CHAPTER 1

Introduction

1.1. Outline

We will examine the theory of large deviations through three concrete examples. We will work them out in some detail and in the process develop the subject.

The first example is the exit problem. Let $G \subset \mathbb{R}^d$ be a bounded, open domain with smooth boundary ∂G. We consider the solution $u = u_\epsilon$ of the Dirichlet problem

$$\frac{\epsilon}{2}\Delta u + b(x)\cdot\nabla u = 0, \quad x \in G,$$

with boundary condition $u = f$ on ∂G. The vector field b will be of the form $b = -\nabla V$ for some smooth function V. As $\epsilon \to 0$, the limiting behavior of the solution u_ϵ will depend on the behavior of the solutions of the ODE

$$\frac{dx}{dt} = b(x(t)). \tag{1.1}$$

The solution $u_\epsilon(x)$ has the representation

$$u_\epsilon(x) = E_x[f(x(\tau_G))]$$

in terms of the expectation with respect to the distribution $P_{\epsilon,x}$ of the solution $x(t) = x_\epsilon(t)$ of the stochastic integral equation

$$x(t) = x + \int_0^t b(x(s))ds + \sqrt{\epsilon}\beta(t),$$

where $\beta(t)$ is the standard Brownian motion in d dimensions, $\tau_G = \inf\{t : x(t) \notin G\}$ is the exit time from G, and $x(\tau_G)$ is the exit place on ∂G. One expects the limit $u(x) = \lim_{\epsilon\to 0} u_\epsilon(x)$ to exist and be given by

$$u(x) = f(x(\tau_G))$$

where $x(t)$ is the solution of the ODE (1.1).

The difficult case is when the solution of the ODE does not exit from G and therefore $\tau(G) = \infty$. Then large-deviation theory can provide the answer. Assuming that there is a unique stable equilibrium point inside G and that all trajectories starting from $x \in G$ converge to it without leaving G, one can show that

$$\lim_{\epsilon\to 0} u_\epsilon(x) = f(y)$$

provided the minimum of $V(\cdot)$ on the boundary ∂G is attained uniquely at $y \in \partial G$; i.e., for all $y' \in \partial G$ and $y' \neq y$, we have $V(y') > V(y)$.

The second example is about the simple random walk in d dimensions. Let $e_1, \dots, e_d$ be the unit vectors in the d coordinate directions of $\mathbb{Z}^d$ and let $X_1, \dots, X_n, \dots$ be a sequence of independent identically distributed random variables with $P[X_i = \pm e_j] = \frac{1}{2d}$ for $j = 1, \dots, d$. We denote by $S_n = X_1 + \dots + X_n$ the resulting random walk and by D_n

the range of $S_1, \dots, S_n$. Then $|D_n|$ is the number of distinct sites visited by the random walk. The question is the behavior of

$$E[e^{-\nu|D_n|}]$$

for large n. Contribution comes mainly from paths that do not visit too many sites. We can insist that the random walk be confined to a ball of radius $R = R(n)$. Then the number of sites visited is at most the number of lattice points inside the ball, which is approximately $v(d)R^d$ for large R where $v(d)$ is the volume of the unit ball in $\mathbb{R}^d$. On the other hand, confining a random walk to the region for n steps has exponentially small probability $p(n) \simeq \exp[-n\lambda_d(R)] = \exp[-n\frac{\lambda_d}{R^2}]$. Here $-\lambda_d$ is the ground state eigenvalue of the Laplace operator

$$\frac{1}{2d}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}$$

in the unit ball of $\mathbb{R}^d$ with the Dirichlet boundary condition. The contribution from these paths is at least

$$\exp\left[-\nu v(d)R^d - n\frac{\lambda(d)}{R^2}\right],$$

and $R = R(n)$ can be chosen to maximize this contribution. One can fashion a proof that establishes this as a lower bound. But to show that the optimal lower bound obtained in this manner is actually a true upper bound requires a theory.

The third example that we will consider is the symmetric simple exclusion process. On the periodic d-dimensional integer lattice $\mathbf{Z}_N^d$ of size N^d we have $k(N) = \rho N^d$ particles (with at most one particle per site) doing simple random walks independently with rate 1. However jumps to occupied sites are forbidden. The Markov process is defined through the generator

$$(\mathcal{A}_N u)(x_1, \dots, x_{k(N)}) =$$
$$\frac{1}{2d}\sum_{i=1}^{k(N)}\sum_{e}[1-\eta(x_i+e)][u(x_1,\dots,x_i+e,\dots,x_{k(N)}) - u(x_1,\dots,x_{k(N)})]$$

acting on functions u defined on $(\mathbf{Z}_N^d)^{k(N)}$. Here e runs over the units in the $2d$ directions $\{\pm e_j\}$ and $\eta(x) = \sum_i \mathbb{1}_{x_i = x}$ is the particle count at x, which is either 0 or 1. We do a diffusive rescaling of space and time and consider the random measure γ_N on the path space $D[[0,T];\mathcal{T}^d]$,

$$\gamma_N = \frac{1}{N^d}\sum_{1\le i\le k(N)} \delta_{\frac{x_i(N^2\cdot)}{N}}.$$

We want to study the behavior of γ_n as $N \to \infty$. The theory of large deviations is needed even to prove a law of large numbers for γ_N.

1.2. Supplementary Material

Large-deviation theorems in some generality were first established by Cramér in [**2**]. He considered deviations from the law of large numbers for sums of independent identically distributed random variables and showed that the rate function is given by the convex conjugate of the logarithm of the moment-generating function of the underlying common distribution. The subject has evolved considerably over time, and several texts are now available offering different perspectives. The exit problem was studied by Wentzell and

Freidlin in their work [**12**]. They go on to study in [**13**] the long-time behavior of small random perturbations of dynamical systems when there are several equilibrium points.

The problem of counting the number of distinct sites comes up in the discussion of a random walk on $\mathbb{Z}^d$ in the presence of randomly located traps. The estimation of the probability of avoiding traps for a long time reduces to the calculation described in the second example. This problem was proposed by Mark Kac [**16**], along with a similar problem for a Brownian path avoiding traps in $\mathbb{R}^d$. The traps are balls of some fixed radius δ with their centers located randomly as a Poisson point process of intensity ρ. Now the role of the number of distinct sites of the random walk is replaced by the volume $|\bigcup_{0\le s\le t} B(x(s),\delta)|$ of the "Wiener sausage," i.e., the δ-neighborhood of the range of the Brownian path $x(\cdot)$ in $[0,t]$.

The use of large-deviation techniques in the study of hydrodynamic scaling limits began with the work of Guo, Papanicolaou, and Varadhan in [**15**], and the results presented here started with the work of Kipnis, Olla, and Varadhan in [**18**] followed by the study of nongradient systems in [**31**], the Ph.D. thesis of Quastel [**19**], and the work of Quastel, Rezakhanlou, and Varadhan in [**20, 21, 23**].