Foreword

The theory of hyperbolic equations is a large subject, and its applications are many: fluid dynamics and aerodynamics, the theory of elasticity, optics, electromagnetic waves, direct and inverse scattering, and the general theory of relativity.

The first seven chapters of this book, based on notes of lectures delivered at Stanford in the spring and summer of 1963, deal with basic theory: the relation of hyperbolicity to the finite propagation of signals, the concept and role of characteristic surfaces and rays, energy, and energy inequalities.

The structure of solutions of equations with constant coefficients is explored with the help of the Fourier and Radon transforms. The existence of solutions of equations with variable coefficients with prescribed initial values is proved using energy inequalities. The propagation of singularities is studied with the help of progressing waves.

Chapter 8 of the second part describes finite difference approximations of hyperbolic equations. This subject is obviously of great importance for applications, but also intriguing for the theorist. The proof of stability of difference schemes is analogous to the derivation of energy estimates, but much more sophisticated.

Chapter 9 presents a streamlined version of the Lax-Phillips scattering theory. The last section describes the Pavlov-Faddeev analysis of automorphic waves, and their mysterious connection to the Riemann hypothesis.

Chapter 10, the only one dealing with nonlinear waves, is about hyperbolic systems of conservation laws, an active research area today. We present the basic concepts and results.

Five brief appendices sketch topics that are important or amusing, such as Huygens' principle, a theory of mixed initial and boundary value problems, and the use of nonstandard energy identities.

I hope that this book will serve well as an introduction to the multifaceted subject of hyperbolic equations.

Peter Lax
New York
February 2006

CHAPTER 1

Basic Notions

The wave equation is the prototype of a hyperbolic equation

$$u_{tt} - ku_{xx} = 0\,, \quad k \text{ positive.} \tag{1.1}$$

To put the positivity of k into evidence, we set $k = c^2$; then (1.1) becomes

$$u_{tt} - c^2 u_{xx} = 0\,. \tag{1.1$'$}$$

This equation governs the transverse motion of a flexible elastic string, the constant k being the ratio of the tension T and the linear density ρ. Observe that c has the dimension of velocity.

We expect, in analogy with the motion of finite systems of particles, that the motion of the string is determined once we specify its initial position and velocity:

$$u(x,0) = a(x), \quad u_t(x,0) = b(x)\,. \tag{1.2}$$

This is indeed so; to find the solution, we write the wave equation in operator form,

$$Lu = 0\,,$$

and then factor the operator L. We get

$$L = D_t^2 - c^2 D_x^2 = (D_t + cD_x)(D_t - cD_x)\,.$$

Each linear factor on the right is directional differentiation, along the lines $x = x_0 \pm ct$, respectively. Integrating along these lines successively and making use of the initial conditions (1.2), we get, after a brief calculation, the following explicit expression for u:

$$u(x,t) = \frac{a(x+ct)}{2} + \frac{a(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} b(s)ds\,. \tag{1.3}$$

This derivation shows that the lines $x = x_0 \pm ct$ (called *rays*) play a special role for the wave equation.

Suppose we prescribe u and u_t not at $t = 0$ but along some curve $t = p(x)$. Then the above procedure can still be used to determine the solution u *as long as every ray of both families cuts the curve in exactly one point*. We call such curves *spacelike*; the analytic condition[1] sufficient for a curve $t = p(x)$ to be spacelike is that $|p'|$ be less than $1/c$ at every point.

[1]The condition $|p'| > 1/c$ also guarantees this; but since it turns out to be a special feature of the one-dimensional situation, we leave it aside.

We shall now use formula (1.3) to study the manner in which the solution depends on the initial data. The following features—some qualitative, some quantitative—are of importance:

(1) The motion is uniquely determined by the initial data.
(2) The initial data can be prescribed as arbitrary, infinitely differentiable functions with compact support, and the corresponding u is infinitely differentiable in x and t.
(3) The principle of superposition holds.
(4) Influence propagates with speed $\le c$.

We shall show that properties (1)–(4) imply the following further properties:

(5) Motion is governed by a partial differential equation.
(6) Sharp signals propagate along rays.
(7) Energy is conserved.
(8) Spacelike manifolds have the same properties as the manifolds $t = \text{const.}$

We shall further show that motions depend continuously on their initial data; this implies that the governing equation is of a special type called *hyperbolic*.

The first two properties follow immediately from formula (1.3). The third property follows from the linearity of the wave equation. Property (4) means that, as evidenced by formula (1.3), the value of u at x, t is not influenced by the initial values outside the interval $(x - ct, x + ct)$. The formula also shows that, as asserted under (6), the influence of the endpoints is stronger than that of the interior; this will be made more precise later. To give meaning to (7) we have to define energy. In analogy with mechanics, we define

$$E_{\text{kinetic}} = \frac{1}{2}\rho \int u_t^2\, dx\,, \quad E_{\text{potential}} = \frac{1}{2}T \int u_x^2\, dx\,.$$

The total energy is then (using $T/\rho = c^2$)

$$E_{\text{total}} = \frac{1}{2}\rho \int u_t^2 + c^2 u_x^2\, dx.$$

From the explicit formula for u we can verify that the *energy density* $u_t^2 + c^2 u_x^2$ *is the sum of a function of* $x + ct$ *and of* $x - ct$; the integral of such a function is indeed independent of time.

We shall give now a derivation of the law of conservation of energy for arbitrary spacelike surfaces; this second method is applicable in rather general situations.

Let P_1 and P_2 be two spacelike curves; multiply equation (1.1′) by u_t and integrate over the domain contained between P_1 and P_2. We get

$$0 = \iint_{P_1}^{P_2} u_t u_{tt} - c^2 u_t u_{xx}\, dx\, dt\,.$$

Integrating by parts with respect to x in the second term, we get

$$0 = \iint_{P_1}^{P_2} u_t u_{tt} + c^2 u_{tx} u_x\, dx\, dt - c^2 \int u_t u_x \xi\, ds \Big|_{P_1}^{P_2},$$

where ξ, τ denote the x, t components of the unit normal drawn in the positive t direction. The remaining double integrals are both perfect t derivatives and so they can be integrated with respect to t. The result is

$$E(P_1) = E(P_2)\,,$$

where we define the energy $E(P)$ contained in u on the curve P as

$$E(S) = \frac{1}{2}\int_P \left(\tau u_t^2 - 2c^2\xi u_t u_x + c^2\tau u_x^2\right)ds\,.$$

We recall now that P is spacelike if

$$\tau > c|\xi|\,.$$

From the above form of E we deduce that *the energy density* (*and thus the total energy*) *along P is positive definite if and only if P is spacelike*.

The law of conservation of energy gives another proof of the result that initial data along a spacelike curve uniquely determine the motion. We shall see later that energy conservation is the basic tool for constructing solutions of general hyperbolic equations.

Property (8) follows easily from an explicit representation for u in terms of the values of u and u_t along a spacelike curve.

In the next chapter we shall investigate a class of media whose motions have properties (1)–(4). We shall show that such motions are governed by partial differential equations satisfying a certain algebraic condition. We shall show that, conversely, solutions of differential equations satisfying that algebraic property have properties (1)–(8). It is perhaps surprising that the qualitative assumptions (1)–(4) have such quantitative consequences as (5), (6), and (7).