theorem 1.8, we conclude that harmonic functions are smooth and satisfy the mean value property. Hence harmonic functions satisfy the maximum principle, a consequence of which is the uniqueness of solution to the following Dirichlet problem in a bounded domain

$$\Delta u = f \quad \text{in } \Omega,$$
$$u = \varphi \quad \text{on } \partial\Omega,$$

for $f \in C(\Omega)$ and $\varphi \in C(\partial\Omega)$. In general uniqueness does not hold for an unbounded domain. Consider the following Dirichlet problem in the unbounded domain Ω

$$\Delta u = 0 \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

First consider the case $\Omega = \{x \in \mathbb{R}^n; |x| > 1\}$. For $n = 2, u(x) = \log|x|$ is a solution. Note $u \to \infty$ as $r \to \infty$. For $n \geq 3$, $u(x) = |x|^{2-n} - 1$ is a solution. Note $u \to -1$ as $r \to \infty$. Hence u is bounded. Next, consider the upper half space $\Omega = \{x \in \mathbb{R}^n; x_n > 0\}$. Then $u(x) = x_n$ is a nontrivial solution, which is unbounded.

In the following we discuss the gradient estimates.

LEMMA 1.10 *Suppose $u \in C(\bar{B}_R)$ is harmonic in $B_R = B_R(x_0)$. Then there holds*

$$|Du(x_0)| \leq \frac{n}{R} \max_{\bar{B}_R} |u|.$$

PROOF: For simplicity we assume $u \in C^1(\bar{B}_R)$. Since u is smooth, then $\Delta(D_{x_i} u) = 0$, that is, $D_{x_i} u$ is also harmonic in B_R. Hence $D_{x_i} u$ satisfies the mean value property. By the divergence theorem we have

$$D_{x_i} u(x_0) = \frac{n}{\omega_n R^n} \int_{B_R(x_0)} D_{x_i} u(y) dy = \frac{n}{\omega_n R^n} \int_{\partial B_R(x_0)} u(y) \nu_i \, dS_y,$$

which implies

$$|D_{x_i} u(x_0)| \leq \frac{n}{\omega_n R^n} \max_{\partial B_R} |u| \cdot \omega_n R^{n-1} \leq \frac{n}{R} \max_{\bar{B}_R} |u|.$$

□

LEMMA 1.11 *Suppose $u \in C(\bar{B}_R)$ is a nonnegative harmonic function in $B_R = B_R(x_0)$. Then there holds*

$$|Du(x_0)| \le \frac{n}{R} u(x_0).$$

PROOF: As before by the divergence theorem and the nonnegativeness of u we have

$$|D_{x_i} u(x_0)| \le \frac{n}{\omega_n R^n} \int_{\partial B_R(x_0)} u(y) dS_y = \frac{n}{R} u(x_0)$$

where in the last equality we used the mean value property. □

COROLLARY 1.12 *A harmonic function in $\mathbb{R}^n$ bounded from above or below is constant.*

PROOF: Suppose u is a harmonic function in $\mathbb{R}^n$. We will prove that u is a constant if $u \ge 0$. In fact, for any $x \in \mathbb{R}^n$ we apply Lemma 1.11 to u in $B_R(x)$ and then let $R \to \infty$. We conclude that $Du(x) = 0$ for any $x \in \mathbb{R}^n$. □

PROPOSITION 1.13 *Suppose $u \in C(\bar{B}_R)$ is harmonic in $B_R = B_R(x_0)$. Then there holds for any multi-index α with $|\alpha| = m$*

$$|D^\alpha u(x_0)| \le \frac{n^m e^{m-1} m!}{R^m} \max_{\bar{B}_R} |u|.$$

PROOF: We prove by induction. It is true for $m = 1$ by Lemma 1.10. Assume it holds for m. Consider $m + 1$. For $0 < \theta < 1$, define $r = (1 - \theta)R \in (0, R)$. We apply Lemma 1.10 to u in B_r and get

$$|D^{m+1} u(x_0)| \le \frac{n}{r} \max_{\bar{B}_r} |D^m u|.$$

By the induction assumption we have

$$\max_{\bar{B}_r} |D^m u| \le \frac{n^m \cdot e^{m-1} \cdot m!}{(R - r)^m} \max_{\bar{B}_R} |u|.$$

Hence we obtain

$$|D^{m+1} u(x_0)| \le \frac{n}{r} \cdot \frac{n^m e^{m-1} m!}{(R - r)^m} \max_{B_R} |u| = \frac{n^{m+1} e^{m-1} m!}{R^{m+1} \theta^m (1 - \theta)} \max_{\bar{B}_R} |u|.$$

Take $\theta = \frac{m}{m+1}$. This implies

$$\frac{1}{\theta^m (1 - \theta)} = \left(1 + \frac{1}{m}\right)^m (m + 1) < e(m + 1).$$

Hence the result is established for any single derivative. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we have

$$\alpha_1! \cdots \alpha_n! \le (|\alpha|)!.$$

□

THEOREM 1.14 *Harmonic function is analytic.*

PROOF: Suppose u is a harmonic function in Ω. For fixed $x \in \Omega$, take $B_{2R}(x) \subset \Omega$ and $h \in \mathbb{R}^n$ with $|h| \le R$. We have by Taylor expansion

$$u(x+h) = u(x) + \sum_{i=1}^{m-1} \frac{1}{i!}\left[\left(h_1 \frac{\partial}{\partial x_1} + \cdots + h_n \frac{\partial}{\partial x_n}\right)^i u\right](x) + R_m(h)$$

where

$$R_m(h) = \frac{1}{m!}\left[\left(h_1 \frac{\partial}{\partial x_1} + \cdots + h_n \frac{\partial}{\partial x_n}\right)^m u\right](x_1 + \theta h_1, \ldots, x_n + \theta h_n)$$

for some $\theta \in (0,1)$. Note $x + h \in B_R(x)$ for $|h| < R$. Hence by Proposition 1.13 we obtain

$$|R_m(h)| \le \frac{1}{m!}|h|^m \cdot n^m \cdot \frac{n^m e^{m-1} m!}{R^m} \max_{\bar{B}_{2R}} |u| \le \left(\frac{|h| n^2 e}{R}\right)^m \max_{\bar{B}_{2R}} |u|.$$

Then for any h with $|h| n^2 e < \frac{R}{2}$ there holds $R_m(h) \to 0$ as $m \to \infty$. □

Next we prove the Harnack inequality.

THEOREM 1.15 *Suppose u is harmonic in Ω. Then for any compact subset K of Ω there exists a positive constant $C = (\Omega, K)$ such that if $u \ge 0$ in Ω, then*

$$\frac{1}{C} u(y) \le u(x) \le C u(y) \quad \textit{for any } x, y \in K.$$

PROOF: By mean value property, we can prove if $B_{4R}(x_0) \subset \Omega$, then

$$\frac{1}{c} u(y) \le u(x) \le c u(y) \quad \text{for any } x, y \in B_R(x_0)$$

where c is a positive constant depending only on n. Now for the given compact subset K, take $x_1, \ldots, x_N \in K$ such that $\{B_R(x_i)\}$ covers K with $4R < \text{dist}(K, \partial\Omega)$. Then we can choose $C = c^N$. □

We finish this section by proving a result, originally due to Weyl. Suppose u is harmonic in Ω. Then integrating by parts we have

$$\int_\Omega u \Delta\varphi = 0 \quad \text{for any } \varphi \in C_0^2(\Omega).$$

The converse is also true.

THEOREM 1.16 *Suppose $u \in C(\Omega)$ satisfies*

$$\int_\Omega u \Delta\varphi = 0 \quad \textit{for any } \varphi \in C_0^2(\Omega). \tag{1.1}$$

Then u is harmonic in Ω.

PROOF: We claim for any $B_r(x) \subset \Omega$ there holds

$$r \int_{\partial B_r(x)} u(y) dS_y = n \int_{B_r(x)} u(y) dy. \tag{1.2}$$

Then we have

$$\begin{aligned}
&\frac{d}{dr}\left(\frac{1}{\omega_n r^{n-1}}\int\limits_{\partial B_r(x)} u(y)dS_y\right)\\
&= \frac{n}{\omega_n}\frac{d}{dr}\left(\frac{1}{r^n}\int\limits_{B_r(x)} u(y)dy\right)\\
&= \frac{n}{\omega_n}\left\{-\frac{n}{r^{n+1}}\int\limits_{B_r(x)} u(y)dy + \frac{1}{r^n}\int\limits_{\partial B_r(x)} u(y)dS_y\right\} = 0.
\end{aligned}$$

This implies

$$\frac{1}{\omega_n r^{n-1}}\int\limits_{\partial B_r(x)} u(y)dS_y = \text{const.}$$

This constant is $u(x)$ if we let $r \to 0$. Hence we have

$$u(x) = \frac{1}{\omega_n r^{n-1}}\int\limits_{\partial B_r(x)} u(y)dS_y \quad \text{for any } B_r(x) \subset \Omega.$$

Next we prove (1.2) for $n \geq 3$. For simplicity we assume that $x = 0$. Set

$$\varphi(y,r) = \begin{cases}(|y|^2 - r^2)^n, & |y| \leq r,\\ 0, & |y| > r,\end{cases}$$

and then $\varphi_k(y,r) = (|y|^2 - r^2)^{n-k}(2(n-k+1)|y|^2 + n(|y|^2 - r^2))$ for $|y| \leq r$ and $k = 2, 3, \ldots, n$. Direct calculation shows $\varphi(\cdot, r) \in C_0^2(\Omega)$ and

$$\Delta_y\varphi(y,r) = \begin{cases}2n\varphi_2(y,r), & |y| \leq r,\\ 0, & |y| > r.\end{cases}$$

By assumption (1.1) we have

$$\int\limits_{B_r(0)} u(y)\varphi_2(y,r)dy = 0.$$

Now we prove if for some $k = 2, 3, \ldots, n-1$,

$$\int\limits_{B_r(0)} u(y)\varphi_k(y,r)dy = 0, \tag{1.3}$$

then

$$\int\limits_{B_r(0)} u(y)\varphi_{k+1}(y,r)dy = 0. \tag{1.4}$$

In fact, we differentiate (1.3) with respect to r and get

$$\int\limits_{\partial B_r(0)} u(y)\varphi_k(y,r)dy + \int\limits_{B_r(0)} u(y)\frac{\partial\varphi_k}{\partial r}(y,r)dy = 0.$$

For $2 \le k < n$, $\varphi_k(y,r) = 0$ for $|y| = r$. Then we have

$$\int_{B_r(0)} u(y) \frac{\partial \varphi_k}{\partial r}(y,r)dy = 0.$$

Direct calculation yields $\frac{\partial \varphi_k}{\partial r}(y,r) = (-2r)(n-k+1)\varphi_{k+1}(y,r)$. Hence we have (1.4). Therefore by taking $k = n-1$ in (1.4) we conclude

$$\int_{B_r(0)} u(y)((n+2)|y|^2 - nr^2)dy = 0.$$

Differentiating with respect to r again we get (1.2). □

1.3. Fundamental Solutions

We begin this section by seeking a harmonic function u, that is, $\Delta u = 0$ in $\mathbb{R}^n$, which depends only on $r = |x-a|$ for some fixed $a \in \mathbb{R}^n$. We set $v(r) = u(x)$. This implies

$$v'' + \frac{n-1}{r}v' = 0$$

and hence

$$v(r) = \begin{cases} c_1 + c_2 \log r, & n = 2, \\ c_3 + c_4 r^{2-n}, & n \ge 3, \end{cases}$$

where c_i are constants for $i = 1, 2, 3, 4$. We are interested in a function with a singularity such that

$$\int_{\partial B_r} \frac{\partial v}{\partial r} dS = 1 \quad \text{for any } r > 0.$$

Hence we set for any fixed $a \in \mathbb{R}^n$

$$\Gamma(a,x) = \frac{1}{2\pi} \log |a-x| \qquad \text{for } n = 2$$
$$\Gamma(a,x) = \frac{1}{\omega_n(2-n)} |a-x|^{2-n} \quad \text{for } n \ge 3.$$

To summarize, we have that for fixed $a \in \mathbb{R}^n$, $\Gamma(a,x)$ is harmonic at $x \ne a$, that is,

$$\Delta_x \Gamma(a,x) = 0 \quad \text{for any } x \ne a$$

and has a singularity at $x = a$. Moreover, it satisfies

$$\int_{\partial B_r(a)} \frac{\partial \Gamma}{\partial n_x}(a,x)dS_x = 1 \quad \text{for any } r > 0.$$

Now we prove the Green's identity.

THEOREM 1.17 *Suppose Ω is a bounded domain in $\mathbb{R}^n$ and that $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Then for any $a \in \Omega$ there holds*

$$u(a) = \int_\Omega \Gamma(a,x)\Delta u(x)dx - \int_{\partial\Omega} \left(\Gamma(a,x)\frac{\partial u}{\partial n_x}(x) - u(x)\frac{\partial \Gamma}{\partial n_x}(a,x)\right)dS_x.$$

REMARK 1.18.

(i) For any $a \in \Omega$, $\Gamma(a,\cdot)$ is integrable in Ω although it has a singularity.
(ii) For $a \notin \overline{\Omega}$, the expression in the right side gives 0.
(iii) By letting $u = 1$ we have $\int_{\partial\Omega} \frac{\partial \Gamma}{\partial n_x}(a,x)dS_x = 1$ for any $a \in \Omega$.

PROOF: We apply Green's formula to u and $\Gamma(a,\cdot)$ in the domain $\Omega \setminus B_r(a)$ for small $r > 0$ and get

$$\int_{\Omega\setminus B_r(a)} (\Gamma\Delta u - u\Delta\Gamma)dx = \int_{\partial\Omega}\left(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n}\right)dS_x - \int_{\partial B_r(a)}\left(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n}\right)dS_x.$$

Note $\Delta\Gamma = 0$ in $\Omega \setminus B_r(a)$. Then we have

$$\int_\Omega \Gamma\Delta u\, dx = \int_{\partial\Omega}\left(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n}\right)dS_x - \lim_{r\to 0}\int_{\partial B_r(a)}\left(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n}\right)dS_x.$$

For $n \geq 3$, we get by definition of Γ

$$\left|\int_{\partial B_r(a)} \Gamma\frac{\partial u}{\partial n}\, dS\right| = \left|\frac{1}{(2-n)\omega_n}r^{2-n}\int_{\partial B_r(a)}\frac{\partial u}{\partial n}\, dS\right| \leq \frac{r}{n-2}\sup_{\partial B_r(a)}|Du| \to 0 \qquad \text{as } r \to 0,$$

$$\int_{\partial B_r(a)} u\frac{\partial\Gamma}{\partial n}\, dS = \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(a)} u\, dS \to u(a) \qquad \text{as } r \to 0.$$

We get the same conclusion for $n = 2$ in the same way. □

REMARK 1.19. We may employ the local version of the Green's identity to get gradient estimates without using the mean value property. Suppose $u \in C(\overline{B}_1)$ is harmonic in B_1. For any fixed $0 < r < R < 1$ choose a cutoff function $\varphi \in C_0^\infty(B_R)$ such that $\varphi = 1$ in B_r and $0 \leq \varphi \leq 1$. Apply the Green's formula to u and $\varphi\Gamma(a,\cdot)$ in $B_1 \setminus B_\rho(a)$ for $a \in B_r$ and ρ small enough. We proceed as in the proof of Theorem 1.17 and obtain

$$u(a) = -\int_{r<|x|<R} u(x)\Delta_x(\varphi(x)\Gamma(a,x))dx \quad \text{for any } a \in B_r(0).$$

Hence one may prove (without using the mean value property)

$$\sup_{B_{1/2}} |u| \le c\Big(\int_{B_1} |u|^p\Big)^{1/p} \quad \text{and} \quad \sup_{B_{1/2}} |Du| \le c \max_{B_1} |u|$$

where c is a constant depending only on n.

Now we begin to discuss the Green's functions. Suppose Ω is a bounded domain in $\mathbb{R}^n$. Let $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. We have by Theorem 1.17 for any $x \in \Omega$

$$u(x) = \int_{\Omega} \Gamma(x,y)\Delta u(y)dy - \int_{\partial\Omega} \Big(\Gamma(x,y)\frac{\partial u}{\partial n_y}(y) - u(y)\frac{\partial \Gamma}{\partial n_y}(x,y)\Big)dS_y.$$

If u solves the Dirichlet boundary value problem

$$(*) \qquad \begin{cases} \Delta u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

for some $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$, then u can be expressed in terms of f and φ, with one *unknown term*. We want to eliminate this term by adjusting Γ.

For any fixed $x \in \Omega$, consider

$$\gamma(x,y) = \Gamma(x,y) + \Phi(x,y)$$

for some $\Phi(x,\cdot) \in C^2(\bar{\Omega})$ with $\Delta_y \Phi(x,y) = 0$ in Ω. Then Theorem 1.17 can be expressed as follows for any $x \in \Omega$

$$u(x) = \int_{\Omega} \gamma(x,y)\Delta u(y)dy - \int_{\partial\Omega} \Big(\gamma(x,y)\frac{\partial u}{\partial n_y}(y) - u(y)\frac{\partial \gamma}{\partial n_y}(x,y)\Big)dS_y$$

since the extra $\Phi(x,\cdot)$ is harmonic. Now by choosing Φ appropriately, we are led to the important concept of Green's function.

For each fixed $x \in \Omega$ choose $\Phi(x,\cdot) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} \Delta_y \Phi(x,y) = 0 & \text{for } y \in \Omega, \\ \Phi(x,y) = -\Gamma(x,y) & \text{for } y \in \partial\Omega. \end{cases}$$

Denote the resulting $\gamma(x,y)$ by $G(x,y)$, which is called Green's function. If such a G exists, then the solution u to the Dirichlet problem $(*)$ can be expressed as

$$u(x) = \int_{\Omega} G(x,y)f(y)dy + \int_{\partial\Omega} \varphi(y)\frac{\partial G}{\partial n_y}(x,y)dS_y.$$

Note that Green's function $G(x,y)$ is defined as a function of $y \in \bar{\Omega}$ for each fixed $x \in \Omega$.

Now we discuss some properties of G as a function of x and y. Our first observation is that the Green's function is unique. This is proved by the maximum principle since the difference of two Green's functions are harmonic in Ω with zero boundary value. In fact, we have more.

PROPOSITION 1.20 *Green's function $G(x, y)$ is symmetric in $\Omega \times \Omega$; that is, $G(x, y) = G(y, x)$ for $x \neq y \in \Omega$.*

PROOF: Pick $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$. Choose $r > 0$ small such that $B_r(x_1) \cap B_r(x_2) = \varnothing$. Set $G_1(y) = G(x_1, y)$ and $G_2(y) = G(x_2, y)$. We apply Green's formula in $\Omega \setminus B_r(x_1) \cup B_r(x_2)$ and get

$$\begin{aligned}\int_{\Omega \setminus B_r(x_1) \cup B_r(x_2)} (G_1 \Delta G_2 - G_2 \Delta G_1) = &\int_{\partial\Omega} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) dS \\ &- \int_{\partial B_r(x_1)} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) dS \\ &- \int_{\partial B_r(x_2)} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) dS.\end{aligned}$$

Since G_i is harmonic for $y \neq x_i$, $i = 1, 2$, and vanishes on $\partial\Omega$, we have

$$\int_{\partial B_r(x_1)} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) dS + \int_{\partial B_r(x_2)} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) dS = 0.$$

Note the left side has the same limit as the left side in the following as $r \to 0$:

$$\int_{\partial B_r(x_1)} \left(\Gamma \frac{\partial G_2}{\partial n} - G_2 \frac{\partial \Gamma}{\partial n}\right) dS + \int_{\partial B_r(x_2)} \left(G_1 \frac{\partial \Gamma}{\partial n} - \Gamma \frac{\partial G_1}{\partial n}\right) dS = 0.$$

Since

$$\int_{\partial B_r(x_1)} \Gamma \frac{\partial G_2}{\partial n} \, dS \to 0, \qquad \int_{\partial B_r(x_2)} \Gamma \frac{\partial G_1}{\partial n} \, dS \to 0 \quad \text{as } r \to 0,$$

$$\int_{\partial B_r(x_1)} G_2 \frac{\partial \Gamma}{\partial n} \, dS \to G_2(x_1), \qquad \int_{\partial B_r(x_2)} G_1 \frac{\partial \Gamma}{\partial n} \, dS \to G_1(x_2) \quad \text{as } r \to 0,$$

we obtain $G_2(x_1) - G_1(x_2) = 0$ or equivalently $G(x_2, x_1) = G(x_1, x_2)$. □

PROPOSITION 1.21 *There holds for $x, y \in \Omega$ with $x \neq y$*

$$\begin{aligned} 0 > G(x, y) > \Gamma(x, y) & \qquad \textit{for } n \geq 3 \\ 0 > G(x, y) > \Gamma(x, y) - \frac{1}{2\pi} \log \operatorname{diam}(\Omega) & \qquad \textit{for } n = 2. \end{aligned}$$

PROOF: Fix $x \in \Omega$ and write $G(y) = G(x, y)$. Since $\lim_{y \to x} G(y) = -\infty$ then there exists an $r > 0$ such that $G(y) < 0$ in $B_r(x)$. Note that G is harmonic in $\Omega \setminus B_r(x)$ with $G = 0$ on $\partial\Omega$ and $G < 0$ on $\partial B_r(x)$. The maximum principle implies $G(y) < 0$ in $\Omega \setminus B_r(x)$ for such $r > 0$. Next, consider the other part of the inequality. Recall the definition of the Green's function

$$G(x, y) = \Gamma(x, y) + \Phi(x, y) \quad \text{where } \begin{cases} \Delta\Phi = 0 & \text{in } \Omega, \\ \Phi = -\Gamma & \text{on } \partial\Omega. \end{cases}$$

For $n \geq 3$, we have

$$\Gamma(x, y) = \frac{1}{(2-n)\omega_n}|x-y|^{2-n} < 0 \quad \text{for } y \in \partial\Omega,$$

which implies $\Phi(x, \cdot) > 0$ on $\partial\Omega$. By the maximum principle, we have $\Phi > 0$ in Ω. For $n = 2$ we have

$$\Gamma(x, y) = \frac{1}{2\pi}\log|x-y| \leq \frac{1}{2\pi}\log \operatorname{diam}(\Omega) \quad \text{for } y \in \partial\Omega.$$

Hence the maximum principle implies $\Phi > -\frac{1}{2\pi}\log\operatorname{diam}(\Omega)$ in Ω. □

We may calculate the Green's functions for some special domains.

PROPOSITION 1.22 *The Green's function for the ball $B_R(0)$ is given by*

(i) $$G(x, y) = \frac{1}{(2-n)\omega_n}\left(|x-y|^{2-n} - \left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|^{2-n}\right) \quad \textit{for } n \geq 3,$$

(ii) $$G(x, y) = \frac{1}{2\pi}\left(\log|x-y| - \log\left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|\right) \quad \textit{for } n = 2.$$

PROOF: Fix $x \neq 0$ with $|x| < R$. Consider $X \in \mathbb{R}^n \setminus \bar{B}_R$ with X the multiple of x and $|X| \cdot |x| = R^2$, that is, $X = \frac{R^2}{|x|^2}x$. In other words, X and x are reflexive with respect to the sphere ∂B_R. Note the map $x \longmapsto X$ is conformal; that is, it preserves angles. If $|y| = R$, we have by similarity of triangles

$$\frac{|x|}{R} = \frac{R}{|X|} = \frac{|y-x|}{|y-X|},$$

which implies

$$|y-x| = \frac{|x|}{R}|y-X| = \left|\frac{|x|}{R}y - \frac{R}{|x|}x\right| \quad \text{for any } y \in \partial B_R. \tag{1.5}$$

Therefore, in order to have zero boundary value, we take for $n \geq 3$

$$G(x, y) = \frac{1}{(2-n)\omega_n}\left(\frac{1}{|x-y|^{n-2}} - \left(\frac{R}{|x|}\right)^{n-2}\frac{1}{|y-X|^{n-2}}\right).$$

The case $n = 2$ is similar. □

Next, we calculate the normal derivative of Green's function on the sphere.

COROLLARY 1.23 *Suppose G is the Green's function in $B_R(0)$. Then there holds*

$$\frac{\partial G}{\partial n}(x, y) = \frac{R^2 - |x|^2}{\omega_n R|x-y|^n} \quad \textit{for any } x \in B_R \textit{ and } y \in \partial B_R.$$

PROOF: We just consider the case $n \geq 3$. Recall with $X = R^2 x/|x|^2$

$$G(x, y) = \frac{1}{(2-n)\omega_n}\left(|x-y|^{2-n} - \left(\frac{R}{|x|}\right)^{n-2}|y-X|^{2-n}\right)$$
for $x \in B_R$, $y \in \partial B_R$.

Hence we have for such x and y

$$D_{y_i}G(x,y) = -\frac{1}{\omega_n}\left(\frac{x_i - y_i}{|x-y|^n} - \left(\frac{R}{|x|}\right)^{n-2} \cdot \frac{X_i - y_i}{|X-y|^n}\right) = \frac{y_i}{\omega_n R^2}\frac{R^2 - |x|^2}{|x-y|^n}$$

by (1.5) in the proof of Proposition 1.22. We obtain with $n_i = \frac{y_i}{R}$ for $|y| = R$

$$\frac{\partial G}{\partial n}(x,y) = \sum_{i=1}^{n} n_i D_{y_i}G(x,y) = \frac{1}{w_n R}\cdot\frac{R^2 - |x|^2}{|x-y|^n}.$$

□

Denote by $K(x,y)$ the function in Corollary 1.23 for $x \in \Omega, y \in \partial\Omega$. It is called a Poisson kernel and has the following properties:

(i) $K(x,y)$ is smooth for $x \neq y$;
(ii) $K(x,y) > 0$ for $|x| < R$;
(iii) $\int_{|y|=R} K(x,y)dS_y = 1$ for any $|x| < R$.

The following result gives the existence of harmonic functions in balls with prescribed Dirichlet boundary value.

THEOREM 1.24 (Poisson Integral Formula) *For $\varphi \in C(\partial B_R(0))$, the function u defined by*

$$u(x) = \begin{cases} \int_{\partial B_R(0)} K(x,y)\varphi(y)dS_y, & |x| < R, \\ \varphi(x), & |x| = R, \end{cases}$$

satisfies $u \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ and

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

For the proof, see [**9**, pp. 107–108].

REMARK 1.25. In the Poisson integral formula, by letting $x = 0$, we have

$$u(0) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(0)} \varphi(y)dS_y,$$

which is the mean value property.

LEMMA 1.26 (Harnack's Inequality) *Suppose u is harmonic in $B_R(x_0)$ and $u \geq 0$. Then there holds*

$$\left(\frac{R}{R+r}\right)^{n-2}\frac{R-r}{R+r}u(x_0) \leq u(x) \leq \left(\frac{R}{R-r}\right)^{n-2}\frac{R+r}{R-r}u(x_0)$$

where $r = |x - x_0| < R$.

PROOF: We may assume $x_0 = 0$ and $u \in C(\bar{B}_R)$. Note that u is given by the Poisson integral formula

$$u(x) = \frac{1}{\omega_n R}\int_{\partial B_R} \frac{R^2 - |x|^2}{|y-x|^n}u(y)dS_y.$$

Since $R - |x| \le |y - x| \le R + |x|$ for $|y| = R$, we have

$$\frac{1}{\omega_n R} \cdot \frac{R-|x|}{R+|x|}\left(\frac{1}{R+|x|}\right)^{n-2} \int_{\partial B_R} u(y) dS_y \le u(x) \le \frac{1}{\omega_n R} \cdot \frac{R+|x|}{R-|x|}\left(\frac{1}{R-|x|}\right)^{n-2} \int_{\partial B_R} u(y) dS_y.$$

The mean value property implies

$$u(0) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R} u(y) dS_y.$$

This finishes the proof. □

COROLLARY 1.27 *If harmonic function u in $\mathbb{R}^n$ is bounded above or below, then $u \equiv$ const.*

PROOF: We assume $u \ge 0$ in $\mathbb{R}^n$. Take any point $x \in \mathbb{R}^n$ and apply Lemma 1.26 to any ball $B_R(0)$ with $R > |x|$. We obtain

$$\left(\frac{R}{R+|x|}\right)^{n-2} \frac{R-|x|}{R+|x|} u(0) \le u(x) \le \left(\frac{R}{R-|x|}\right)^{n-2} \frac{R+|x|}{R-|x|} u(0),$$

which implies $u(x) = u(0)$ by letting $R \to +\infty$. □

Next we prove a result concerning the removable singularity.

THEOREM 1.28 *Suppose u is harmonic in $B_R \setminus \{0\}$ and satisfies*

$$u(x) = \begin{cases} o(\log|x|), & n = 2, \\ o(|x|^{2-n}), & n \ge 3, \end{cases} \quad \text{as } |x| \to 0.$$

Then u can be defined at 0 so that it is C^2 and harmonic in B_R.

PROOF: Assume u is continuous in $0 < |x| \le R$. Let v solve

$$\begin{cases} \Delta v = 0 & \text{in } B_R, \\ v = u & \text{on } \partial B_R. \end{cases}$$

We will prove $u = v$ in $B_R \setminus \{0\}$. Set $w = v - u$ in $B_R \setminus \{0\}$ and $M_r = \max_{\partial B_r} |w|$. We prove for $n \ge 3$. It is obvious that

$$|w(x)| \le M_r \cdot \frac{r^{n-2}}{|x|^{n-2}} \quad \text{on } \partial B_r.$$

Note w and $\frac{1}{|x|^{n-2}}$ are harmonic in $B_R \setminus B_r$. Hence the maximum principle implies

$$|w(x)| \le M_r \cdot \frac{r^{n-2}}{|x|^{n-2}} \quad \text{for any } x \in B_R \setminus B_r$$

where $M_r = \max_{\partial B_r} |v - u| \le \max_{\partial B_r} |v| + \max_{\partial B_r} |u| \le M + \max_{\partial B_r} |u|$ with $M = \max_{\partial B_R} |u|$. Hence we have for each fixed $x \neq 0$

$$|w(x)| \le \frac{r^{n-2}}{|x|^{n-2}} M + \frac{1}{|x|^{n-2}} r^{n-2} \max_{\partial B_r} |u| \to 0 \quad \text{as } r \to 0,$$

that is $w = 0$ in $B_R \setminus \{0\}$. □

1.4. Maximum Principles

In this section we will use the maximum principle to derive the interior gradient estimate and the Harnack inequality.

THEOREM 1.29 *Suppose $u \in C^2(B_1) \cap C(\bar{B}_1)$ is a subharmonic function in B_1; that is, $\triangle u \ge 0$. Then there holds*

$$\sup_{B_1} u \le \sup_{\partial B_1} u.$$

PROOF: For $\varepsilon > 0$ we consider $u_\varepsilon(x) = u(x) + \varepsilon|x|^2$ in B_1. Then simple calculation yields

$$\triangle u_\varepsilon = \triangle u + 2n\varepsilon \ge 2n\varepsilon > 0.$$

It is easy to see, by a contradiction argument, that u_ε cannot have an interior maximum, in particular,

$$\sup_{B_1} u_\varepsilon \le \sup_{\partial B_1} u_\varepsilon.$$

Therefore we have

$$\sup_{B_1} u \le \sup_{B_1} u_\varepsilon \le \sup_{\partial B_1} u + \varepsilon.$$

We finish the proof by letting $\varepsilon \to 0$. □

REMARK 1.30. The result still holds if B_1 is replaced by any bounded domain.

The next result is the interior gradient estimate for harmonic functions. The method is due to Bernstein back in 1910.

PROPOSITION 1.31 *Suppose u is harmonic in B_1. Then there holds*

$$\sup_{B_{1/2}} |Du| \le c \sup_{\partial B_1} |u|$$

where $c = c(n)$ is a positive constant. In particular, for any $\alpha \in [0, 1]$ there holds

$$|u(x) - u(y)| \le c|x - y|^\alpha \sup_{\partial B_1} |u| \quad \textit{for any } x, y \in B_{1/2}$$

where $c = c(n, \alpha)$ is a positive constant.

PROOF: Direct calculation shows that

$$\triangle(|Du|^2) = 2 \sum_{i,j=1}^{n} (D_{ij}u)^2 + 2\sum_{i=1}^{n} D_i u D_i(\triangle u) = 2 \sum_{i,j=1}^{n} (D_{ij}u)^2$$

where we used $\triangle u = 0$ in B_1. Hence $|Du|^2$ is a subharmonic function. To get interior estimates we need a cutoff function. For any $\varphi \in C_0^1(B_1)$ we have

$$\triangle(\varphi|Du|^2) = (\triangle\varphi)|Du|^2 + 4\sum_{i,j=1}^n D_i\varphi D_j u D_{ij}u + 2\varphi\sum_{i,j=1}^n (D_{ij}u)^2.$$

By taking $\varphi = \eta^2$ for some $\eta \in C_0^1(B_1)$ with $\eta \equiv 1$ in $B_{1/2}$, we obtain by the Hölder inequality

$$\begin{aligned}\triangle(\eta^2|Du|^2) &= 2\eta\triangle\eta|Du|^2 + 2|D\eta|^2|Du|^2 \\ &\quad + 8\eta\sum_{i,j=1}^n D_i\eta D_j u D_{ij}u + 2\eta^2\sum_{i,j=1}^n (D_{ij}u)^2 \\ &\ge (2\eta\triangle\eta - 6|D\eta|^2)|Du|^2 \ge -C|Du|^2\end{aligned}$$

where C is a positive constant depending only on η. Note that $\triangle(u^2) = 2|Du|^2 + 2u\triangle u = 2|Du|^2$ since u is harmonic. By taking α large enough we get

$$\triangle(\eta^2|Du|^2 + \alpha u^2) \ge 0.$$

We may apply Theorem 1.29 (the maximum principle) to get the result. $\square$

Next we derive the Harnack inequality.

LEMMA 1.32 *Suppose u is a nonnegative harmonic function in B_1. Then there holds*

$$\sup_{B_{1/2}} |D\log u| \le C$$

where $C = C(n)$ is a positive constant.

PROOF: We may assume $u > 0$ in B_1. Set $v = \log u$. Then direct calculation shows

$$\triangle v = -|Dv|^2.$$

We need the interior gradient estimate on v. Set $w = |Dv|^2$. Then we get

$$\triangle w + 2\sum_{i=1}^n D_i v D_i w = 2\sum_{i,j=1}^n (D_{ij}v)^2.$$

As before we need a cutoff function. First note

$$\sum_{i,j=1}^n (D_{ij}v)^2 \ge \sum_i (D_{ii}v)^2 \ge \frac{1}{n}(\triangle v)^2 = \frac{|Dv|^4}{n} = \frac{w^2}{n}. \tag{1.6}$$

Take a nonnegative function $\varphi \in C_0^1(B_1)$. We obtain by the Hölder inequality

$$\begin{aligned}
&\triangle(\varphi w) + 2\sum_{i=1}^n D_i v D_i(\varphi w) \\
&= 2\varphi \sum_{i,j=1}^n (D_{ij}v)^2 + 4\sum_{i,j=1}^n D_i\varphi D_j v D_{ij} v + 2w\sum_{i=1}^n D_i\varphi D_i v + (\triangle\varphi) w \\
&\geq \varphi \sum_{i,j=1}^n (D_{ij}v)^2 - 2|D\varphi||Dv|^3 - \left(|\triangle\varphi| + C\frac{|D\varphi|^2}{\varphi}\right)|Dv|^2
\end{aligned}$$

if φ is chosen such that $|D\varphi|^2/\varphi$ is bounded in B_1. Choose $\varphi = \eta^4$ for some $\eta \in C_0^1(B_1)$. Hence for such fixed η we obtain by (1.1)

$$\begin{aligned}
&\triangle(\eta^4 w) + 2\sum_{i=1}^n D_i v D_i(\eta^4 w) \\
&\geq \frac{1}{n}\eta^4|Dv|^4 - C\eta^3|D\eta||Dv|^3 - 4\eta^2(\eta\triangle\eta + C|D\eta|^2)|Dv|^2 \\
&\geq \frac{1}{n}\eta^4|Dv|^4 - C\eta^3|Dv|^3 - C\eta^2|Dv|^2
\end{aligned}$$

where C is a positive constant depending only on n and η. Hence we get by the Hölder inequality

$$\triangle(\eta^4 w) + 2\sum_{i=1}^n D_i v D_i(\eta^4 w) \geq \frac{1}{n}\eta^4 w^2 - C$$

where C is a positive constant depending only on n and η.

Suppose $\eta^4 w$ attains its maximum at $x_0 \in B_1$. Then $D(\eta^4 w) = 0$ and $\triangle(\eta^4 w) \leq 0$ at x_0. Hence there holds

$$\eta^4 w^2(x_0) \leq C(n,\eta).$$

If $w(x_0) \geq 1$, then $\eta^4 w(x_0) \leq C(n)$. Otherwise $\eta^4 w(x_0) \leq w(x_0) \leq \eta^4(x_0)$. In both cases we conclude

$$\eta^4 w \leq C(n,\eta) \quad \text{in } B_1.$$

□

COROLLARY 1.33 *Suppose u is a nonnegative harmonic function in B_1. Then there holds*

$$u(x_1) \leq C u(x_2) \quad \textit{for any } x_1, x_2 \in B_{1/2}$$

where C is a positive constant depending only on n.

PROOF: We may assume $u > 0$ in B_1. For any $x_1, x_2 \in B_{1/2}$ by simple integration we obtain with Lemma 1.32

$$\log\frac{u(x_1)}{u(x_2)} \leq |x_1 - x_2| \int_0^1 |D\log u(tx_2 + (1-t)x_1)|dt \leq C|x_1 - x_2|.$$

□

Next we prove a quantitative Hopf lemma.

PROPOSITION 1.34 *Suppose $u \in C(\bar{B}_1)$ is a harmonic function in $B_1 = B_1(0)$. If $u(x) < u(x_0)$ for any $x \in \bar{B}_1$ and some $x_0 \in \partial B_1$, then there holds*

$$\frac{\partial u}{\partial n}(x_0) \geq C(u(x_0) - u(0))$$

where C is a positive constant depending only on n.

PROOF: Consider a positive function v in B_1 defined by

$$v(x) = e^{-\alpha|x|^2} - e^{-\alpha}.$$

It is easy to see

$$\triangle v(x) = e^{-\alpha|x|^2}(-2\alpha n + 4\alpha^2|x|^2) > 0 \quad \text{for any } |x| \geq \tfrac{1}{2}$$

if $\alpha \geq 2n + 1$. Hence for such fixed α the function v is subharmonic in the region $A = B_1 \setminus B_{1/2}$. Now define for $\varepsilon > 0$

$$h_\varepsilon(x) = u(x) - u(x_0) + \varepsilon v(x).$$

This is also a subharmonic function, that is, $\triangle h_\varepsilon \geq 0$ in A. Obviously $h_\varepsilon \leq 0$ on ∂B_1 and $h_\varepsilon(x_0) = 0$. Since $u(x) < u(x_0)$ for $|x| = \frac{1}{2}$ we may take $\varepsilon > 0$ small such that $h_\varepsilon(x) < 0$ for $|x| = \frac{1}{2}$. Therefore by Theorem 1.29 h_ε assumes at the point x_0 its maximum in A. This implies

$$\frac{\partial h_\varepsilon}{\partial n}(x_0) \geq 0 \quad \text{or} \quad \frac{\partial u}{\partial n}(x_0) \geq -\varepsilon\frac{\partial v}{\partial n}(x_0) = 2\alpha\varepsilon e^{-\alpha} > 0.$$

Note that so far we have only used the subharmonicity of u. We estimate ε as follows. Set $w(x) = u(x_0) - u(x) > 0$ in B_1. Obviously w is a harmonic function in B_1. By Corollary 1.33 (the Harnack inequality) there holds

$$\inf_{B_{1/2}} w \geq c(n)w(0) \quad \text{or} \quad \max_{B_{1/2}} u \leq u(x_0) - c(n)(u(x_0) - u(0)).$$

Hence we may take

$$\varepsilon = \delta c(n)(u(x_0) - u(0))$$

for δ small, depending on n. This finishes the proof. □

To finish this section we prove a global Hölder continuity result.

LEMMA 1.35 *Suppose $u \in C(\bar{B}_1)$ is a harmonic function in B_1 with $u = \varphi$ on ∂B_1. If $\varphi \in C^\alpha(\partial B_1)$ for some $\alpha \in (0, 1)$, then $u \in C^{\alpha/2}(\bar{B}_1)$. Moreover, there holds*

$$\|u\|_{C^{\alpha/2}(\bar{B}_1)} \leq C\|\varphi\|_{C^\alpha(\partial B_1)}$$

where C is a positive constant depending only on n and α.

PROOF: First the maximum principle implies that $\inf_{\partial B_1} \varphi \leq u \leq \sup_{\partial B_1} \varphi$ in B_1. Next we claim that for any $x_0 \in \partial B_1$ there holds

$$\sup_{x \in B_1} \frac{|u(x) - u(x_0)|}{|x - x_0|^{\alpha/2}} \leq 2^{\alpha/2} \sup_{x \in \partial B_1} \frac{|\varphi(x) - \varphi(x_0)|}{|x - x_0|^\alpha}. \tag{1.7}$$

Lemma 1.35 follows easily from (1.7). For any $x, y \in B_1$, set $d_x = \operatorname{dist}(x, \partial B_1)$ and $d_y = \operatorname{dist}(y, \partial B_1)$. Suppose $d_y \le d_x$. Take $x_0, y_0 \in \partial B_1$ such that $|x - x_0| = d_x$ and $|y - y_0| = d_y$. Assume first that $|x - y| \le d_x/2$. Then $y \in \overline{B}_{d_x/2}(x) \subset B_{d_x}(x) \subset B_1$. We apply Proposition 1.31 (scaled version) to $u - u(x_0)$ in $B_{d_x}(x)$ and get by (1.7)

$$d_x^{\alpha/2} \frac{|u(x) - u(y)|}{|x - y|^{\alpha/2}} \le C|u - u(x_0)|_{L^\infty(B_{d_x}(x))} \le C d_x^{\alpha/2} \|\varphi\|_{C^\alpha(\partial B_1)}.$$

Hence we obtain

$$|u(x) - u(y)| \le C|x - y|^{\alpha/2} \|\varphi\|_{C^\alpha(\partial B_1)}.$$

Assume now that $d_y \le d_x \le 2|x - y|$. Then by (1.7) again we have

$$\begin{aligned} |u(x) - u(y)| &\le |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y_0) - u(y)| \\ &\le C(d_x^{\alpha/2} + |x_0 - y_0|^{\alpha/2} + d_y^{\alpha/2}) \|\varphi\|_{C^\alpha(\partial B_1)} \\ &\le C|x - y|^{\alpha/2} \|\varphi\|_{C^\alpha(\partial B_1)} \end{aligned}$$

since $|x_0 - y_0| \le d_x + |x - y| + d_y \le 5|x - y|$.

In order to prove (1.7) we assume $B_1 = B_1((1, 0, \dots, 0))$, $x_0 = 0$, and $\varphi(0) = 0$. Define $K = \sup_{x \in \partial B_1} |\varphi(x)|/|x|^\alpha$. Note $|x|^2 = 2x_1$ for $x \in \partial B_1$. Therefore for $x \in \partial B_1$ there holds

$$\varphi(x) \le K|x|^\alpha \le 2^{\alpha/2} K x_1^{\alpha/2}.$$

Define $v(x) = 2^{\alpha/2} K x_1^{\alpha/2}$ in B_1. Then we have

$$\triangle v(x) = 2^{\alpha/2} K \cdot \frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right) x_1^{\alpha/2 - 2} < 0 \quad \text{in } B_1.$$

Theorem 3.1 implies

$$u(x) \le v(x) = 2^{\alpha/2} K x_1^{\alpha/2} \le 2^{\alpha/2} K |x|^{\alpha/2} \quad \text{for any } x \in B_1.$$

Considering $-u$ similarly, we get

$$|u(x)| \le 2^{\alpha/2} K |x|^{\alpha/2} \quad \text{for any } x \in B_1.$$

This proves (1.7). □

1.5. Energy Method

In this section we discuss harmonic functions by using the energy method. In general we assume throughout this section that $a_{ij} \in C(B_1)$ satisfies

$$\lambda |\xi|^2 \le a_{ij}(x) \xi_i \xi_j \le \Lambda |\xi|^2 \quad \text{for any } x \in B_1 \text{ and } \xi \in \mathbb{R}^n$$

for some positive constants λ and Λ. We consider the function $u \in C^1(B_1)$ satisfying

$$\int_{B_1} a_{ij} D_i u D_j \varphi = 0 \quad \text{for any } \varphi \in C_0^1(B_1).$$

It is easy to check by integration by parts that the harmonic functions satisfy the above equation for $a_{ij} = \delta_{ij}$.

LEMMA 1.36 (Cacciopolli's Inequality) *Suppose $u \in C^1(B_1)$ satisfies*

$$\int_{B_1} a_{ij} D_i u D_j \varphi = 0 \quad \textit{for any } \varphi \in C_0^1(B_1).$$

Then for any function $\eta \in C_0^1(B_1)$, we have

$$\int_{B_1} \eta^2 |Du|^2 \le C \int_{B_1} |D\eta|^2 u^2$$

where C is a positive constant depending only on λ and Λ.

PROOF: For any $\eta \in C_0^1(B_1)$ set $\varphi = \eta^2 u$. Then we have

$$\lambda \int_{B_1} \eta^2 |Du|^2 \le \Lambda \int_{B_1} \eta |u||D\eta||Du|.$$

We obtain the result by the Hölder inequality. □

COROLLARY 1.37 *Let u be as in Lemma* 1.36. *Then for any* $0 \le r < R \le 1$ *there holds*

$$\int_{B_r} |Du|^2 \le \frac{C}{(R-r)^2} \int_{B_R} u^2$$

where C is a positive constant depending only on λ and Λ.

PROOF: Take η such that $\eta = 1$ on B_r, $\eta = 0$ outside B_R, and $|D\eta| \le 2(R-r)^{-1}$. □

COROLLARY 1.38 *Let u be as in Lemma* 1.36. *Then for any* $0 < R \le 1$ *there hold*

$$\int_{B_{R/2}} u^2 \le \theta \int_{B_R} u^2 \quad \textit{and} \quad \int_{B_{R/2}} |Du|^2 \le \theta \int_{B_R} |Du|^2$$

where $\theta = \theta(n, \lambda, \Lambda) \in (0,1)$.

PROOF: Take $\eta \in C_0^1(B_R)$ with $\eta = 1$ on $B_{R/2}$ and $|D\eta| \le 2R^{-1}$. Then Lemma 1.36 yields

$$\int_{B_R} |D(\eta u)|^2 \le C \int_{B_R} |D\eta|^2 u^2 \le \frac{C}{R^2} \int_{B_R \setminus B_{R/2}} u^2$$

by noting $D\eta = 0$ in $B_{R/2}$. Hence by the Poincaré inequality we get

$$\int_{B_R} (\eta u)^2 \le c(n) R^2 \int_{B_R} |D(\eta u)|^2.$$

Therefore we obtain

$$\int_{B_{R/2}} u^2 \le C \int_{B_R \setminus B_{R/2}} u^2, \quad \text{which implies} \quad (C+1) \int_{B_{R/2}} u^2 \le C \int_{B_R} u^2.$$

For the second inequality, observe that Lemma 1.36 holds for $u-a$ for arbitrary constant a. Then as before we have

$$\int_{B_R} \eta^2 |Du|^2 \le C \int_{B_R} |D\eta|^2 (u-a)^2 \le \frac{C}{R^2} \int_{B_R \setminus B_{R/2}} (u-a)^2.$$

The Poincaré inequality implies with $a = |B_R \setminus B_{R/2}|^{-1} \int_{B_R \setminus B_{R/2}} u$

$$\int_{B_R \setminus B_{R/2}} (u-a)^2 \le c(n) R^2 \int_{B_R \setminus B_{R/2}} |Du|^2.$$

Hence we obtain

$$\int_{B_{R/2}} |Du|^2 \le C \int_{B_R \setminus B_{R/2}} |Du|^2;$$

in particular,

$$(C+1) \int_{B_{R/2}} |Du|^2 \le C \int_{B_R} |Du|^2.$$

□

REMARK 1.39. Corollary 1.38 implies, in particular, that a harmonic function in $\mathbb{R}^n$ with finite L^2-norm is identically 0 and that a harmonic function in $\mathbb{R}^n$ with finite Dirichlet integral is constant.

REMARK 1.40. By iterating the result in Corollary 1.38, we have the following estimates. Let u be in Lemma 1.36. Then for any $0 < \rho < r \le 1$ there hold

$$\int_{B_\rho} u^2 \le C \left(\frac{\rho}{r}\right)^\mu \int_{B_r} u^2 \quad \text{and} \quad \int_{B_\rho} |Du|^2 \le C \left(\frac{\rho}{r}\right)^\mu \int_{B_r} |Du|^2$$

for some positive constant $\mu = \mu(n, \lambda, \Lambda)$. Later on we will prove that we can take $\mu \in (n-2, n)$ in the second inequality. For harmonic functions we have better results.

LEMMA 1.41 *Suppose* $\{a_{ij}\}$ *is a constant positive definite matrix with*

$$\lambda |\xi|^2 \le a_{ij} \xi_i \xi_j \le \Lambda |\xi|^2 \quad \textit{for any } \xi \in \mathbb{R}^n$$

for some constants $0 < \lambda \le \Lambda$. *Suppose* $u \in C^1(B_1)$ *satisfies*

$$\int_{B_1} a_{ij} D_i u D_j \varphi = 0 \quad \textit{for any } \varphi \in C_0^1(B_1). \tag{1.8}$$

Then for any $0 < \rho \le r$, *there hold*

$$\int_{B_\rho} |u|^2 \le c\left(\frac{\rho}{r}\right)^n \int_{B_r} |u|^2, \tag{1.9}$$

$$\int_{B_\rho} |u-u_\rho|^2 \le c\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |u-u_r|^2, \tag{1.10}$$

where $c = c(\lambda, \Lambda)$ *is a positive constant and* u_r *denotes the average of* u *in* B_r.

PROOF: By dilation, consider $r = 1$. We restrict our consideration to the range $\rho \in (0, \frac{1}{2}]$, since (1.9) and (1.10) are trivial for $\rho \in (\frac{1}{2}, 1]$. □

CLAIM.

$$|u|^2_{L^\infty(B_{1/2})} + |Du|^2_{L^\infty(B_{1/2})} \le c(\lambda, \Lambda) \int_{B_1} |u|^2.$$

Therefore for $\rho \in (0, \frac{1}{2}]$

$$\int_{B_\rho} |u|^2 \le \rho^n |u|^2_{L^\infty(B_{1/2})} \le c\rho^n \int_{B_1} |u|^2$$

and

$$\int_{B_\rho} |u-u_\rho|^2 \le \int_{B_\rho} |u-u(0)|^2 \le \rho^{n+2} |Du|^2_{L^\infty(B_{1/2})} \le c\rho^{n+2} \int_{B_1} |u|^2.$$

If u is a solution of (1.8), so is $u - u_1$. With u replaced by $u - u_1$ in the above inequality, there holds

$$\int_{B_\rho} |u-u_\rho|^2 \le c\rho^{n+2} \int_{B_1} |u-u_1|^2.$$

PROOF: We present two methods.

METHOD 1. By rotation, we may assume $\{a_{ij}\}$ is a diagonal matrix. Hence (1.8) becomes

$$\sum_{i=1}^n \lambda_i D_{ii} u = 0$$

with $0 < \lambda \le \lambda_i \le \Lambda$ for $i = 1, \dots, n$. It is easy to see there exists an $r_0 = r_0(\lambda, \Lambda) \in (0, \frac{1}{2})$ such that for any $x_0 \in B_{1/2}$ the rectangle

$$\left\{x : \frac{|x_i - x_{0i}|}{\sqrt{\lambda_i}} < r_0\right\}$$

is contained in B_1. Change the coordinate

$$x_i \longmapsto y_i = \frac{x_i}{\sqrt{\lambda_i}} \quad \text{and set} \quad v(y) = u(x).$$

Then v is harmonic in $\{y : \sum_{i=1}^{n} \lambda_i y_i^2 < 1\}$. In the ball $\{y : |y - y_0| < r_0\}$ use the interior estimates to yield

$$|v(y_0)|^2 + |Dv(y_0)|^2 \leq c(\lambda, \Lambda) \int_{B_{r_0}(y_0)} v^2 \leq c(\lambda, \Lambda) \int_{\{\sum_{i=1}^{n} \lambda_i y_i^2 < 1\}} v^2.$$

Transform back to u to get

$$|u(x_0)|^2 + |Du(x_0)|^2 \leq c(\lambda, \Lambda) \int_{|x|<1} u^2.$$

METHOD 2. If u is a solution to (1.8), so are any derivatives of u. By applying Corollary 1.37 to derivatives of u we conclude that for any positive integer k

$$\|u\|_{H^k(B_{1/2})} \leq c(k, \lambda, \Lambda)\|u\|_{L^2(B_1)}.$$

If we fix a value of k sufficiently large with respect to n, $H^k(B_{1/2})$ is continuously embedded into $C^1(\bar{B}_{1/2})$ and therefore

$$|u|_{L^\infty(B_{1/2})} + |Du|_{L^\infty(B_{1/2})} \leq c(\lambda, \Lambda)\|u\|_{L^2(B_1)}.$$

This finishes the proof. □