For example, consider a game with $N = 7$ cards. Suppose after shuffling the deck becomes

$$\begin{array}{c} 4 \\ 2 \\ 1 \\ 7 \\ 5 \\ 6 \\ 3 \end{array} \quad \simeq \pi \in S_7.$$

Following the above rules, the game evolves as follows:

$$\begin{array}{c} \\ \\ 4 \end{array} \qquad \begin{array}{c} \\ 2 \\ 4 \end{array} \qquad \begin{array}{c} 1 \\ 2 \\ 4 \end{array} \qquad \begin{array}{ll} 1 & \\ 2 & \\ 4 & 7 \end{array} \qquad \begin{array}{ll} 1 & \\ 2 & 5 \\ 4 & 7 \end{array} \qquad \begin{array}{lll} 1 & & \\ 2 & 5 & \\ 4 & 7 & 6 \end{array} \qquad \begin{array}{lll} 1 & 3 & \\ 2 & 5 & \\ 4 & 7 & 6 \end{array}$$

and hence $q_7(\pi) = 3$. Equip S_N with uniform measure. Then

$$\mathrm{Prob}\{q_N \leq n\} = \frac{\#\{\pi \in S_N : q_N(\pi) \leq n\}}{N!}.$$

How does q_N behave statistically as $N, n \to \infty$? In 1999 Baik, Deift, and Johansson [**8**] found that, after centering and scaling, as $N, n \to \infty$,

> q_N behaves statistically like the largest eigenvalue of a GUE matrix described by the Tracy-Widom distribution [**97**].

Other related problems for permutations (see [**9**] and also [**22**]) are described by GOE and the so-called Gaussian symplectic ensemble (GSE).

Many books on RMT and its applications are now available; see, for example, [**21, 39, 63, 100**].

1.3. Synopsis of the Book

In Part 1 we consider the algebraic aspects of invariant matrix ensembles with general weights $w(x)$ following the method of Tracy and Widom [**99**]. The only restriction here is that $w(x)$ has finite moments, $\int_{-\infty}^{\infty} |x|^k w(x) dx < \infty$, $k = 0, 1, 2, \dots$.

In the case that $w'(x)/w(x)$ is rational, Widom [**103**] showed that the formulae in [**99**] could be placed in a compact form that is particularly amenable to asymptotic analysis. In Part 2 we apply the method of Widom [**103**] to weights of the form $w(x) = e^{-Q(x)}$ where $Q(x)$ is a polynomial of even degree, $Q(x) = \kappa_{2m} x^{2m} + \cdots$, $\kappa_{2m} > 0$, $m = 1, 2, \dots$. For such weights, we show how to use Widom's formulae, together with results on the asymptotics of orthogonal polynomials from [**27**], to prove universality in the bulk for orthogonal and symplectic ensembles. For pedagogical reasons, we present full details only in the monomial case $Q(x) = \kappa_{2m} x^{2m}$, $m = 1, 2, \dots$, where all the technical difficulties are already present, but the formulae simplify. The analysis extends in a straightforward way, however, to the general case $Q(x) = \kappa_{2m} x^{2m} + \cdots$ (see [**24**]).

It is an interesting, open problem to prove universality for orthogonal and symplectic ensembles with varying weights $w(x) = w_N(x) = e^{-NQ(x)}$, where $Q(x)$

is a polynomial of even degree as above and N is the dimension of the matrix ensemble. Here the equilibrium measure for the orthogonal polynomials associated to $w_N(x)$ (see [**82**], see also (6.28) below) may be supported on more than one interval. The main technical difficulty is to prove the analogue of Theorem 6.51 below for such weights. Universality for orthogonal and symplectic ensembles with varying weights $e^{-NQ(x)}$ with $Q(x)$ an even, quartic (two-interval) potential was proven by Stojanovic in the very interesting paper [**93**] (see Remark 6.15 below for more information). Recently Shcherbina [**85, 86**] has introduced some promising new ideas for the varying weights problem, but so far the results in [**85, 86**] are limited to the case where the associated equilibrium measure is supported on a single interval. The general case $w(x) = e^{-NQ(x)}$, $Q(x) = \kappa_{2m}x^{2m} + \cdots$, remains open.

In Chapter 2, we define the ensembles precisely and derive formulae for the eigenvalue distributions. In Chapter 3, we present various algebraic and analytical results that will be used further on in the text. In Chapter 4, we compute the basic statistics for the ensembles using the formulae from Chapter 3. In Chapter 5, we apply Widom's method to weights of the form $w(x) = e^{-Q(x)}$ where $Q(x)$ is a polynomial of even degree, to recast the formulae in Chapter 4 into a form that is amenable to asymptotic analysis. Finally, in Chapter 6, we analyze the formulae in Chapter 5 in the bulk scaling limit as the dimension of the matrices goes to infinity, and prove universality for the orthogonal and symplectic ensembles. (As mentioned above, we only present full details in the case that that $Q(x)$ is a monomial of even degree.) The main results are Theorem 6.7 and Corollaries 6.11 and 6.12. Theorem 6.51 (see Theorem 6.234 for a quantitative version) plays a fundamental role, and Section 6.8.2 is devoted to its proof following [**16**]. The original proof of Theorem 6.234 in [**24**] was more cumbersome and relied on a detailed estimate of numerical errors.

In contrast to [**23, 24**], we prove quantitative versions of the error estimates for the Widom correction terms for the orthogonal and symplectic ensembles. These estimates are of the form $O(N^{-\tau})$ as $N \to \infty$ for some $\tau > 0$. The method of establishing such error estimates in place of just $o(1)$ as in [**23, 24**] was introduced in [**25**]. However, an explicit value of τ was not found in [**25**], and this is a new result in Chapter 6 (see, in particular, Theorem 6.38).

REMARK 1.1 (Technical remark). In Theorem 6.7 and Corollaries 6.11 and 6.12, the asymptotic behavior of the scalings $\lambda_{N,1} = R_{N,1,1}(r)$ and $\lambda_{N,4} = R_{N/2,4,1}(r)$ (see (6.5)) as $N \to \infty$, is not known a priori, and part of the proofs of Theorem 6.7 and Corollaries 6.11 and 6.12 is the evaluation of these asymptotics. This is done (see (6.197) et seq.) by introducing auxiliary scalings q_N with prescribed asymptotic behavior (see (6.211), (6.212)): the behavior of $\lambda_{N,1}$, $\lambda_{N,4}$ is then inferred a posteriori (see (6.220)).

1.4. Some General Remarks

REMARK 1.2. For orthogonal ensembles we always take N to be even (see Section 4.1.3 and Remark 6.3 below).