

# Contents

Preface	ix
Organization of the Text	x
Acknowledgments	x
Chapter 1. Vector Spaces over a Field $\mathbb{K}$	1
1.1. Vector Spaces	1
Basic Properties and Examples	1
Multivariate Polynomials and Multi-Index Notation	4
Row Space and Column Space of a Matrix	7
1.2. Vector Subspaces	7
Systems of Linear Equations and Their Solutions	8
$M(n, \mathbb{R})$ and Other Spaces of Matrices	11
1.3. Solving Matrix Equation $A\mathbf{x} = \mathbf{b}$	12
Elementary Operations and Echelon Form of $A\mathbf{x} = \mathbf{b}$	13
The Homogeneous Equation $A\mathbf{x} = \mathbf{0}$	16
Solving Inhomogeneous Equations $A\mathbf{x} = \mathbf{b}$	17
Determining the Linear Span of a Set of Vectors	18
Implicit and Parametric Descriptions of Vector Subspaces	19
More on Elementary Row and Column Operations	20
1.4. Linear Span, Independence, and Bases	21
Existence and Construction of Bases	21
The Dimension $\dim(V)$ of a Vector Space	24
Implicit and Parametric Description of Subspaces (Revisited)	26
The Lagrange Interpolation Formula	29
Rank of a Matrix: Row Rank vs. Column Rank	30
1.5. Quotient Spaces $V/W$	31
Algebraic Structure in $V/W$	32
Finding Bases in $V/W$ and the Dimension Formula	34
Additional Exercises	38
Appendix: The Degree Formula for $K[x_1, \dots, x_N]$	46
Chapter 2. Linear Operators $T : V \rightarrow W$	49
2.1. Definitions and General Facts	49
Dimension Theorems for Kernel $K(T)$ and Range $R(T)$	51
Computing $K(T)$ and $R(T)$	53
2.2. Isomorphisms and Invariant Subspaces	56
Eigenvalues, Eigenspaces, and the Characteristic Polynomial $p_T(x)$	56
Decomposition of Operators	57

Isomorphisms of Vector Spaces	59
More on Quotient Spaces	60
First and Second Isomorphism Theorems for Quotient Spaces	60
2.3. Direct Sums of Vector Spaces	62
Projection Operators and Direct Sums	64
Direct Sums and Diagonalization	67
Independence of the Eigenspaces	67
2.4. Representing Linear Operators as Matrices	69
The Associated Matrices $[T]_{\mathfrak{B}\mathfrak{B}}$	69
The Correspondence Between Matrices and Linear Operators	72
Change of Basis and Similarity Transformations	76
Similarity Classes in Matrix Space	79
Similarity and RST Equivalence Relations	80
Additional Exercises	82
Chapter 3. Duality and the Dual Space $V^*$	89
3.1. Definitions and Examples	89
Lengths and Orthogonality of Vectors in Inner Product Spaces	90
Duality and the Fourier Transform	93
3.2. Dual Bases in the Dual Space $V^*$	94
3.3. The Transpose $T^T : W^* \rightarrow V^*$ of $T : V \rightarrow W$	97
Matrix Description of a Transpose $T^T$	98
Calculating the Transpose of a Projection Operator	99
Reflexivity of Finite-Dimensional Spaces	101
The Annihilator $M^\circ$ of a Subspace $M$ in $V$	103
A Dimension Formula for Annihilators $M^\circ$	103
Row Rank vs. Column Rank (Revisited)	104
Outline of a Proof That $(\text{RowRank}) = (\text{ColRank})$	104
Additional Exercises	105
Chapter 4. Determinants	109
4.1. The Permutation Group $S_n$	109
Cycles and the Cycle Decomposition Theorem	110
Parity of a Permutation	114
(Optional) An Alternative Proof of the Parity Theorem 4.17	117
4.2. Determinants	118
Proving the Basic Properties of $\mathbf{det}(A)$	119
Row Operations, Determinants, and Matrix Inverses	122
Computing Matrix Inverses	124
Computational Issues	126
Proving the Multiplicative Property $\mathbf{det}(AB) = \mathbf{det}(A)\mathbf{det}(B)$	127
Defining Determinants of Linear Operators	129
More on Rank, RowRank, and ColumnRank	130
Expansion by Minors and Cramer's Rule	132
Additional Exercises	134

Chapter 5. The Diagonalization Problem	139
5.1. Eigenvalues, Characteristic Polynomial, and Spectrum	139
Factoring Polynomials	141
Fundamental Theorem of Algebra	142
The Quadratic Formula	144
5.2. Eigenvalues and the Characteristic Polynomial	145
Finding the Eigenspaces of $T : V \rightarrow V$	147
Example: Rotation Matrices in $\mathbb{R}^2$	150
The Case of Distinct Eigenvalues	152
5.3. Diagonalization and Limits of Operators	153
Norms on Finite-Dimensional Spaces	153
Multiplicative Properties of Norms on Matrix Space	154
The Operator Norm $\ T\ _{\text{op}}$ on Linear Operators and Matrices	155
Equivalence of Norms on Finite-Dimensional Spaces	157
Practical Calculations with Matrix Norms	158
Convergence of Sequences and Series in Matrix Space	158
Properties of Matrix Norms	159
5.4. Application: Computing the Exponential $e^A$ of a Matrix	160
The Cauchy Convergence Criterion in Matrix Space	161
Convergence of the Exponential Series	162
Explicit Computation of $e^A$	162
5.5. Application: Linear Systems of Differential Equations	165
Example: Solving $dx/dt = Ax$	167
5.6. Application: Matrix-Valued Geometric Series	169
Convergence of Matrix-Valued Power Series	170
Small Perturbations of Invertible Matrices	171
Geometric Series for Nilpotent $N$	172
Additional Exercises	172
Chapter 6. Inner Product Spaces	181
6.1. Basic Definitions and Examples	181
Euclidean Norms on $\mathbb{R}^n$ and $\mathbb{C}^n$	182
The Cauchy-Schwarz Inequality	184
Hilbert-Schmidt Norm	186
Polarization Identity	187
Orthonormal Bases in Inner Product Spaces	187
Bessel's Inequality	188
6.2. Orthogonal Complements and Projections	190
Orthogonal Projections on Inner Product Spaces	190
The Gram-Schmidt Construction	192
The Legendre Polynomials	195
Fourier Series Expansions	196
A Geometry Problem	199
6.3. Adjoints and Orthonormal Decompositions	200
Diagonalization vs. Orthogonal Diagonalization	200
Dual Spaces of Inner Product Spaces	201

The Adjoint Operator $T^* : W \rightarrow V$	202
Linear Projections vs. Orthogonal Projections	204
Adjoint $T^*$ vs. Transpose $T^T$	206
Computing an Operator Adjoint	206
Self-Adjoint, Unitary, and Normal Operators	207
6.4. Diagonalization in Inner Product Spaces	209
Orthogonal Diagonalization	209
Schur Normal Form	209
Diagonalizing Self-Adjoint and Normal Operators	212
Unitary Equivalence of Operators vs. Similarity	216
The Matrix Groups $\mathbf{U}(n)$ , $\mathbf{SU}(n)$ , $\mathbf{O}(n)$ , $\mathbf{SO}(n)$	219
Change of Orthonormal Basis	221
Diagonalization over $\mathbb{K} = \mathbb{C}$ : A Summary	222
6.5. Reflections, Rotations, and Rigid Motions on $\mathbb{R}^n$	222
The Group of Rigid Motions $\mathbf{M}(n)$	223
Reflections on Inner Product Spaces	224
Euler's Theorem: Rotations on $\mathbb{R}^3$	226
Further Comments on Euler	228
6.6. Spectral Theorem for Vector and Inner Product Spaces	229
Spectral Theorem	230
Functions of Operators: $e^T$ and $\sqrt{T}$ Revisited	234
Computing a Spectral Decomposition	236
Determining the Spectral Projections $P_\lambda$	237
6.7. Positive Operators and Polar Decomposition	239
Positive Square Roots	240
Polar Decompositions $T = UP$ (Invertible $T$ )	241
Computing the Polar Decomposition for Invertible $T : V \rightarrow W$	243
The Singular Value Decomposition	244
Computing a Singular Value Decomposition	245
The General Polar Decomposition	246
Additional Exercises	248
Index	257