CHAPTER 1

The Fluid Continuum

Classical fluid mechanics is concerned with a mathematical idealization of common fluids such as air or water. The main idealization is embodied in the notion of a *continuum*, and our "fluids" will generally be identified with a certain connected set of points in $\mathbb{R}^N$, where we will consider dimension N to be 1, 2, or 3. Of course, the fluids will move, so basically our subject is that of a moving continuum.

This description is an idealization that neglects the molecular structure of real fluids. *Liquids* are fluids characterized by random motions of molecules on the scale of 10^{-7} to 10^{-8} cm, and by a substantial resistance to compression. *Gases* consist of molecules moving over much larger distances, with mean free paths of the order of 10^{-3} cm, and are readily compressed. Both liquids and gases will fall within the scope of the theory of fluid motion that we will develop below. The theory will deal with observable properties such as velocity, density, and pressure. These properties must be understood as averages over volumes that contain many molecules but are small enough to be "infinitesimal" with respect to the length scale of variation of the property. We shall use the term *fluid parcel* to indicate such a small volume. The notion of a *particle* of fluid will also be used. It is a point of the continuum, but it should not be confused with a molecule. For example, the time rate of change of position of a fluid particle will be the *fluid velocity*, which is an average velocity taken over a parcel and is distinct from molecular velocities. The continuum theory has wide applicability to the natural world, but there are certain situations where it is not satisfactory. Usually these will involve small domains where the molecular structure becomes important, such as shock waves or fluid interfaces.

1.1. Eulerian and Lagrangian Descriptions

Let the independent variables (observables) describing a fluid be functions of position $\mathbf{x} = (x_1, \ldots, x_N)$ in Euclidean space and time t. Suppose that at $t = 0$ the fluid is identified with an open set S_0 of $\mathbb{R}^N$. As the fluid moves, the particles of fluid will take up new positions, occupying the set S_t at time t. We can introduce the map $M_t, S_0 \to S_t$ to describe this change, and write $M_t S_0 = S_t$. If $\mathbf{a} = (a_1, \ldots, a_N)$ is a point of S_0, we introduce the function $\mathbf{x} = \boldsymbol{X}(\mathbf{a}, t)$ as the position of a fluid particle at time t, which was located at $\mathbf{a}$ at time $t = 0$. The function $\boldsymbol{X}(\mathbf{a}, t)$ is called the *Lagrangian coordinate* of the fluid particle identified by the point $\mathbf{a}$. We remark that the "coordinate" $\mathbf{a}$ need not in fact be the initial position of a particle, although that is the most common choice and will be generally used here. But any unique labeling of the particles is acceptable.*

The *Lagrangian description* of a fluid emerges from this focus on the fluid properties associated with individual fluid particles. To "think Lagrangian" about a fluid, one must move mentally with the fluid and sample the fluid properties in each moving parcel. The Lagrangian analysis of a fluid has certain conceptual and mathematical advantages, but it is often difficult to apply to realistic fluid flows. Also it is not directly related to experience, since measurements in a fluid tend to be performed at fixed points in space, as the fluid flows past the point.

If we therefore adopt the point of view that we will observe fluid properties at a fixed point $\mathbf{x}$ as a function of time, we must break the association with a given fluid particle and realize that as time flows different fluid particles will occupy the position $\mathbf{x}$. This will make sense as long as $\mathbf{x}$ remains within the set S_t. Once properties are expressed as functions of $\mathbf{x}, t$ we have the *Eulerian description* of a fluid. For example, we might consider the fluid to fill all space and be at rest "at infinity." We then can consider the velocity $\mathbf{u}(\mathbf{x}, t)$ at each point of space, with $\lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x}, t) = 0$. Or we might have a fixed rigid body with fluid flowing over it such that at infinity we have a fixed velocity $\mathbf{U}$. For points outside the body the fluid velocity will be defined and satisfy $\lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x}, t) = \mathbf{U}$.

It is of interest to compare these two descriptions of a fluid and understand their connections. The most obvious is the meaning of velocity: the definition is

$$\mathbf{x}_t = \left.\frac{\partial \boldsymbol{X}}{\partial t}\right|_{\mathbf{a}} = \mathbf{u}(\boldsymbol{X}(\mathbf{a}, t), t). \tag{1.1}$$

That is to say, following the particle we calculate the rate of change of position with respect to time. Given the Eulerian velocity field, the calculation of Lagrangian coordinates is therefore mathematically equivalent to solving the initial value problem for the system (1.1) of ordinary differential equations for the function $\mathbf{x}(t)$, with the initial condition $\mathbf{x}(0) = \mathbf{a}$, the order of the system being the dimension of space. The special case of a *steady flow*, that is, one that is independent of time, leads to a system of *autonomous* ODEs.

EXAMPLE 1.1. In two dimensions ($N = 2$), with fluid filling the plane, we take $\mathbf{u}(\mathbf{x}, t) = (u(x, y, t), v(x, y, t)) = (x, -y)$. This velocity field is independent of time, hence a steady flow. To compute the Lagrangian coordinates of the fluid particle initially at $\mathbf{a} = (a, b)$ we solve

$$\frac{\partial x}{\partial t} = x, \quad x(0) = a, \qquad \frac{\partial y}{\partial t} = -y, \quad y(0) = b, \tag{1.2}$$

so that $\boldsymbol{X} = (ae^t, be^{-t})$. Note that, since $xy = ab$, the *particle paths* are hyperbolas; the curves traced out by the particles are independent of time; see Figure 1.1. If we consider the fluid in $y > 0$ only and take $y = 0$ as a rigid wall, we have a flow that is impinging vertically on a wall. The point at the origin, where the velocity is 0, is called a *stagnation point*. This point is a hyperbolic point relative to particle paths. A flow of this kind occurs at the nose of a smooth body placed

*We shall often use (x, y, z) in place of (x_1, x_2, x_3), and (a, b, c) in place of (a_1, a_2, a_3).

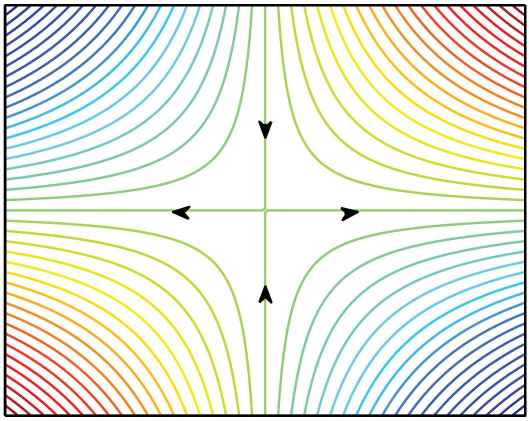

FIGURE 1.1. Stagnation point flow.

in a uniform current. Because this flow is steady, the particle paths are also called *streamlines*.

EXAMPLE 1.2. Again in two dimensions, consider $(u, v) = (y, -x)$. Then $\frac{\partial x}{\partial t} = y$ and $\frac{\partial y}{\partial t} = -x$. Solving, the Lagrangian coordinates are $x = a\cos t + b\sin t$, $y = -a\sin t + b\cos t$, and the particle paths (and streamlines) are the circles $x^2 + y^2 = a^2 + b^2$. The motion on the streamlines is clockwise, and fluid particles located at some time on a ray $x/y =$ const remain on the same ray as it rotates clockwise once for every 2π units of time. This is *solid-body rotation*.

EXAMPLE 1.3. If instead $(u, v) = (y/r^2, -x/r^2), r^2 = x^2 + y^2$, we again have particle paths that are circles, but the velocity becomes infinite at $r = 0$. This is an example of a flow representing a two-dimensional *point vortex* (see Chapter 3).

1.1.1. Particle Paths, Instantaneous Streamlines, and Streak Lines. The present considerations are *kinematic*, meaning that we are assuming knowledge of fluid motion, through an Eulerian velocity field $\mathbf{u}(\mathbf{x}, t)$ or else Lagrangian coordinates $\mathbf{x} = \boldsymbol{X}(\mathbf{a}, t)$, irrespective of the cause of the motion. One useful kinematic characterization of a fluid flow is the pattern of streamlines, as already mentioned in the above examples. In steady flow the streamlines and particle paths coincide. In an unsteady flow this is not the case and the only useful recourse is to consider *instantaneous streamlines*, at a particular time. In three dimensions the instantaneous streamlines are the orbits of the velocity field $\mathbf{u}(\mathbf{x}, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$ at time t. These are the integral curves satisfying

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{1.3}$$

These streamlines will change in an unsteady flow, and the connection with particle paths is not obvious in flows of any complexity.

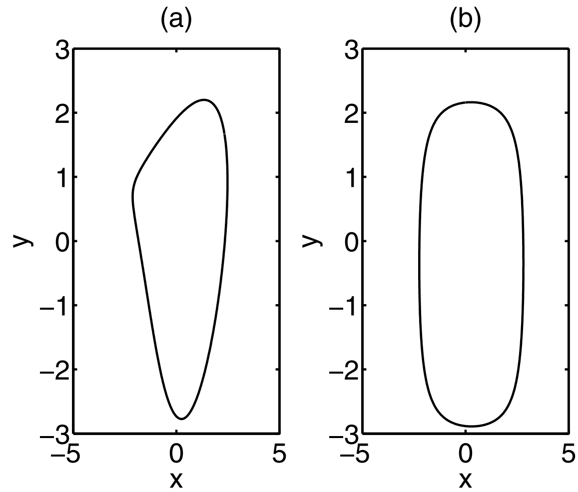

FIGURE 1.2. (a) Particle path and (b) streak line in Example 1.4.

Visualization of flows in water is sometimes accomplished by introducing dye at a fixed point in space. The dye can be thought of as labeling by color the fluid particle found at the point at a given time. As each point is labeled it moves along its particle path. The resulting *streak line* thus consists of all particles that at some time in the past were located at the point of injection of the dye. To describe a streak line mathematically we need to generalize the time of initiation of a particle path. Thus we introduce the *generalized Lagrangian coordinate* $\mathbf{x} = \boldsymbol{X}(\mathbf{a}, t, t_a)$, defined to be the position at time t of a particle that was located at $\mathbf{a}$ at time t_a. A streak line observed at time $t > 0$, which was started at time $t = 0$ say, is given by $\mathbf{x} = \boldsymbol{X}(\mathbf{a}, t, t_a), 0 < t_a < t$. Particle paths, instantaneous streamlines, and streak lines are generally distinct objects in unsteady flows.

EXAMPLE 1.4. Let $(u, v) = (y, -x + \epsilon \cos \omega t)$. For this flow the instantaneous streamlines satisfy $dx/y = dy/(-x + \epsilon \cos \omega t)$, yielding the circles $(x - \epsilon \cos \omega t)^2 + y^2 = \text{const}$. The generalized Lagrangian coordinates can be obtained from the general solution of a second-order ODE and take the form

$$x = -\frac{\epsilon}{\omega^2 - 1} \cos \omega t + A \cos t + B \sin t, \qquad (1.4)$$
$$y = \frac{\epsilon \omega}{\omega^2 - 1} \sin \omega t + B \cos t - A \sin t,$$

where

$$A = -b \sin t_a + \frac{\epsilon \omega}{\omega^2 - 1} \sin \omega t_a \sin t_a + a \cos t_a + \frac{\epsilon}{\omega^2 - 1} \cos \omega t_a \cos t_a, \qquad (1.5)$$

$$B = a \sin t_a + b \cos t_a - \frac{\epsilon}{\omega^2 - 1} \cos \omega t_a \sin t_a + \frac{\epsilon \omega}{\omega^2 - 1} \sin \omega t_a \cos t_a. \qquad (1.6)$$

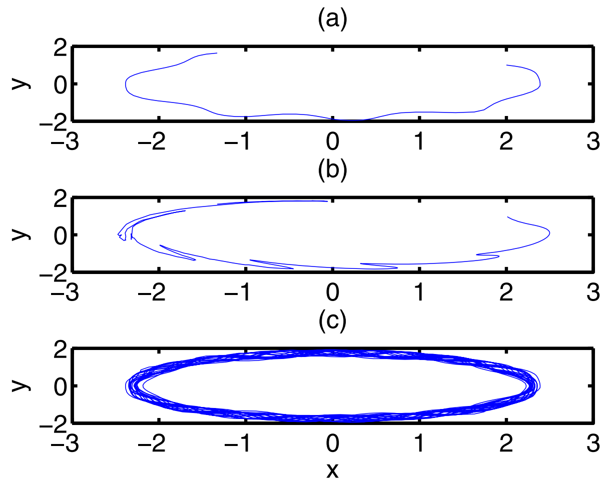

FIGURE 1.3. The oscillating vortex, Example 1.5, $\epsilon = 1.5, \omega = 2$. The lines emanate from $(2, 1)$. (a) Particle path, $0 < t < 20$. (b) Streak line, $0 < t < 20$. (c) Particle path, $0 < t < 500$.

The particle path with $t_a = 0, \omega = 2, \epsilon = 1$ starting at the point $(2, 1)$ is given by

$$x = -\frac{1}{3}\cos 2t + \sin t + \frac{7}{3}\cos t, \quad y = \cos t - \frac{7}{3}\sin t + \frac{2}{3}\sin 2t, \tag{1.7}$$

and is shown in Figure 1.2(a). All particle paths are closed curves. The streak line emanating from $(2, 1)$ over the time interval $0 < t < 2\pi$ is shown in Figure 1.2(b).

This last example is especially simple since the two-dimensional system is linear and can be integrated explicitly. In general, two-dimensional unsteady flows and three-dimensional steady flows can exhibit chaotic particle paths and streak lines.

EXAMPLE 1.5. A nonlinear system exhibiting this complex behavior is the oscillating point vortex: $(u, v) = (y/r^2, -(x - \epsilon\cos\omega t)/r^2)$. We show an example of particle path and streak line in Figure 1.3.

1.1.2. The Jacobian Matrix. We will, with a few obvious exceptions, be taking all of our functions as infinitely differentiable wherever they are defined. In particular, we assume that Lagrangian coordinates will be continuously differentiable with respect to the particle label **a**. Accordingly, we may define the Jacobian of the Lagrangian map M_t by the matrix

$$J_{ij} = \left.\frac{\partial x_i}{\partial a_j}\right|_t. \tag{1.8}$$

Thus $dl_i = J_{ij}\, da_j$ is a differential vector that can be visualized as connecting two nearby fluid particles whose labels differ by da_j.[†] If $da_1 \cdots da_N$ is the volume

[†]Here and elsewhere the summation convention is understood: unless otherwise stated, repeated indices are to be summed from 1 to N.

of a small fluid parcel, then $\mathrm{Det}(\mathbf{J}) da_1 \cdots a_N$ is the volume of that parcel under the map M_t. Fluids that are *incompressible* must have the property that all fluid parcels preserve their volume, so that $\mathrm{Det}(\mathbf{J}) = \text{const} = 1$ when $\mathbf{a}$ denotes initial position, independently of $\mathbf{a}, t$. We may then say that the Lagrangian map is volume preserving. For general compressible fluids $\mathrm{Det}(\mathbf{J})$ will vary in space and time.

Another important assumption that we shall make is that the map M_t is always invertible, $\mathrm{Det}(\mathbf{J}) > 0$. Thus when needed we can invert to express $\mathbf{a}$ as a function of $\mathbf{x}, t$.

1.2. The Material Derivative

Suppose some scalar property P of the fluid can be attached to a certain fluid parcel, e.g., temperature or density. Further, suppose that, as the parcel moves, this property is invariant in time. We can express this fact by the equation

$$\left.\frac{\partial P}{\partial t}\right|_{\mathbf{a}} = 0, \tag{1.9}$$

since this means that the time derivative is taken with particle label fixed, i.e., taken as we move with the fluid particle in question. We will say that such an invariant scalar is *material*. A material invariant is one attached to a fluid particle. We now ask how this property should be expressed in Eulerian variables. That is, we select a point $\mathbf{x}$ in space and seek to express material invariance in terms of properties of the fluid *at this point*. Since the fluid is generally moving at the point, we need to bring in the velocity. The way to do this is to differentiate $P(\mathbf{x}(\mathbf{a}, t), t)$, expressing the property as an Eulerian variable, using the chain rule:

$$\left.\frac{\partial P(\mathbf{x}(\mathbf{a}, t), t)}{\partial t}\right|_{\mathbf{a}} = 0 = \left.\frac{\partial P}{\partial t}\right|_{\mathbf{x}} + \left.\frac{\partial x_i}{\partial t}\right|_{\mathbf{a}} \left.\frac{\partial P}{\partial x_i}\right|_{t} = P_t + \mathbf{u} \cdot \nabla P. \tag{1.10}$$

In fluid dynamics the Eulerian operator $\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ is called the *material derivative* or *substantive* derivative or *convective* derivative. Sometime $\mathbf{u} \cdot \nabla \mathbf{u}$ is called the "convective part" of the derivative. Clearly it is a time derivative "following the fluid" and expresses the Lagrangian time derivative in terms of Eulerian properties of the fluid.

EXAMPLE 1.6. The *acceleration* of a fluid parcel is defined as the material derivative of the velocity $\mathbf{u}$. In Lagrangian variables the acceleration is $\left.\frac{\partial^2 \mathbf{x}}{\partial t^2}\right|_{\mathbf{a}}$, and in Eulerian variables the acceleration is $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}$.

Following a common convention we shall often write

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \tag{1.11}$$

so the acceleration becomes $D\mathbf{u}/Dt$.

EXAMPLE 1.7. We consider the material derivative of the determinant of the Jacobian $\mathbf{J}$. We may divide up the derivative of the determinant into a sum of N

determinants, the first having the first row differentiated, the second having the next row differentiated, and so on. The first term is thus the determinant of the matrix

$$\begin{pmatrix} \frac{\partial u_1}{\partial a_1} & \frac{\partial u_1}{\partial a_2} & \cdots & \frac{\partial u_1}{\partial a_N} \\ \frac{\partial x_2}{\partial a_1} & \frac{\partial x_2}{\partial a_2} & \cdots & \frac{\partial x_2}{\partial a_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_N}{\partial a_1} & \frac{\partial x_N}{\partial a_2} & \cdots & \frac{\partial x_N}{\partial a_N} \end{pmatrix}. \tag{1.12}$$

If we expand the terms of the first row using the chain rule, e.g.,

$$\frac{\partial u_1}{\partial a_1} = \frac{\partial u_1}{\partial x_1}\frac{\partial x_1}{\partial a_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial x_2}{\partial a_1} + \cdots + \frac{\partial u_1}{\partial x_N}\frac{\partial x_N}{\partial a_1}, \tag{1.13}$$

we see that we will get a contribution only from the terms involving $\partial u_1/\partial x_1$, since all other terms involve the determinant of a matrix with two identical rows. Thus the term involving the derivative of the top row gives the contribution

$$\frac{\partial u_1}{\partial x_1}\operatorname{Det}(\mathbf{J}).$$

Similarly, the derivatives of the second row gives the additional contribution

$$\frac{\partial u_2}{\partial x_2}\operatorname{Det}(\mathbf{J}).$$

Continuing, we obtain

$$\frac{D}{Dt}\operatorname{Det}\mathbf{J} = \operatorname{div}(\mathbf{u})\operatorname{Det}(\mathbf{J}). \tag{1.14}$$

Note that, since an incompressible fluid has $\operatorname{Det}(\mathbf{J}) = \text{const} > 0$, such a fluid must satisfy, by (1.14), $\operatorname{div}(\mathbf{u}) = 0$, which is the way an incompressible fluid is defined in Eulerian variables.

1.2.1. Solenoidal Velocity Fields. The adjective *solenoidal* applied to a vector field is equivalent to "divergence free." We will use either $\operatorname{div}(\mathbf{u})$ or $\nabla \cdot \mathbf{u}$ to denote divergence. The incompressibility of a material with a solenoidal vector field means that the Lagrangian map M_t preserves volume and so whatever fluid moves into a fixed region of space is matched by an equal amount of fluid moving out. In two dimensions the equation expressing the solenoidal condition is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.15}$$

If $\psi(x, y)$ possesses continuous second derivatives we may satisfy (1.15) by setting

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{1.16}$$

The function ψ is called the *stream function* of the velocity field. The reason for the term is immediate: The instantaneous streamline passing through x, y has direction $(u(x, y), v(x, y))$ at this point. The normal to the streamline at this point is $\nabla\psi(x, y)$. But we see from (1.16) that $(u, v) \cdot \nabla\psi = 0$ there, so the lines of constant ψ are the instantaneous streamlines of (u, v).

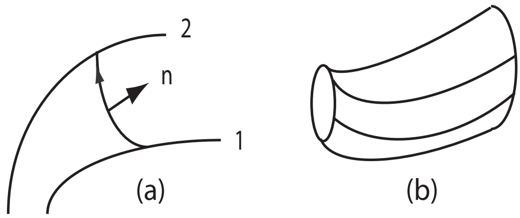

FIGURE 1.4. Solenoidal velocity fields. (a) Two streamlines in two dimensions. (b) A stream tube in three dimensions.

Consider two streamlines $\psi = \psi_i$, $i = 1, 2$ and any oriented simple contour (no self-crossings) connecting one streamline to the other. The claim is then that the flux of fluid across this contour, from left to right seen by an observer facing in the direction of orientation of the contour, is given by the difference of the values of the stream function, $\psi_2 - \psi_1$, if the contour is oriented to go from streamline 1 to streamline 2; see Figure 1.4(a). Indeed, oriented as shown the line integral of flux is just $\int (u, v) \cdot (dy, -dx) = \int d\psi = \psi_2 - \psi_1$. In three dimensions, we similarly introduce a *stream tube*, consisting of a collection of streamlines; see Figure 1.4(b). The flux of fluid across any surface cutting through the tube must be the same. This follows immediately by applying the divergence theorem to the integral of $\operatorname{div} \mathbf{u}$ over the stream tube. Note that we are referring here to the flux of volume of fluid, not to the flux of mass. In three dimensions there are various "stream functions" used when special symmetries allow them. An example of a class of solenoidal flows generated by two scalar functions takes the form $\mathbf{u} = \nabla\alpha \times \nabla\beta$, where the intersections of the surfaces of constant $\alpha(x, y, z)$ and $\beta(x, y, z)$ are the streamlines. Since $\nabla\alpha \times \nabla\beta = \nabla \times (\alpha\nabla\beta)$, we see that these flows are indeed solenoidal.

1.2.2. The Convection Theorem. Suppose that S_t is a region of fluid particles and let $f(\mathbf{x}, t)$ be a scalar function. Forming the volume integral over S_t, $F = \int_{S_t} f \, dV_{\mathbf{x}}$, we seek to compute $\frac{dF}{dt}$. Now

$$dV_{\mathbf{x}} = dx_1 \cdots dx_N = \operatorname{Det}(\mathbf{J}) da_1 \cdots da_N = \operatorname{Det}(\mathbf{J}) dV_{\mathbf{a}}.$$

Thus

$$\begin{aligned}
\frac{dF}{dt} &= \frac{d}{dt} \int_{S_0} f(\mathbf{x}(\mathbf{a}, t), t) \operatorname{Det}(\mathbf{J}) dV_{\mathbf{a}} \\
&= \int_{S_0} \operatorname{Det}(\mathbf{J}) \frac{d}{dt} f(\mathbf{x}(\mathbf{a}, t), t) dV_{\mathbf{a}} + \int_{S_0} f(\mathbf{x}(\mathbf{a}, t), t) \frac{d}{dt} \operatorname{Det}(\mathbf{J}) dV_{\mathbf{a}} \\
&= \int_{S_0} \left[\frac{Df}{Dt} + f \operatorname{div}(\mathbf{u}) \right] \operatorname{Det}(\mathbf{J}) dV_{\mathbf{a}},
\end{aligned}$$

and so

$$\frac{dF}{dt} = \int_{S_t} \left[\frac{Df}{Dt} + f \operatorname{div}(\mathbf{u}) \right] dV_{\mathbf{x}}. \tag{1.17}$$

The result (1.17) is called the *convection theorem.* We can contrast this calculation with one over a fixed finite region R of space with boundary ∂R. In that case the rate of change of f contained in R is just

$$\frac{d}{dt} \int_R f \, dV_{\mathbf{x}} = \int_R \frac{\partial f}{\partial t} dV_{\mathbf{x}}. \tag{1.18}$$

The difference between the two calculations involves the *flux* of f through the boundary of the domain. Indeed, we can write the convection theorem in the form

$$\frac{dF}{dt} = \int_{S_t} \left[\frac{\partial f}{\partial t} + \operatorname{div}(f\mathbf{u}) \right] dV_{\mathbf{x}}. \tag{1.19}$$

Using the divergence (or Gauss's) theorem, and considering the instant when $S_t = R$, we have

$$\frac{dF}{dt} = \int_R \frac{\partial f}{\partial t} dV_{\mathbf{x}} + \int_{\partial R} f\mathbf{u} \cdot \mathbf{n} \, dS_{\mathbf{x}}, \tag{1.20}$$

where $\mathbf{n}$ is the outer normal to the region and $dS_{\mathbf{x}}$ is the area element of ∂R. The second term on the right is flux of f out of the region R. Thus the convection theorem incorporates into the change in f within a region, the flux of f into or out of the region due to the motion of the boundary of the region. Once we identify f with a physical property of the fluid, the convection theorem will be useful for expressing the *conservation* of this property; see Chapter 2.

1.2.3. Material Vector Fields: The Lie Derivative. Certain vector fields in fluid mechanics, and notably the *vorticity field* $\boldsymbol{\omega}(\mathbf{x}, t) = \nabla \times \mathbf{u}$ (see Chapter 3), can in certain cases behave as a *material vector field.* To understand the concept of a material vector one must imagine the direction of the vector to be determined by nearby material points. It is wrong to think of a material vector as attached to a fluid particle and constant there. This would amount to a simple translation of the vector along the particle path.

Instead, the direction of the vector will be that of a differential segment connecting two nearby fluid particles, $dl_i = J_{ij} \, da_j$. Furthermore, the length of the material vector is to be proportional to this differential length as time evolves and the particles move. Consequently, once the particles are selected, the future orientation and length of a material vector will be completely determined by the Jacobian matrix of the flow.

Thus a material vector field will have the form (in Lagrangian variables)

$$v_i(\mathbf{a}, t) = J_{ij}(\mathbf{a}, t) V_j(\mathbf{a}). \tag{1.21}$$

Given the inverse $\mathbf{a}(\mathbf{x}, t)$ we can express v as a function of $\mathbf{x}, t$ to obtain its Eulerian structure.

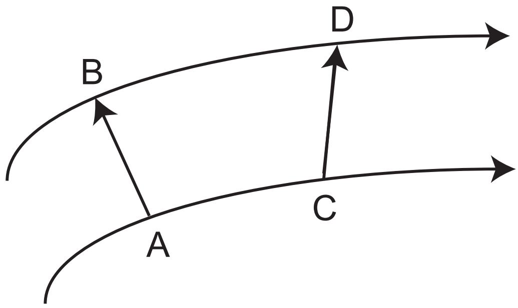

FIGURE 1.5. Computing the time derivative of a material vector.

Consider now the time rate of change of a material vector field following the fluid parcel. We differentiate $v(\mathbf{a}, t)$ with respect to time for fixed $\mathbf{a}$ and develop the result using the chain rule:

$$\begin{aligned} \left.\frac{\partial v_i}{\partial t}\right|_{\mathbf{a}} &= \left.\frac{\partial J_{ij}}{\partial t}\right|_{\mathbf{a}} V_j(\mathbf{a}) = \frac{\partial u_i}{\partial a_j} V_j \\ &= \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial a_j} V_j = v_k \frac{\partial u_i}{\partial x_k}. \end{aligned} \tag{1.22}$$

Introducing the material derivative, a material vector field is seen to satisfy the following equation in Eulerian variables:

$$\frac{D\mathbf{v}}{Dt} = \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{x}} + \mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{u} \equiv v_t + L_{\mathbf{u}}\mathbf{v} = 0. \tag{1.23}$$

In differential geometry $L_{\mathbf{u}}$ is called the Lie derivative of the vector field $\mathbf{v}$ with respect to the vector field $\mathbf{u}$.

The way this works can be understood by moving neighboring points along particle paths. Let $\mathbf{v} = \overline{AB} = \Delta\mathbf{x}$ be a small material vector at time t; see Figure 1.5. At time Δt later, the vector has become $\overline{CD}$. The curved lines are the particle paths through A, B of the vector field $\mathbf{u}(\mathbf{x}, t)$. Selecting A as $\mathbf{x}$, we see that after a small time interval Δt the point C is $A + \mathbf{u}(\mathbf{x}, t)\Delta t$ and D is the point $B + \mathbf{u}(\mathbf{x} + \Delta\mathbf{x}, t)\Delta t$. Consequently,

$$\frac{\overline{CD} - \overline{AB}}{\Delta t} = \mathbf{u}(\mathbf{x} + \Delta\mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t). \tag{1.24}$$

The left-hand side of (1.24) is approximately $\frac{D\mathbf{v}}{Dt}$, and the right-hand side is approximately $\mathbf{v} \cdot \nabla\mathbf{u}$, so in the limit $\Delta\mathbf{x}, \Delta t \to 0$ we get (1.23). A material vector field has the property that its magnitude can change by the stretching properties of the underlying flow, and its direction can change by the rotation of the fluid parcel.

Problem Set 1

(1.1) Consider the flow in the (x, y) plane given by $u = -y$, $v = x + t$. (a) What is the instantaneous streamline through the origin at $t = 1$? (b) What is the path of the fluid particle initially at the origin, $0 < t < 6\pi$? (c) What is the streak line emanating form the origin, $0 < t < 6\pi$?

(1.2) The "point vortex " flow in two dimensions has the velocity field

$$(u, v) = UL\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \quad x^2+y^2 \neq 0,$$

where U, L are reference values of speed and length. (a) Show that the Lagrangian coordinates for this flow may be written

$$x(a, b, t) = R_0 \cos(\omega t + \theta_0), \quad y(a, b, t) = R_0 \sin(\omega t + \theta_0)$$

where $R_0^2 = a^2 + b^2$, $\theta_0 = \arctan(\frac{b}{a})$, and $\omega = UL/R_0^2$. (b) Consider at $t = 0$ a small rectangle of marked fluid particles determined by the points $A(L, 0)$, $B(L + \Delta x, 0)$, $C(L + \Delta x, \Delta y)$, and $D(L, \Delta y)$. If the points move with the fluid, once point A returns to its initial position what is the shape of the marked region? Since $(\Delta x, \Delta y)$ are small, you may assume the region remains a parallelogram. Do this, first, by computing the entry $\frac{\partial y}{\partial a}$ in the Jacobian, evaluated at $A(L, 0)$. Then verify your result by considering the "lag" of particle B as it moves on a slightly larger circle at a slightly slower speed relative to particle A for a time taken by A to complete one revolution.

(1.3) We have noted that Lagrangian coordinates can use any unique labeling of fluid particles. To illustrate this, consider the Lagrangian coordinates in two dimensions

$$x(a, b, t) = a + \frac{1}{k}e^{kb}\sin k(a + ct), \quad y = b - \frac{1}{k}e^{kb}\cos k(a + ct),$$

where k, c are constants. Note here a, b are *not* equal to (x, y) for any t_0. By examining the determinant of the Jacobian, verify that this gives a unique labeling of fluid particles provided that $b \neq 0$. What is the situation if $b = 0$? These waves, which were discovered by Gerstner in 1802, represent gravity waves if $c^2 = \frac{g}{k}$ where g is the acceleration of gravity. They do not have any simple Eulerian representation.

(1.4) In one dimension, the Eulerian velocity is given to be $u(x, t) = \frac{2x}{1+t}$. (a) Find the Lagrangian coordinate $x(a, t)$. (b) Find the Lagrangian velocity as a function of a, t. (c) Find the Jacobean $\frac{\partial x}{\partial a} = J$ as a function of a, t.

(1.5) For the stagnation point flow $\mathbf{u} = (u, v) = \frac{U}{L(x, -y)}$, show that a fluid particle in the first quadrant that crosses the line $y = L$ at time $t = 0$, crosses the line $x = L$ at time $t = \frac{L}{U}\log(\frac{UL}{\psi})$ on the streamline $\frac{Uxy}{L} = \psi$. Do this in two ways. First, consider the line integral of $\mathbf{u} \cdot \vec{ds}/(u^2 + v^2)$ along a streamline. Second, use Lagrangian variables.

(1.6) Let S be the surface of a deformable body in three dimensions, and let $I = \int_S f\mathbf{n}\, dS$ for some scalar function f, $\mathbf{n}$ being the outward normal. Show that

$$\frac{d}{dt}\int f\mathbf{n}\, dS = \int_S \frac{\partial f}{\partial t}\mathbf{n}\, dS + \int_S (\mathbf{u}_b \cdot \mathbf{n})\nabla f\, dS \tag{1.25}$$

where $\mathbf{u}_b$ is the velocity of the surface of the body.

(Hint: First convert to a volume integral between S and an outer surface S' that is *fixed*. Then differentiate and apply the convection theorem. Finally, convert back to a surface integral.)