CHAPTER 4

# Magnetostatics

Magnetic forces act on charged particles, just as electric forces do. But a magnetic force does not depend solely on the charge of the particle on which it acts; it also depends on the particle's velocity. In addition, this force depends on a "magnetic field vector," the counterpart of the electric vector. The magnetic field is assumed to be present everywhere in the space considered, just as that was assumed of the electric field. Denoting the charge by $e$, the velocity vector by $\mathbf{v}$, and the magnetic vector by $\mathbf{B}$, we can write the force exerted by the magnetic field on the particle as the vector

$$\mathbf{F} = e(\mathbf{v} \times \mathbf{B}).$$

Here $\mathbf{a} \times \mathbf{b}$ is the vector product of the vectors $\mathbf{a}$ and $\mathbf{b}$; hence

$$F_1 = e(v_2 B_3 - v_3 B_2), \quad F_2 = e(v_3 B_1 - v_1 B_3), \quad F_3 = e(v_1 B_2 - v_2 B_1).$$

The relationship between $\mathbf{F}$, $\mathbf{v}$, and $\mathbf{B}$ for $e > 0$ is indicated in the diagram (Figure 4.1).

It is seen that the magnetic force acts in a direction perpendicular to both the magnetic field and the velocity of the particle.

The force $e(\mathbf{v}\times\mathbf{B})$, proposed by Lorentz after 1890, is referred to as the "Lorentz force." We have called the vector $\mathbf{B}$ the "magnetic field vector"; actually, the "official" name of this vector is "magnetic induction," while the name "magnetic field" is used for another vector, $\mathbf{H}$, which is related to $\mathbf{B}$. The term "magnetic induction" is rather awkward and somewhat misleading. Since we shall never use a name for the vector $\mathbf{H}$, we shall feel free to use the term "magnetic field" for the vector $\mathbf{B}$.

It should also be mentioned that in some presentations of the theory of magnetism, in particular in earlier ones, the vector $\mathbf{B}$ is written as $c^{-1}\mathbf{B}$, where $c$ is the speed of light. We shall discuss this and other notational matters at the end of this chapter.

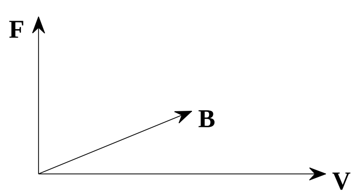


FIGURE 4.1

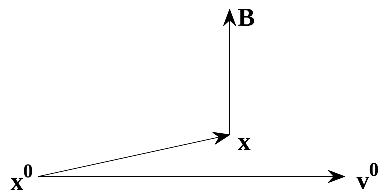


FIGURE 4.2

A charged stationary particle in its active role produces an electric field according to Coulomb's law. A magnetic field is also produced by a charged particle, provided this particle moves. Let $e^0$ be the charge of the active particle located at a point $\mathbf{x}^0$, and let $\mathbf{v}^0$ be its velocity assumed to be constant. Then the magnetic field $\mathbf{B}(\mathbf{x})$ at a point $\mathbf{x}$, produced instantaneously by this particle is

$$\mathbf{B}(\mathbf{x}) = \kappa_B e^0 \mathbf{v}^0 \times (\mathbf{x} - \mathbf{x}^0)|\mathbf{x} - \mathbf{x}^0|^{-3}.$$

Here $\kappa_B$ is a certain positive constant. Evidently, this law is the counterpart of Coulomb's law; it also involves the vector $(\mathbf{x} - \mathbf{x}^0)|\mathbf{x} - \mathbf{x}^0|^{-3}$; but it is the vector product of this vector with the velocity vector that produces the magnetic field. The relationship between $\mathbf{B}$, $\mathbf{v}$, and $\mathbf{x} - \mathbf{x}^0$, for $e^0 > 0$ is indicated in Figure 4.2.

Inasmuch as the active particle moves, the magnetic field produced by this particle cannot be stationary. A stationary field can however be produced by an assembly of charged particles moving in various directions. We assume that these particles form a steady current with the density

$$\mathbf{j}^0(\mathbf{x}') = \eta^1 \mathbf{v}^1(\mathbf{x}') + \eta^2 \mathbf{v}^2(\mathbf{x}') + \cdots .$$

Then the natural extension of the law given above is

$$\mathbf{B}(\mathbf{x}) = \kappa_B \int \mathbf{j}^0(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}\, d\mathbf{x}'.$$

Actually, a stationary magnetic field $\mathbf{B}$ will be produced according to this law, provided the active current is divergence-free,

$$\nabla' \cdot \mathbf{j}^0(\mathbf{x}') = 0,$$

so that no charges are built up from the current. The law so formulated is that of Biot and Savart. The fact that a current produces a magnetic field was discovered earlier (1820) by Oersted. We shall give examples of magnetic fields produced by a given distribution of currents only after we have derived the differential form of this law; at present we discuss facts that follow immediately from its present form that involves action at a distance.

First we shall express explicitly the force exerted by a magnetic field produced by a divergence-free steady current distribution $\mathbf{j}^0(\mathbf{x}')$. Actually, we shall not determine the force exerted on a single particle but that exerted on a distributed assembly of moving particles, i.e., on a current $\mathbf{j}(\mathbf{x})$. Clearly, the force per unit volume exerted by the field $\mathbf{B}$ on this current is

$$\mathbf{f}(\mathbf{x}) = (\mathbf{j}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})).$$

Employing the Biot-Savart law and the identity

$$\mathbf{j} \times (\mathbf{j}^0 \times (\mathbf{x} - \mathbf{x}')) = \mathbf{j}^0(\mathbf{j} \cdot (\mathbf{x} - \mathbf{x}')) - (\mathbf{j} \cdot \mathbf{j}^0)(\mathbf{x} - \mathbf{x}'),$$

we obtain the relation

$$\begin{aligned}\mathbf{f}(\mathbf{x}) = \kappa_B \int \mathbf{j}^0(\mathbf{x}') \frac{\mathbf{j}(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, d\mathbf{x}' \\ - \kappa_B \int (\mathbf{j}^0(\mathbf{x}') \cdot \mathbf{j}(\mathbf{x})) \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}\, d\mathbf{x}'.\end{aligned}$$

For the total force

$$F = \int \mathbf{f}(\mathbf{x}) d\mathbf{x}$$

exerted on the distributed currents $\mathbf{j}$, we then obtain the relation

$$\begin{aligned}\mathbf{F} = \kappa_B \iint \mathbf{j}^0(\mathbf{x}')(\mathbf{j}(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}')) \frac{d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}\, d\mathbf{x} \\ - \kappa_B \iint (\mathbf{j}^0(\mathbf{x}') \cdot \mathbf{j}(\mathbf{x})) \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}\, d\mathbf{x}'\, d\mathbf{x}.\end{aligned}$$

We now assume that the current $\mathbf{j}(\mathbf{x})$ is divergence-free; then, we claim, the first integral is zero.

We assume that the functions $\mathbf{j}^0(\mathbf{x}')$ and $\mathbf{j}(\mathbf{x})$ are sufficiently smooth and die out sufficiently at $\infty$; then we know that we can interchange the order of integration. Using the formula

$$\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = -\nabla |\mathbf{x} - \mathbf{x}'|^{-1},$$

already used in electrostatics, we can carry out integration by parts obtaining the expression

$$\kappa_B \iint \mathbf{j}^0(\mathbf{x}')(\nabla \cdot \mathbf{j}(\mathbf{x})) |\mathbf{x} - \mathbf{x}'|^{-1}\, d\mathbf{x}'\, d\mathbf{x}$$

for the first integral. Evidently it equals zero. Thus *the force exerted by divergence-free currents on a divergence-free current* is given by

$$\mathbf{F} = -\kappa_B \iint (\mathbf{j}^0(\mathbf{x}') \cdot \mathbf{j}(\mathbf{x})) \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}\, d\mathbf{x}'\, d\mathbf{x}.$$

Clearly, this expression for $\mathbf{F}$ goes over into that for $-\mathbf{F}$ if the roles of $\mathbf{j}^0$ and $\mathbf{j}$ are interchanged; in other words, the law of *action* and *reaction* holds for the magnetic forces exerted and experienced by divergence-free currents.

Of course, we cannot expect this reaction law to hold if one current is required to be divergence-free and the other one is allowed to be not divergence-free.

It is interesting that the force $\mathbf{F}$ just determined has the same form as it has for the electrostatic interaction; the role of $e^0 e$ is taken by the inner product $\mathbf{j}^0 \cdot \mathbf{j}$. But we also see that two parallel currents attract each other, while two equal charges repel each other.

Using an argument previously used in electrostatics, we can conclude that *a divergence-free current distribution exerts no force on itself.*

The term $(\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|^3$, which is the gradient of $-|\mathbf{x} - \mathbf{x}'|^{-1}$, was used in electrostatics to express the electric field vector as the negative gradient of a potential. Similarly, we can express the magnetic field vector **B** as the curl of a "vector potential," namely the vector

$$\chi(\mathbf{x}) = \kappa_B \int \mathbf{j}^0(\mathbf{x}')|\mathbf{x} - \mathbf{x}'|^{-1}\, d\mathbf{x}';$$

that is to say, the relation

$$B(\mathbf{x}) = \nabla \times \chi(\mathbf{x})$$

holds. This relation is readily verified; it will be helpful in many ways.

It is clear that the Biot-Savart formula applies also to limiting cases. In an obvious way it yields the magnetic fields produced by sheet currents and by line currents. Of particular significance is the limit of the magnetic field produced by a ring current, if one lets the ring shrink to a point while allowing the current strength to increase suitably.

Suppose the ring is a circle

$$\hat{\mathbf{x}} = (\hat{r}\cos\theta, \hat{r}\sin\theta, 0)$$

and the line current in it is given by

$$(-J\sin\theta, J\cos\theta, 0).$$

Then the vector potential of the resulting magnetic field is

$$\chi(\mathbf{x}) = \kappa_B \hat{r} J \int (-\sin\theta, \cos\theta, 0) \cdot [|\mathbf{x}|^2 - 2\hat{r}(x_1\cos\theta + x_2\sin\theta) + \hat{r}^2]^{-1/2}\, d\theta.$$

We now let $\hat{r}$, the radius of the ring, tend to zero while $J$ grows such that the number $\pi\hat{r}^2 J$ tends to a finite constant $K^0$. Expansion with respect to $\hat{r}$ then yields the expression

$$\chi(\mathbf{x}) = \kappa_B K^0 |\mathbf{x}|^{-3}(-x_2, x_1, 0)$$

for the limit vector potential.

More generally, let $\mathbf{u}^0$ be any unit vector at a point $\mathbf{x}^0$, and let the current ring be in the plane perpendicular to $\mathbf{u}^0$ with the center at $\mathbf{x}^0$. Then the same limit process yields the vector potential

$$\chi(\mathbf{x}) = \kappa_B[\mathbf{K}^0 \times (\mathbf{x} - \mathbf{x}^0)]|\mathbf{x} - \mathbf{x}^0|^{-3},$$

where $\mathbf{K}^0$ is the limit of the vector $\pi\hat{r}^2 J\mathbf{u}^0$ . The limit vector potential $\chi$ is ascribed to a "magnetic dipole" with strength and direction given by the vector $\mathbf{K}^0$ . Taking a continuous distribution of the dipoles with the density $\mathbf{k}^0(\mathbf{x}')$ we can build up the vector potential

$$\chi(\mathbf{x}) = \kappa_B \int \frac{\mathbf{k}^0(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, d\mathbf{x}',$$

which, after integration by parts, can be written as

$$\chi(\mathbf{x}) = \kappa_B \int \frac{\nabla' \times \mathbf{k}^0(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d\mathbf{x}.$$

Then we see that the dipole vector potential can be generated from the current distribution

$$\mathbf{j}^0(\mathbf{x}') = \nabla' \times \mathbf{k}^0(\mathbf{x}').$$

Evidently, $\mathbf{j}^0(\mathbf{x}')$ is divergence-free. Actually, every divergence-free current distribution can be derived from a dipole distribution.

If we allow that the dipole vector $\mathbf{k}^0(\mathbf{x}')$ jumps across a surface $\mathcal{S}$ the additional term

$$\kappa_B \int_{\mathcal{S}} \frac{[\mathbf{n}'(\mathbf{x}') \times \mathbf{k}^0(\mathbf{x}')]}{|\mathbf{x}-\mathbf{x}'|}\, d\mathcal{S}$$

is to be added to the expression for $\chi(\mathbf{x})$ after integration by parts. Thus the sheet current

$$\mathcal{F}^0(\mathbf{x}') = [\mathbf{n}'(\mathbf{x}') \times \mathbf{k}^0(\mathbf{x}')]$$

also contributes to the vector potential $\chi(\mathbf{x})$.

Materials that carry a distribution of magnetic dipoles will be referred to as "magnets," since actual magnets can be so regarded. We proceed to evaluate the total force and the total moment of the forces exerted by a magnetic field on such a magnet carrying a dipole distribution.

Employing the relation $\mathbf{j} = \nabla \times \mathbf{k}$, we can express the total force $\mathbf{F}$,

$$\mathbf{F} = \int \mathbf{j}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) d\mathbf{x},$$

acting on the magnet, as

$$\mathbf{F} = \int (\nabla \times \mathbf{k}(\mathbf{x})) \times \mathbf{B}(\mathbf{x}) d\mathbf{x}.$$

In case the dipole strength $\mathbf{k}(\mathbf{x})$ jumps across a surface $\mathcal{S}$, the term

$$\int [\mathbf{n} \times \mathbf{k}(\mathbf{x})] \times \mathbf{B}(\mathbf{x}) d\mathcal{S}$$

should be added to this expression. In any case we assume that $\mathbf{k}(\mathbf{x})$ dies out at infinity sufficiently strongly to allow us to carry out integration by parts.

Assume at first that $\mathbf{B}(\mathbf{x})$ is constant. Then integration by parts yields zero. Thus we can say: *A constant magnetic field exerts no force on a magnet.* It follows that the force exerted by any magnetic field on a magnet depends on the deviation of the field from constancy.

Consider two cylindrical magnets formed by constant magnetic dipoles, placed at some distance from each other with poles aligned. Specifically, let them occupy the regions

$$\mathrm{Re}_{\pm}\colon \quad x_1^2 + x_2^2 < R^2, \quad 0 < a \le \pm x_3 \le b, \text{ respectively,}$$

with dipoles $\mathbf{k} = (0, 0, k_3)$ in $\mathrm{Re}_{\pm}$ and $= 0$ otherwise. Then the resulting magnetic field will, in some approximation, be constant in the $x_3$-direction, thus pointing from one magnet to the other. One might then be inclined to compute the force with which these magnets attract each other approximately by employing this constant approximating field **B**. We now know that the force so computed would be zero. In other words, the force with which the magnets attract each other stems from the small deviation of the magnetic field from constancy.

To determine this force from these deviations consider a magnet with a constant dipole vector **k** covering a region $\mathscr{G}$. Then the current $\mathbf{j} = \nabla \times \mathbf{k}$ is zero inside $\mathscr{G}$; the only actual current is a sheet current $-\mathbf{n} \times \mathbf{k}$ on the boundary of $\mathscr{S}$. Assume $\mathscr{G}$ to be a cylinder with mantle $\mathscr{M}$ and two end plates with normals in the direction of **k**. Then the sheet current runs only over the mantle. The total force **F** exerted by a magnetic field is then

$$\begin{aligned}\mathbf{F} &= -\int\limits_{\mathscr{M}} (\mathbf{n} \times \mathbf{k}) \times \mathbf{B}\, d\mathscr{S} \\ &= \int\limits_{\mathscr{M}} (\mathbf{n} \cdot \mathbf{B}) d\mathscr{S}\, \mathbf{k} - \int\limits_{\mathscr{M}} (\mathbf{k} \cdot \mathbf{B}) \mathbf{n}\, d\mathscr{S}.\end{aligned}$$

Under certain circumstances the last term is zero.

For example, assume $\mathscr{M}$ to be given by $x_{1,2} = x_{1,2}(s)$, $|x_3| \le b$, so that $d\mathscr{S} = ds\, dx_3$. Then the last term in the expression for **F** is zero if $\mathbf{k} \cdot \mathbf{B}$ is constant on each ring $x_3 = \text{const}$. For in that case

$$\int\limits_{\mathscr{M}} (\mathbf{k} \cdot \mathbf{B}) \mathbf{n}\, d\mathscr{S} = \int_{-b}^{b} \mathbf{k} \cdot \mathbf{B} \int \mathbf{n}\, ds\, dx_3 = 0,$$

since $\int \mathbf{n}\, ds = 0$. In this case then *the force exerted by the field **B** on the magnet* is given by the simple formula

$$\mathbf{F} = \int\limits_{\mathscr{M}} \mathbf{n} \cdot \mathbf{B}\, d\mathscr{S}\, \mathbf{k}.$$

This formula shows that the force **F** exerted on the magnet depends only on the component of the magnetic field in the direction normal to the mantle and not on the component in the direction of **k**. Note that under the circumstances described earlier, the main direction of the magnetic field would be that of **k**.

Next we determine the *moment of the forces* exerted on a magnet, also called the "torque." Here we assume again the magnetic field that exerts these forces to be constant. We can write

$$\begin{aligned}\mathbf{M} &= \int \mathbf{x} \times (\mathbf{j}(\mathbf{x}) \times \mathbf{B}) d\mathbf{x} \\ &= \int (\mathbf{x} \cdot \mathbf{B}) \mathbf{j}(\mathbf{x}) d\mathbf{x} - \mathbf{B} \int \mathbf{x} \cdot \mathbf{j}(\mathbf{x}) d\mathbf{x}.\end{aligned}$$

The current **j** is assumed to derive as $\mathbf{j} = \nabla \times \mathbf{k}$ from a dipole distribution **k**, assumed to die out strongly at infinity.

We now write $\mathbf{x} = \frac{1}{2}\nabla|\mathbf{x}|^2$ in the second integral, integrate by parts, and use $\nabla \cdot \mathbf{j} = 0$. It follows that this second integral is zero. In the first integral we set $\mathbf{j}(\mathbf{x}) = \nabla \times \mathbf{k}(\mathbf{x})$ and then integrate by parts. The result is

$$\mathbf{M} = \mathbf{K} \times \mathbf{B}$$

with $\mathbf{K} = \int \mathbf{k}\, d\mathbf{x}$ being the "total dipole strength." Thus we have expressed the torque exerted on the magnet by the constant field $\mathbf{B}$ in terms of the total dipole strength of the magnet.

It is interesting that in the history of magnetic theory the torque exerted on a magnet was described very early, while the Lorentz force, from which it was derived here, was formulated only much later.

In electrostatics we have shown that Coulomb's "long distance" law can be replaced by *differential laws*; the same can be done with Biot-Savart's law of magnetism.

First of all, since the magnetic field $\mathbf{B}$ can be expressed as a curl,

$$\mathbf{B}(\mathbf{x}) = \nabla \times \chi(\mathbf{x}),$$

and since a curl is divergence-free, the relation

$$\text{II:} \quad \nabla \cdot \mathbf{B}(\mathbf{x}) = 0$$

holds. Next we determine the curl of $\mathbf{B}$ from the relation

$$\nabla \times \nabla \times \chi = \nabla\nabla \cdot \chi - \nabla \cdot \nabla \chi.$$

We now know that $\nabla \cdot \chi = 0$ by virtue of the form

$$\chi(\mathbf{x}) = \kappa_B \int \mathbf{j}^0(\mathbf{x}')|\mathbf{x} - \mathbf{x}'|^{-1}\, d\mathbf{x}$$

of the vector potential, and the assumption that the active current $\mathbf{j}^0(\mathbf{x}')$ is divergence-free. The term $\nabla \cdot \nabla \chi$ can be evaluated just as in electrostatics with each component of $\mathbf{j}^0(\mathbf{x}')$ taking the place of $\eta^0(\mathbf{x}')$. Hence the retrieval formula

$$-\nabla \cdot \nabla \chi(\mathbf{x}) = 4\pi\kappa_B \mathbf{j}^0(\mathbf{x})$$

holds. We now introduce the constant

$$\nu_0 = \frac{1}{4\pi\kappa_B}.$$

We prefer working with this constant instead of the "magnetic permeability"

$$\mu_0 = 4\pi\kappa_B,$$

since $\nu_0$ is the counterpart of the permittivity $\epsilon_0$ used in electrostatics. We then can write the last equation as

$$\text{I:} \quad \nu_0 \nabla \times \mathbf{B}(\mathbf{x}) = \mathbf{j}^0(\mathbf{x}).$$

The laws I and II are the basic differential laws of magnetostatics.

It is clear how to extend these laws to sheet currents, and also how to derive integral forms for them. The local law for a sheet current $\tilde{\mathbf{j}}^0(\mathbf{x})$ on a surface $\mathcal{S}$ is

$$\tilde{\text{I}}: \quad \nu_0[\mathbf{n}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})] = \tilde{\mathbf{j}}^0(\mathbf{x}),$$

where $\mathbf{n}(\mathbf{x})$ is the normal on $\mathscr{S}$. One may verify that this sheet current $\tilde{\mathbf{j}}(\mathbf{x})$ is also divergence-free on the surface $\mathscr{S}$ provided no current crossing this surface gets lost in it or emerges from it, i.e., if $[\mathbf{j}_n] = 0$ on $\mathscr{S}$. The global law comprising both laws I and $\tilde{\text{I}}$ is

$$\text{I}^{\text{I}}: \quad \nu_0 \int_{\mathscr{L}} (\mathbf{n}_{\mathscr{S}} \times \mathbf{n}_{\mathscr{L}}) \cdot \mathbf{B}\, ds = \int_{\mathscr{S}} \mathbf{n}_{\mathscr{S}} \cdot \mathbf{j}\, d\mathscr{S}$$

when $\mathscr{L}$ is the boundary of the surface $\mathscr{S}$ and $\mathbf{n}_{\mathscr{L}}$ is the normal on $\mathscr{L}$ that is perpendicular to the normal $\mathbf{n}_{\mathscr{S}}$ on $\mathscr{S}$.

Analogous laws hold also for line currents. We shall not derive these laws; nevertheless, we shall rely on them in connection with special problems.

With the aid of the integral

$$\int |\nabla \chi|^2\, d\mathbf{x},$$

we could readily prove, as in electrostatics, that the vector potential $\chi$ is uniquely determined by the current distribution. The relation $\nabla \cdot \chi = 0$ readily follows from the relation $\nabla \cdot \mathbf{j}^0 = 0$.

One could also show, by integration by parts, that

$$\int \|\nabla \times \chi\|^2\, d\mathbf{x} = \int \mathbf{B} \cdot \mathbf{B}\, d\mathbf{x}.$$

The expression $\frac{1}{2}\nu_0 \int \mathbf{B} \cdot \mathbf{B}\, d\mathbf{x}$ can be interpreted as the magnetic energy.

We describe two typical magnetic fields. They refer to the hollow cylinder

$$0 \le R_0 < r < R_1, \quad r^2 = x_1^2 + x_2^2.$$

The various quantities are given in the three regions:

| | $0 \le r \le R_0$ | $R_0 \le r < R_1$ | $R_1 \le r$ |
|---|---|---|---|
| I: | $j_3 = 0$ | $2b$ | $0$ |
| | $\gamma_0 B_{1,2} = 0$ | $\mp b(1 - R_0^2 r^{-2}) x_{2,1}$ | $\mp b(1 - R_1^2 - R_0^2) r^{-2} x_{2,1}$ |
| | $f_{1,2} = 0$ | $-2b^2(1 - R_0^2 r^{-2}) x_{1,2}$ | $0$ |

Note that the total current is $J = 2\pi b(R_1^2 - R_0^2)$ so that the outer field depends only on it.

| | $0 \le r \le R_0$ | $R_0 \le r < R_1$ | $R_1 \le r$ |
|---|---|---|---|
| II: | $j_{1,2} = 0$ | $\mp 2a x_{2,1}$ | $0$ |
| | $\nu_0 B_3 = a(R_1^2 - R_0^2)$ | $a(R_1^2 - r^2)$ | $0$ |
| | $\mu f_{1,2} = 0$ | $2a^2(R_1^2 - r^2) x_{1,2}$ | $0$ |

Here the total current per unit length is $\tilde{J} = 2a(R_1 - R_0)$. The limits for $R_0 \to 0$ are easily determined.

Just as in a dielectric medium an electric field arouses a dipole distribution; also, *magnetic fields will arouse a magnetic dipole distribution* in certain materials. Under certain circumstances the aroused dipole strength is proportional to the magnetic field

$$\mathbf{k}(\mathbf{x}) = (\nu_0 - \nu(\mathbf{x}))\mathbf{B}(\mathbf{x}),$$

where $\nu(\mathbf{x})$ is a given function of $\mathbf{x}$, the inverse of the "permeability." The nonaroused part of the current, the "introduced" or "true" one, denoted by $\mathbf{j}(\mathbf{x})$, is then related to the field $\mathbf{B}$ by the formula

$$\mathbf{j}(\mathbf{x}) = \nabla \times [\nu(\mathbf{x})\mathbf{B}(\mathbf{x})] = \nabla \times \mathbf{H}(\mathbf{x}).$$

The vector

$$\mathbf{H} = \nu\mathbf{B},$$

which yields the introduced current, is the one that was originally called the "magnetic field" vector.

An "aroused magnet" must be distinguished from a permanent magnet, whose dipole field $\mathbf{k}$ is present without an introduced magnetic field. For such a permanent magnet one can define the vector $\mathbf{H}$ by

$$\mathbf{H} = \nu_0\mathbf{B} - \mathbf{k};$$

then again the relation $\mathbf{j} = \nabla \times \mathbf{H}$ holds for the introduced current.

Finally, we shall describe a *magnetic analogue of Ohm's law*. In electrostatics this law governs the formation of a current in a wire under the influence of the forces exerted by an electric field. Suppose magnetic, rather than electric, forces are effective in a wire. We can go through the same steps that led to the formulation of Ohm's law, replacing the electric by magnetic forces, and we will automatically be led to a counterpart of Ohm's law.

Magnetic forces will be exerted on the charged particles in a wire if this wire moves. With $\mathbf{v}$ being the *constant* velocity of the wire, the magnetic force exerted on a particle with the charge $e$ is

$$e[\mathbf{v} \times \mathbf{B}].$$

We now simply substitute $[\mathbf{v} \times \mathbf{B}]$ for $\mathbf{E}$ in the steps that led to Ohm's law; then we arrive at the relations

$$\sigma[\mathbf{v} \times \mathbf{B}] = \mathbf{j}_{\mathscr{W}}, \quad [\mathbf{v} \times \mathbf{B}] = r\mathbf{j}_{\mathscr{W}},$$

in which $\mathbf{j}_{\mathscr{W}}$ is the density of the current relative to the wire. Inasmuch as the wire moves, the motion of charged particles with the wire will also contribute to the total current. If $\eta$ is the density of these charges, this purely convective contribution to the total current density is $\eta\mathbf{v}$. Hence the total current density is

$$\mathbf{j} = \mathbf{j}_{\mathscr{W}} + \eta\mathbf{v}.$$

The magnetic form of Ohm's law can now be written as

$$\sigma[\mathbf{v} \times \mathbf{B}] = \mathbf{j} - \eta\mathbf{v}, \quad [\mathbf{v} \times \mathbf{B}] = r(\mathbf{j} - \eta\mathbf{v}).$$

We may say that just as a current flows through a wire connecting two charged conductors, such a current also flows if a magnetic field is present and the wire moves. Many, if not most, machines that generate electric current with the aid of magnetic fields are based on this fact.

A particular case of the process here described, namely the case in which the wire forms a loop, is commonly said to be a variety of "magnetic induction." We shall discuss this interpretation when we deal with the other variety of magnetic induction. The description here presented is mentioned in the literature but kept somewhat in the background. It is certainly the most direct one, once the Lorentz force is accepted as the basic magnetic force.

We end this chapter by making some *remarks about different definitions of the notion of magnetic field and the question of units*. In doing this we shall employ the customary term "magnetic induction" for the vector **B**; also we replace $\nu_0$ by $\mu_0^{-1}$. We have said at the beginning of this section that *in some presentations of the theory of magnetism the vector* $c\mathbf{B}$*, rather than* **B***, is regarded as the "magnetic induction"* and therefore denoted by **B**. The connection between magnetic induction and current density then takes the form

$$(c\mu_0)^{-1}\nabla \times \mathbf{B} = \mathbf{j},$$

while the density of the magnetic force is written as

$$\mathbf{f} = c^{-1}\mathbf{j} \times \mathbf{B}.$$

The velocity of light, which enters here, is given by

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}},$$

as we shall show later on. It is claimed that this is advantageous in the treatment of electromagnetism within the framework of the theory of relativity. This claim is not quite convincing as we shall argue when we shall treat that theory.

The arguments in favor of one or the other definition of "**B**" are customarily tied in with *the question of units*: the Gaussian or other older units versus the units proposed in 1901 by Giorgi. Actually, *the choice of the one or the other definition of magnetic induction has nothing to do with the choice of units*; one can assign units to either "**B**" which are consistent with any system of basic units. The connection between the choice of "**B**" and the choice of basic units has developed somehow historically and is at present maintained by somewhat emotional bonds. Which basic quantities and basic units to adopt is a matter of expediency, and on that opinions and taste may differ.

Although these two questions are logically independent we might just as well make a few remarks about units. The Gaussian electromagnetic units result by stipulating that

$$\epsilon_0 = \frac{1}{4\pi}.$$

Then the coulomb as unit of charge can be expressed in terms of the basic units gram, centimeter, and second of mass, length, and time, but in a rather awkward way. The Giorgi system employs the coulomb (C) as an independent unit (actually now the ampere (A) is the basic unit and the coulomb is defined as the product $C = A \cdot \text{sec}$). The unit of voltage, the volt (V), is a derived one; it must be so defined that CV has the dimension of energy since charge times potential is an energy. Actually CV is expressed very directly in terms of kilogram, meter, and second as

$$\mathrm{CV} = \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{sec}^2.$$

This unit of energy is also called a joule.

The units of **E** and **B** then are V/m and $\mathrm{V}\cdot\mathrm{sec}/\mathrm{m}^2$; the unit of $c\mathbf{B}$ is $3\times 10^8$ V/m. The units of $\epsilon_0$ and $\mu_0$ are $\mathrm{C}/\mathrm{V}\cdot\mathrm{m}$ and $\mathrm{V}\cdot\mathrm{sec}^2/\mathrm{C}\cdot\mathrm{m}$. The values of $\epsilon_0$, $\mu_0$, and $\nu_0$ are roughly given by

$$\epsilon_0 \sim .88\times 10^{-12}\ \mathrm{C}/\mathrm{V}\cdot\mathrm{m},$$
$$\mu_0 \sim 1.25\times 10^{-6}\ \mathrm{V}\cdot\mathrm{sec}^2/\mathrm{C}\cdot\mathrm{m},$$
$$\nu_0 \sim .8\times 10^{6}\ \mathrm{C}\cdot\mathrm{m}/\mathrm{V}\cdot\mathrm{sec}^2.$$

In the following we shall rarely have occasion to be concerned with units.