CHAPTER 1

The Classical WKB Method

1.1. Introduction

Various problems of mathematical and theoretical physics involve partial differential equations with a small parameter in the highest derivative terms. For constructing approximate solutions to these equations, the asymptotic method has long been used. In this chapter, we shall consider a typical example of this type of problem: the Cauchy problem with rapidly oscillating initial data for the Schrödinger equation:

$$\begin{cases} ih\partial_t\psi^h = -\frac{h^2}{2}\triangle\psi^h + V\psi^h & \text{for } (t,x)\in\mathbb{R}\times\mathbb{R}^d, \\ \psi^h(t=0,x) = a_0^h(x)e^{i\phi_0(x)/h}. \end{cases} \tag{1.1}$$

Here both the potential function $V(t,x)$ and phase function $\phi_0(x)$ are real-valued and smooth functions. Problem (1.1) originates from quantum mechanics, where ψ^h is called a "wave function" and h can be explained as a scaled Planck's constant. One may check [**54**] for details.

We are looking for an asymptotic solution to the problem (1.1) as $h\to 0$ within a positive time interval $[0,T]$. The corresponding asymptotic formulas are said to be a "semiclassical approximation" or "semiclassical asymptotics." The scheme sketched in what follows was first introduced by Wentzel, Kramers, and Brillouin in quantum mechanical problems; this method is now referred to as the "WKB" method.

The aim of the WKB method is to describe ψ^h in the limit as $h\to 0$ when ϕ_0 does not depend on h; a_0^h has an asymptotic expansion of the form

$$a_0^h(x)\sim a(x)+ha_1(x)+h^2a_2(x)+\cdots. \tag{1.2}$$

The WKB method consists in seeking an approximate solution to (1.1) of the form

$$\psi^h(t,x)\sim (a(t,x)+ha_1(t,x)+h^2a_2(t,x)+\cdots)e^{i\Phi(t,x)/h}. \tag{1.3}$$

One must not expect this approach to be valid when the caustics, where the solution to (1.4) blows up, are formed even for the linear Schrödinger equation. Near the caustic, all the terms $\Phi, a, a_1,\dots$ become singular. Past the caustic, several phases are necessary in general to describe the asymptotic behavior of the solution (see [**73**] for the canonical operator approach to the linear Schrödinger equation). In this chapter, we shall restrict ourselves to times preceding this breakup and only study the asymptotic behavior of ψ^h at leading order.

1.2. WKB Approximation to the Linear Schrödinger Equation

In this case, the potential function $V(t,x)$ in (1.1) does not depend on ψ^h. We assume that the potential function V and the initial phase function ϕ_0 are smooth and subquadratic. More precisely,

(A1) $V \in C^\infty(R \times \mathbb{R}^d)$ and $\partial_x^\alpha V \in L^\infty_{\text{loc}}(\mathbb{R}; L^\infty(\mathbb{R}^d))$ for $|\alpha| \geq 2$;
(A2) $\phi_0 \in C^\infty(\mathbb{R}^d)$ and $\partial_x^\alpha \phi_0 \in L^\infty(\mathbb{R}^d)$ for $|\alpha| \geq 2$;
(A3) $a_0 \in H^\infty(\mathbb{R}^d) \stackrel{\text{def}}{=} \bigcap_{s\geq 0} H^s(\mathbb{R}^d)$ such that, for all s as $h \to 0$, a_0^h converges to a_0 in $H^s(\mathbb{R}^d)$

We substitute (1.3) into (1.1), and compare both sides of the resulting equation. Then canceling the term of order $O(1)$, we find that $\Phi(t,x)$ satisfies

$$\partial_t \Phi + \frac{1}{2}|\nabla_x \Phi|^2 + V = 0, \quad \Phi|_{t=0} = \phi_0. \tag{1.4}$$

This equation is called the eikonal equation in the WKB method or geometric optics. Canceling the term of order $O(h)$, we get the transport equation of a,

$$\partial_t a + \nabla\Phi \cdot \nabla a + \frac{1}{2} a \Delta\Phi = 0, \quad a|_{t=0} = a_0. \tag{1.5}$$

LEMMA 1.1 *Under assumptions* (A1) *and* (A2)*, there exists a positive time $T > 0$ such that*

(i) (1.4) *has a unique solution $\Phi \in C^\infty([0,T] \times \mathbb{R}^d)$. Furthermore, this solution Φ is subquadratic: $\partial_x^\alpha \Phi \in L^\infty([0,T] \times \mathbb{R}^d)$ for $|\alpha| \geq 2$.*
(ii) *Let $X(t,x)$ be determined by*

$$\begin{cases} \partial_t X(t,x) = \Xi(t,x), & X(0,x) = x, \\ \partial_t \Xi(t,x) = -\nabla_x V(t, X(t,x)), & \Xi(0,x) = \nabla\phi_0(x). \end{cases} \tag{1.6}$$

We denote by $X_t^{-1}(x)$ its inverse on $[0,T]$ and $J_t(x) \stackrel{\text{def}}{=} \det(\nabla_x X(t,x))$. Then (1.5) *has a unique solution of the form*

$$a(t,x) = \frac{1}{\sqrt{J_t(X_t^{-1}(x))}} a_0(X_t^{-1}(x)). \tag{1.7}$$

PROOF:

(i) We are going to use the characteristic method and the inverse function theorem to solve (1.4). We first get by taking ∇_x to (1.4) that

$$\begin{cases} \partial_t \nabla_x \Phi + (\nabla_x \Phi \cdot \nabla_x)\nabla_x \Phi + \nabla_x V = 0, \\ \nabla\Phi|_{t=0} = \nabla\phi_0; \end{cases}$$

from this and from (1.6), we deduce by the standard characteristic method that

$$\Xi(t,x) = \nabla_x \Phi(t, X(t,x)). \tag{1.8}$$

Therefore, to solve (1.4), we only need to solve (1.6). In fact, thanks to (A1) and (A2), for each fixed $x \in \mathbb{R}^d$, the classical ODE theory ensures that (1.6) has a unique solution on $[0, T_x]$ for some $T_x > 0$. In what follows, we are going to prove that $T_x = \infty$ via an a priori estimate method. In order to do so, we differentiate (1.6) with respect to x to obtain

$$\begin{cases} \partial_t \partial_x X(t,x) = \partial_x \Xi(t,x), \quad \partial_x X(0,x) = \mathrm{Id}, \\ \partial_t \partial_x \Xi(t,x) = -\nabla_x^2 V(t, X(t,x))\partial_x X(t,x), \quad \partial_x \Xi(0,x) = \nabla^2 \phi_0(x), \end{cases} \tag{1.9}$$

For any $t \le T_x$, it follows by integrating the above system in time over $[0,t]$ and making using of (A1) and (A2) that

$$|\partial_x X(t,x)| + |\partial_x \Xi(t,x)| \le C + C\int_0^t (|\partial_x X(s,x)| + |\partial_x \Xi(s,x)|)ds.$$

Applying the Gronwall lemma gives

$$\|\partial_x X(t,\cdot)\|_{L^\infty} + \|\partial_x \Xi(t,\cdot)\|_{L^\infty} \le Ce^{Ct} \quad \text{for } t \le T_x.$$

A similar procedure yields

$$\begin{gathered} \|\partial_x^\alpha X\|_{L^\infty([0,T_x]\times\mathbb{R}^d)} + \|\partial_x^\alpha \Xi\|_{L^\infty([0,T_x]\times\mathbb{R}^d)} \le C(\alpha, T_x), \\ \forall \alpha \in \mathbb{N}^d,\ |\alpha| \ge 1, \end{gathered} \tag{1.10}$$

with $C(\alpha, T_x)$ being independent of x and finite as long as $T_x < \infty$. This shows that indeed $T_x = \infty$.

On the other hand, we get by integrating the first equation of (1.9) in time that

$$\det(\nabla_x X(t,x)) = \det\left(\mathrm{Id} + \int_0^t \nabla_x \Xi(s,x)dx\right).$$

From this and (1.10), we deduce that there exist a $T > 0$ and $C_0 > 0$ such that

$$|\det(\nabla_x X(t,x))| \ge C_0 \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d. \tag{1.11}$$

Therefore, we can take the inverse of $y \mapsto X(t,x)$ for $t \in [0,T]$ and denote it by $X_t^{-1}(x)$. Furthermore, the classical inverse function theorem guarantees that $\nabla_x X_t^{-1} \in L^\infty([0,T]\times\mathbb{R}^d)$, which together with (1.8) gives

$$\nabla_x \Phi(t,x) = \Xi(t, X_t^{-1}(x)) \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^d.$$

With $\nabla_x \Phi(t,x)$ being determined on $[0,T]\times\mathbb{R}^d$, we can uniquely solve (1.4) for Φ. The property that $\partial_x^\alpha \Phi \in L^\infty([0,T]\times\mathbb{R}^d)$ for $|\alpha| \ge 2$ stems from (1.10) and (1.11).

(ii) Set $A(t,x) \stackrel{\text{def}}{=} a(t, X(t,x))\sqrt{J_t(x)}$. It is easy to observe from (1.8) and (1.9) that

$$\partial_t J_t(x) = \Delta\Phi(t, X(t,x))J_t(x),$$

which together with (1.5) implies that

$$\partial_t A = 0.$$

This gives (1.7). Finally, the uniqueness of solutions to (1.4) and (1.5) follows from the uniqueness of the solution to (1.6), which is rather standard from ODE theory. This completes the proof of the lemma.

□

Now we are in a position to justify the WKB approximation for the solutions of (1.1) under assumptions (A1)–(A3).

THEOREM 1.2 *Under assumptions* (A1)–(A3)*, let* T*,* Φ*, and* a *be determined in Lemma* 1.1. *Then* (1.1) *has a unique solution* $\psi^h \in C^\infty([0,T] \times \mathbb{R}^d) \cap C([0,T]; H^s(\mathbb{R}^d))$ *for all* $s > \frac{d}{2}$*, such that*

$$\|\psi^h - a e^{i\Phi/h}\|_{L^\infty([0,T];L^2\cap L^\infty)} \to 0 \quad \text{as } h \to 0. \tag{1.12}$$

PROOF: Define

$$a^h(t,x) \stackrel{\text{def}}{=} \psi^h(t,x) e^{-i\Phi(t,x)/h}.$$

Then thanks to (1.4), ψ^h solves (1.1) if and only if a^h solves

$$\begin{cases} \partial_t a^h + \nabla_x \Phi \cdot \nabla a^h + \frac{1}{2} a^h \Delta\Phi = i\,\frac{h}{2}\,\Delta a^h, \\ a^h\big|_{t=0} = a_0^h. \end{cases} \tag{1.13}$$

As is well-known, the existence of a solution to a PDE essentially follows from constructing the approximate solution sequence and obtaining the uniform estimates for such approximate solutions. For simplicity, we just present the a priori estimate to a smooth enough solution of (1.13).

Let $s > \frac{d}{2}$ be an integer and let $\alpha \in \mathbb{N}^d$ with $|\alpha| = s$. Then we get by applying ∂_x^α to (1.13) that

$$\partial_t \partial_x^\alpha a^h + \nabla\Phi \cdot \nabla \partial_x^\alpha a^h = i\,\frac{h}{2}\,\Delta \partial_x^\alpha a^h + R_\alpha^h, \tag{1.14}$$

with

$$R_\alpha^h = [\nabla\Phi \cdot \nabla; \partial_x^\alpha] a^h - \frac{1}{2}\,\partial_x^\alpha (a^h \Delta\Phi).$$

Here and in what follows, we shall always denote by $[\mathcal{A};\mathcal{B}]$ the commutator between $\mathcal{A}$ and $\mathcal{B}$. Taking the L^2 inner product of (1.14) with $\partial_x^\alpha a^h$ and considering the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha a^h\|_{L^2}^2 + \mathrm{Re} \int_{\mathbb{R}^d} \overline{\partial_x^\alpha a^h} \nabla\Phi \cdot \nabla \partial_x^\alpha a^h \, dx \le \|R_\alpha^h\|_{L^2} \|\partial_x^\alpha a^h\|_{L^2}.$$

Notice that

$$\begin{aligned} \left| \mathrm{Re} \int_{\mathbb{R}^d} \overline{\partial_x^\alpha a^h} \nabla\Phi \cdot \nabla \partial_x^\alpha a^h \, dx \right| &= \frac{1}{2} \left| \int_{\mathbb{R}^d} \nabla\Phi \cdot \nabla |\partial_x^\alpha a^h|^2 \, dx \right| \\ &\le \frac{1}{2} \left| \int_{\mathbb{R}^d} \Delta\Phi |\partial_x^\alpha a^h|^2 \, dx \right| \le C \|a^h\|_{H^s}^2, \end{aligned}$$

where we used the fact that

$$\Delta\Phi \in L^\infty([0,T]; H^\infty(\mathbb{R}^d))$$

so that $\Delta\Phi \in L^\infty([0,T] \times \mathbb{R}^d)$. Applying Moser-type inequalities gives

$$\|R_\alpha^h\|_{L^2} \le C \|\partial^2 \Phi\|_{H^s} \|a^h\|_{H^s}.$$

Therefore, we obtain

$$\frac{d}{dt}\|a^h(t)\|_{H^s}^2 \leq C\|\partial^2\Phi\|_{H^s}\|a^h(t)\|_{H^s}^2.$$

From this and the Gronwall inequality, we deduce that

$$\|a^h\|_{L^\infty([0,T];H^s)} \leq e^{C_s T\|\partial^2\Phi\|_{L^\infty([0,T];H^s)}}\|a_0^h\|_{H^s}. \tag{1.15}$$

This shows the local existence of smooth solutions to (1.13).

Now let us compare the solutions of (1.5) and (1.13). Set $w^h \stackrel{\text{def}}{=} a^h - a$; it follows from (1.5) and (1.13) that

$$\begin{cases} \partial_t w^h + \nabla\Phi\cdot\nabla w^h + \frac{1}{2}w^h\Delta\Phi = i\frac{h}{2}\Delta a^h, \\ w^h|_{t=0} = a_0^h - a_0. \end{cases}$$

Then a similar proof to that of (1.15) yields

$$\frac{d}{dt}\|w^h(t)\|_{H^s} \leq C_s(\|\partial^2\Phi\|_{H^s}\|w^h\|_{H^s} + h\|\Delta a^h\|_{H^s}).$$

From this, (1.15), (A3), and Gronwall's inequality, we deduce that

$$\|a^h - a\|_{L^\infty([0,T];H^s)} = \|w^h\|_{L^\infty([0,T];H^s)} \to 0 \quad \text{as } h\to 0. \tag{1.16}$$

On the other hand, note that

$$\|\psi^h - ae^{i\Phi/h}\|_{L^\infty([0,T];L^2\cap L^\infty)} = \|a^h - a\|_{L^\infty([0,T];L^2\cap L^\infty)},$$

which together with (1.16) concludes the proof of the Theorem. □

1.3. The WKB Approximation to Nonlinear Schrödinger Equation

In this section, we shall study the semiclassical approximation to the following nonlinear Schrödinger equation:

$$\begin{cases} ih\partial_t\psi^h + \frac{h^2}{2}\Delta\psi^h = f(|\psi^h|^2)\psi^h, \\ \psi^h(t=0,x) = a_0^h e^{i\phi_0(x)/h}. \end{cases} \tag{1.17}$$

We assume further that

(A4) $f' > 0$. In addition, $\phi_0 \in H^\infty(\mathbb{R}^d)$, and there exists $a_0, a_1 \in H^\infty(\mathbb{R}^d)$ such that

$$a_0^h = a_0 + ha_1 + o(h) \quad \text{in } H^s,\ \forall s\geq 0.$$

Then we seek the exact solution of (1.17) of the form

$$\psi^h(t,x) = a^h(t,x)e^{i\Phi^h(t,x)/h}, \tag{1.18}$$

where Φ^h is real-valued while the amplitude a^h is complex-valued. Putting (1.18) in (1.17), we obtain

$$-ih\partial_t a^h + \partial_t\Phi^h a^h - \frac{h^2}{2}\Delta a^h - ih\nabla\Phi^h\cdot\nabla a^h \\ - \frac{ih}{2}a^h\Delta\Phi^h + \frac{1}{2}a^h|\nabla\Phi^h|^2 + a^h f(|a^h|^2) = 0.$$

The historical approach [**54, 68**] suggests that

$$\begin{cases} \partial_t \Phi^h + \frac{1}{2}|\nabla\Phi^h|^2 + f(|a^h|^2) = h^2 \frac{\Delta a^h}{2a^h}, & \Phi^h\big|_{t=0} = \phi_0, \\ \partial_t a^h + \nabla\Phi^h \cdot \nabla a^h + \frac{1}{2}\, a^h \Delta\Phi^h = 0, & a^h\big|_{t=0} = a_0^h. \end{cases}$$

Of course, this choice is not adapted when the amplitude a^h vanishes, so it must be left out when $a_0^h \in L^2(\mathbb{R}^d)$. The approach introduced by Grenier in [**40**] consists in imposing

$$\begin{cases} \partial_t \Phi^h + \frac{1}{2}|\nabla\Phi^h|^2 + f(|a^h|^2) = 0, & \Phi^h|_{t=0} = \phi_0, \\ \partial_t a^h + \nabla\Phi^h \cdot \nabla a^h + \frac{1}{2}\, a^h \Delta\Phi^h = i\,\frac{h}{2}\Delta a^h, & a^h|_{t=0} = a_0^h. \end{cases} \tag{1.19}$$

We expect the amplitude a^h and the phase Φ^h to have the asymptotic expansions, as $h \to 0$, of

$$a^h \sim a + ha_1 + h^2 a_2 + \cdots, \qquad \Phi^h \sim \Phi + h\Phi_1 + h^2\Phi_2 + \cdots.$$

THEOREM 1.3 *Let $s > 2 + \frac{d}{2}$ be an integer and* (A4) *be satisfied. Then there exist $T_s > 0$ independent of $h \in (0,1]$ and $\psi^h = a^h e^{i\Phi^h/h}$, a solution to* (1.17) *on* $[0, T_s]$. *Moreover, a^h and Φ^h are bounded in $L^\infty([0,T_s]; H^s(\mathbb{R}^d))$ uniformly in $h \in (0,1]$.*

PROOF: Let $v^h \stackrel{\text{def}}{=} \nabla\Phi^h$; then it follows from (1.19) that

$$\begin{cases} \partial_t v^h + v^h \cdot \nabla v^h + 2f'(|a^h|^2)\operatorname{Re}(\overline{a^h}\nabla a^h) = 0, & v^h\big|_{t=0} = \nabla\phi_0, \\ \partial_t a^h + v^h \cdot \nabla a^h + \frac{1}{2}\, a^h \operatorname{div} v^h = i\,\frac{h}{2}\,\Delta a^h, & a^h\big|_{t=0} = a_0^h. \end{cases} \tag{1.20}$$

Separating real and imaginary parts of $a^h = a_1^h + i a_2^h$, we obtain

$$\partial_t u^h + \sum_{j=1}^d A_j(u^h)\partial_j u^h = \frac{h}{2}\, L u^h, \tag{1.21}$$

where

$$u^h = (a_1^h\ a_2^h\ v_1^h\ \cdots\ v_d^h)^\mathsf{T} \qquad L = \begin{pmatrix} 0 & -\Delta & 0 & \cdots & 0 \\ \Delta & 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0_{d\times d} & \end{pmatrix}$$

and

$$A(u,\xi) = \sum_{j=1}^d A_j(u)\xi_j = \begin{pmatrix} v\cdot\xi & 0 & \frac{a_1}{2}{}^t\xi \\ 0 & v\cdot\xi & \frac{a_2}{2}{}^t\xi \\ 2f'a_1\xi & 2f'a_2\xi & v\cdot\xi\,\mathrm{Id}_{d\times d} \end{pmatrix},$$

where f' stands for $f'(|a_1|^2 + |a_2|^2)$. The matrix $A(u,\xi)$ can be symmetrized by $S(|a_1|^2 + |a_2|^2)$ with

$$S(\eta) \stackrel{\text{def}}{=} \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & \frac{1}{4f'(\eta)}\,\mathrm{Id}_{d\times d} \end{pmatrix},$$

which is symmetric and positive definite since $f' > 0$. To proceed further, we denote by $(\cdot,\cdot)$ the L^2 inner product and $S^h \stackrel{\text{def}}{=} S(|a_1^h|^2 + |a_2^h|^2)$. Then for any integer $s > 2 + \frac{d}{2}$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq s$, by applying the standard energy estimate to (1.21) we get that

$$(S^h \partial_t \partial_x^\alpha u^h, \partial_x^\alpha u^h) = h(S^h L(\partial_x^\alpha u^h), \partial_x^\alpha u^h) - \left(S^h \partial_x^\alpha \left(\sum_{j=1}^d A_j(u^h)\partial_j u^h\right), \partial_x^\alpha u^h\right). \tag{1.22}$$

But notice that

$$(S^h L(\partial_x^\alpha u^h), \partial_x^\alpha u^h) = -\int_{\mathbb{R}^d} \partial_x^\alpha a_1^h \Delta_x \partial_x^\alpha a_2^h - \partial_x^\alpha a_2^h \Delta_x \partial_x^\alpha a_1^h \, dx = 0.$$

The second term of (1.22) can be written in the form

$$\begin{aligned} &\left(S^h \partial_x^\alpha \left(\sum_{j=1}^d A_j(u^h)\partial_j u^h\right), \partial_x^\alpha u^h\right) \\ &= \left(S^h \sum_{j=1}^d A_j(u^h)\partial_j \partial_x^\alpha u^h, \partial_x^\alpha u^h\right) \\ &\quad + \left(S^h \left[\partial_x^\alpha \left(\sum_{j=1}^d A_j(u^h)\partial_j u^h\right) - \sum_{j=1}^d A_j(u^h)\partial_j \partial_x^\alpha u^h\right], \partial_x^\alpha u^h\right). \end{aligned}$$

But as $S^h A_j(u^h)$ is a symmetric matrix,

$$\left(S^h \sum_{j=1}^d A_j(u^h)\partial_j \partial_x^\alpha u^h, \partial_x^\alpha u^h\right) = -\frac{1}{2}\sum_{j=1}^d (\partial_j(S^h A_j(u^h))\partial_x^\alpha u^h, \partial_x^\alpha u^h),$$

from which we deduce by Moser-type inequalities that

$$\left|\left(S^h \sum_{j=1}^d A_j(u^h)\partial_j \partial_x^\alpha u^h, \partial_x^\alpha u^h\right)\right| \leq C(\|u^h\|_{L^\infty})\|\nabla_x u^h\|_{L^\infty}\|\partial_x^\alpha u^h\|_{L^2}^2$$

and

$$\left|\left(S^h [\partial_x^\alpha \left(\sum_{j=1}^d A_j(u^h)\partial_j u^h\right) - \sum_{j=1}^d A_j(u^h)\partial_j \partial_x^\alpha u^h], \partial_x^\alpha u^h\right)\right| \leq C(\|u^h\|_{H^s})\|u^h\|_{H^s}^2.$$

On the other hand, again since S^h is symmetric,

$$\frac{d}{dt}(S^h \partial_x^\alpha u^h, \partial_x^\alpha u^h) = (\partial_t S^h \partial_x^\alpha u^h, \partial_x^\alpha u^h) + 2(S^h \partial_t \partial_x^\alpha u^h, \partial_x^\alpha u^h). \tag{1.23}$$

For the first term in (1.23), we only need to consider the lower $d \times d$ block of S^h:

$$(\partial_t S \partial_x^\alpha u^h, \partial_x^\alpha u^h) \leq \left\| \frac{1}{f'} \partial_t \big(f'(|a_1^h|^2 + |a_2^h|^2)\big) \right\|_{L^\infty} (S \partial_x^\alpha u^h, \partial_x^\alpha u^h).$$

So long as $\|u^h\|_{L^\infty} \le 2\|a_0^h\|_{L^\infty}$, we have

$$f'(|a_1^h|^2 + |a_2^h|^2) \ge \inf\{f'(y) : 0 \le y \le 4 \sup_{0<h\le 1} \|a_0^h\|_{L^\infty}^2\} \stackrel{\text{def}}{=} \delta > 0.$$

From this and (1.20), we infer that

$$\left\| \frac{1}{f'}\, \partial_t \big(f'(|a_1^h|^2 + |a_2^h|^2)\big) \right\|_{L^\infty} \le C\|u^h\|_{H^s}.$$

Consequently, thanks to (1.22) and (1.23), we find

$$\frac{d}{dt} \sum_{|\alpha|\le s} (S^h \partial_x^\alpha u^h, \partial_x^\alpha u^h) \le C(\|u^h\|_{H^s}) \sum_{|\alpha|\le s} (S^h \partial_x^\alpha u^h, \partial_x^\alpha u^h).$$

Then the Gronwall lemma along with a continuity argument yields the theorem. □

Formally taking $h \to 0$ in (1.19), we obtain

$$\begin{cases} \partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + f(|a|^2) = 0, & \Phi\big|_{t=0} = \phi_0, \\ \partial_t a + \nabla\Phi \cdot \nabla a + \frac{1}{2}\, a\Delta\Phi = 0, & a\big|_{t=0} = a_0. \end{cases} \tag{1.24}$$

Under assumption (A4), (1.24) has a unique solution

$$(a, \Phi) \in L^\infty([0, T_*]; H^m(\mathbb{R}^d))$$

for all $m > 0$ and for some T_* independent of m (see [**69**], for instance).

PROPOSITION 1.4 *Let $s > \frac{d}{2}$ be an integer. Then $T_s \ge T_*$, and there exists a positive constant C_s independent of h such that*

$$\|a^h(t) - a(t)\|_{H^s} \le C_s h, \quad \|\Phi^h - \Phi(t)\|_{H^s} \le C_s h t. \tag{1.25}$$

PROOF: We first split a into the real and imaginary parts, $a = a_1 + i a_2$, and $v \stackrel{\text{def}}{=} \nabla_x \Phi$. Then $u \stackrel{\text{def}}{=} (a_1, a_2, v)$ solves

$$\partial_t u + \sum_{j=1}^d A_j(u)\partial_j u = 0, \tag{1.26}$$

with A_j being the same as the correspondence in (1.21). Comparing (1.21) with (1.26) results in

$$\partial_t(u^h - u) + \sum_{j=1}^d A_j(u^h)\partial_j(u^h - u) + \sum_{j=1}^d (A_j(u^h) - A_j(u))\partial_j u = \frac{h}{2}\, L u^h.$$

Note that both u^h and u are bounded in $L^\infty([0, \min(T_s, T_*)]; H^s(\mathbb{R}^d))$. We denote $w^h \stackrel{\text{def}}{=} u^h - u$ and write $Lu^h = Lw^h + Lu$. Then similar to the proof of

Theorem 1.3, the term Lw^h disappears from the energy estimate, and we obtain

$$\frac{d}{dt}\sum_{|\alpha|\le s}(S^h\partial_x^\alpha w^h,\partial_x^\alpha w^h)\le C(\|u^h\|_{H^s},\|u\|_{H^{s+2}})\sum_{|\alpha|\le s}(S^h\partial_x^\alpha w^h,\partial_x^\alpha w^h)+h\|u\|_{H^{s+2}}\|w^h(t)\|_{H^s}.$$

Gronwall's inequality and a continuity argument show that w^h and hence u^h are defined in $[0,T_*]$. Furthermore, thanks to (A4), we get

$$\|w^h\|_{L^\infty([0,T_*];H^s)}=O(h).$$

The estimate for the difference between phases Φ^h and Φ then follows from the above estimate and the integration in time of (1.19) and (1.24). □

Proposition 1.4 yields an approximation of u^h for small time only:

$$\begin{aligned}&\left\|\psi^h(t)-a(t)e^{i\Phi(t)/h}\right\|_{L^2}\\&=\left\|a^h(t)e^{i\Phi^h(t)/h}-a(t)e^{i\Phi(t)/h}\right\|_{L^2}\\&\le\left\|a^h(t)-a(t)\right\|_{L^2}+\left\|e^{i\Phi^h(t)/h}-e^{i\Phi(t)/h}\right\|_{L^\infty}\left\|a(t)\right\|_{L^2}\\&\le C(h+t).\end{aligned}$$

For times of order $O(1)$, the corrector a_1 must be taken into account.

PROPOSITION 1.5 *Under assumption* (A4)*, we define* (a_1,Φ_1) *by*

$$\begin{cases}\partial_t\Phi_1+\nabla\Phi\cdot\nabla\Phi_1+2\,\mathrm{Re}(\bar{a}a_1)f'(|a|^2)=0,\\ \partial_t a_1+\nabla\Phi\cdot\nabla a_1+\nabla\Phi_1\cdot\nabla a+\frac{1}{2}a_1\Delta\Phi+\frac{1}{2}a\Delta\Phi_1=\frac{i}{2}\Delta a,\\ \Phi_1\big|_{t=0}=0,\quad a_1\big|_{t=0}=a_1.\end{cases}$$

Then $a_1,\Phi_1\in L^\infty([0,T_*];H^s(\mathbb{R}^d))$ *for every* $s\ge 0$*, and*

$$\|a^h-a-ha_1\|_{L^\infty([0,T_*];H^s)}+\|\Phi^h-\Phi-h\Phi_1\|_{L^\infty([0,T_*];H^s)}\le C_s h^2\quad\forall s\ge 0.$$

The proof of this proposition is a straightforward consequence of the proof to Proposition 1.4. We omit the details here; one may check [**19, 40**] for the detailed proof. An immediate corollary of the above proposition is:

COROLLARY 1.6 *Under the assumptions of Propositions* 1.4 *and* 1.18*, we have*

$$\|\psi^h-ae^{i\Phi_1}e^{i\Phi/h}\|_{L^\infty([0,T_*];L^2\cap L^\infty)}\le Ch.$$

PROOF: Actually, thanks to Proposition 1.4 and Proposition 1.5, we have

$$\begin{aligned}&\|\psi^h(t)-ae^{i\Phi_1}e^{i\Phi/h}\|_{L^2\cap L^\infty}\\&=\|a^h(t)e^{i\Phi^h(t)/h}-ae^{i\Phi_1}e^{i\Phi/h}\|_{L^2\cap L^\infty}\\&\le\|a^h(t)-a(t)\|_{L^2\cap L^\infty}+\|e^{i\Phi^h(t)/h}-e^{i[\Phi(t)+h\Phi_1]/h}\|_{L^\infty}\|a(t)\|_{L^2\cap L^\infty}\\&\le Ch,\end{aligned}$$

which ensures the corollary. □

1.4. References and Remarks

When V in (1.1) is a given C^∞ function depending only on the x-variables, the canonical operator approach was used by Maslov and Fedoriuk [**73**] to construct global asymptotics for solutions of (1.1). Section 1.2 follows mainly from [**19**].

Concerning the semiclassical approximation of (1.17), Gérard [**34**] treated the case of initial data with analytic regularities. Theorem 1.3 was proved by Grenier [**40**], and Carles extended this result for nonlinear Schrödinger equations with subquadratic potential in [**19**]. Most recently, Alazard and Carles [**3**] dealt with the case when $f(\rho) = \rho^\sigma$, which does not satisfy $f' > 0$, by using the modulated energy estimate method from [**62**]. One may check the recent book by Carles [**20**] for a complete set of references on this approach.

Notice that even for the semiclassical approximation to the linear Schrödinger equation, the eikonal equation (1.4) with smooth initial data blows up in finite time; that is, the WKB method works only within a finite time interval. To get a global semiclassical approximation to (1.1) is actually one of the motivations for us to introduce the tool of Wigner measure in Chapter 2.