CHAPTER 1

Fourier Series

1.1. Introduction

Fourier was interested in solving the heat equation on $[0, \infty) \times \mathbf{R}$

$$u_t = u_{xx}$$

with initial condition $u(0, x) = u_0(x)$, a periodic function of period 1. We look for special solutions of the form $u(t, x) = f(x)g(t)$. In order to satisfy the equation, we need

$$fg_t = f_{xx}g \quad \text{or} \quad \frac{g_t}{g} = \frac{f_{xx}}{f}.$$

The left-hand side is a function of t while the right hand side is a function of x. Therefore, they both must be equal to a constant λ. $g_t = \lambda g$ yields $g(t) = e^{\lambda t}$, and $f_{xx} = \lambda f$ yields $f(x) = A \sin cx + B \cos cx$. The equation is satisfied if $\lambda = -c^2$. For the function to be periodic of period 1, we need c to be an integer multiple of 2π. We have a class of solutions

$$u(t, x) = e^{-4n^2\pi^2 t}[A \cos(2\pi nx) + B \sin(2\pi nx)],$$

or if we allow complex solutions,

$$u(t, x) = Ae^{-4n^2\pi^2 t} e^{2\pi inx}$$

for every $n \in \mathbf{Z}$. Since the equation is linear,

$$\sum_{n=-\infty}^{\infty} a_n e^{-4n^2\pi^2 t} e^{2\pi inx}$$

are solutions with

$$u(0, x) = u_0(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi inx}.$$

If any periodic function $u_0(x)$ can be represented as a sum of periodic exponentials with the same period, then the heat equation can be solved.

It is more convenient to consider periodic functions of period 2π. They are $\sin nx$ and $\cos nx$. We will consider complex-valued periodic functions on $\mathbf{R}$ with period 2π. We will denote by $\mathbf{T}$ the interval $[-\pi, \pi]$ with endpoints identified. Addition in $\mathbf{T}$ is modulo 2π. Functions on $\mathbf{T}$ need to match at $\pm\pi$, along with the required number of derivatives in order to be smooth. In particular, integration by parts of smooth functions on $\mathbf{T}$ produces no boundary terms.We can view periodic functions as functions defined on the circumference of the unit circle in the complex plane which has no boundary. Since boundary points have measure 0, integration on $\mathbf{T}$ is no different from integration on $[-\pi,\pi]$. We will also view $\mathbf{T}$ as group, which viewed on $[-\pi,\pi]$, is addition modulo 2π.

The *Fourier coefficients* of a periodic function $f \in L_1[-\pi,\pi]$ are defined by

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \tag{1.1}$$

If a function f has the representation as a *Fourier series*, then

$$f(x) = \sum_{-\infty<n<\infty} a_n e^{inx} \tag{1.2}$$

with $\sum_{-\infty<n<\infty} |a_n| < \infty$. Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}e^{-inx}dx = \delta_{n,m},$$

i.e., equals 1 if $n = m$ and 0 otherwise, we see that the coefficients a_n can be recovered from f by formula (1.1). If we assume that $f \in L_1[-\pi,\pi]$, then clearly a_n is well-defined by formula (1.1) and

$$|a_n| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|dx.$$

It is not clear that the series on the right hand side of equation (1.2) converges, and even if it does, it is not immediately clear why the sum of the series is actually equal to the function $f(x)$. It is relatively easy to find conditions on $f(\cdot)$ so that the series (1.2) is convergent. If $f(x)$ is assumed to be k times continuously differentiable as a periodic function on $[-\pi,\pi]$, integrating by parts k times, we obtain for $n \ne 0$,

$$|a_n| = \frac{1}{|n|^k}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f^{(k)}(x)e^{-inx}\,dx\right| \le \frac{1}{|n|^k}\sup_x |f^{\{k\}}(x)| \tag{1.3}$$

From the estimate (1.3), it is easily seen that the series is convergent if f is twice continuously differentiable.

An important but elementary fact is the Riemann-Lebesgue lemma.

THEOREM 1.1. *For every $f \in L_1[-\pi,\pi]$ with $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}f(x)dx$,*

$$\lim_{n\to\pm\infty} |a_n| = 0. \tag{1.4}$$

PROOF. Let $f \in L_1[-\pi,\pi]$ and $\epsilon > 0$ be given. Since smooth functions are dense in $L_1[-\pi,\pi]$, given any $\epsilon > 0$, we can approximate f by a function g_ϵ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - g_\epsilon(x)|dx \le \epsilon$$

and g_ϵ is continuously differentiable on $\mathbf{T}$. Then

$$\begin{aligned}
|a_n| &\le \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} g_\epsilon e^{-inx}\,dx\right| + \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - g_\epsilon(x)|dx \\
&\le \frac{1}{|n|}\frac{1}{2\pi}\int_{-\pi}^{\pi} |g_\epsilon'(x)|dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - g_\epsilon(x)|dx \\
&\le \frac{1}{|n|}\sup_x |g_\epsilon'(x)| + \epsilon.
\end{aligned}$$

Therefore $\limsup_{n\to\pm\infty} |a_n| \le \epsilon$. Since $\epsilon > 0$ is arbitrary, $\limsup_{n\to\pm\infty} |a_n| = 0$. □

REMARK 1.2. The functions $\sin nx$ and $\cos nx$ are highly oscillatory around 0, and when integrated, $\int_a^b \sin nx\, dx$ and $\int_a^b \cos nx\, dx$ tend to 0 as $n \to \infty$. Any function f is well approximated in $L_1[-\pi,\pi]$ by functions that are piecewise constant.

1.2. Convergence of Fourier series

Let us define the partial sums

$$(s_N f)(x) = s_N(f,x) = \sum_{|n|\le N} a_n e^{inx} \tag{1.5}$$

and the Fejér sum

$$(S_N f)(x) = S_N(f,x) = \frac{1}{N+1} \sum_{0\le n\le N} s_n(f,x). \tag{1.6}$$

We can calculate

$$\begin{aligned}(s_n f)(x) =& \frac{1}{2\pi}\sum_{|j|\le n} e^{ijx}\int_{\mathbf{T}} e^{-ijy} f(y)dy\\ =& \frac{1}{2\pi}\int_{\mathbf{T}} f(y)[\sum_{|j|\le n} e^{ij(x-y)}]dy\\ =& \frac{1}{2\pi}\int_{\mathbf{T}} f(y)\frac{e^{-in(x-y)}(e^{i(2n+1)(x-y)}-1)}{e^{i(x-y)}-1}dy\\ =& \int_{\mathbf{T}} f(y)k_n(x-y)dy\\ =& (f * k_n)(x),\end{aligned} \tag{1.7}$$

where

$$k_n(z) = \frac{1}{2\pi}\frac{e^{-inz}(e^{i(2n+1)z}-1)}{e^{iz}-1} = \frac{1}{2\pi}\frac{\sin(n+\frac{1}{2})z}{\sin\frac{z}{2}}S \tag{1.8}$$

and the convolution $f * g$ of two functions f, g in $L_1[\mathbf{T}]$ is defined as

$$(f * g)(x) = \int_{\mathbf{T}} f(y)g(x-y)dy = \int_{\mathbf{T}} f(x-y)g(y)dy. \tag{1.9}$$

A similar calculation reveals

$$(S_N f)(x) = \int_{\mathbf{T}} f(y)K_N(x-y)dy = (f * K_N)(x), \tag{1.10}$$

where

$$\begin{aligned}K_N(x) =& \frac{1}{2\pi}\frac{1}{(N+1)}\frac{1}{\sin\frac{x}{2}}\sum_{0\le n\le N}[\sin(n+\frac{1}{2})x]\\ =& \frac{1}{2\pi}\frac{1}{(N+1)}\left[\frac{\sin\frac{(N+1)x}{2}}{\sin\frac{x}{2}}\right]^2.\end{aligned} \tag{1.11}$$

Since

$$k_n(x) = \sum_{|j| \le n} a_j e^{ijx},$$

$$K_N(x) = \frac{1}{N+1}\sum_0^N k_n(x)$$

$$= \sum_{|n| \le N} (1 - \frac{|n|}{N+1}) a_n e^{inx}.$$

Notice that for every N,

$$\int_{\mathbf{T}} k_N(x)dx = \int_{\mathbf{T}} K_N(x)dx = 1. \tag{1.12}$$

The following observations are now easy to make.

(1) Nonnegativity:

$$K_N(x) \ge 0.$$

(2) For any $\delta > 0$,

$$\lim_{N\to\infty} \sup_{|x|\ge\delta} K_N(x) = 0.$$

(3) Therefore,

$$\lim_{N\to\infty} \int_{|x|\ge\delta} K_N(x)dx = 0.$$

It is now a simple exercise to prove the following.

THEOREM 1.3. *For any f that is bounded and continuous on $\mathbf{T}$,*

$$\lim_{N\to\infty} \sup_{x\in\mathbf{T}} |S_N(f,x) - f(x)| = 0.$$

PROOF. Let $\delta > 0$ be given. Then

$$|S_N(f,x) - f(x)| = \left| \int [f(x-z) - f(x)] K_N(z)dz \right|$$

$$\le \int_{|z|\le\delta} |f(x-z) - f(x)| K_N(z)dz + \int_{|z|\ge\delta} |f(x-z) - f(x)| K_N(z)dz$$

$$\le \sup_x \sup_{|z|\le\delta} |f(x-z) - f(x)| + 2\sup_x |f(x)| \int_{|z|\ge\delta} K_N(z)dz.$$

If we let $N \to \infty$ and then $\delta \to 0$

$$\limsup_{N\to\infty} \sup_x |S_N(f,x) - f(x)| \le \sup_x \sup_{|z|\le\delta} |f(x-z) - f(x)| \to 0$$

as $\delta \to 0$. □

We next explore the convergence properties of the Fejér sum in $L_p[-\pi,\pi]$.

THEOREM 1.4. *For $1 \le p < \infty$ and $f \in L_p[-\pi,\pi]$,*

$$\|S_N(f,\cdot)\|_p \le \|f\|_p.$$

Therefore,

$$\lim_{N\to\infty} \|S_N f - f\|_p = 0$$

PROOF. By Hölder's inequality, since $\int K_N(z)dz = 1$, for any x,

$$|S_N(f,x)|^p \le \int_{\mathbf{T}} |f(z)|^p K_N(x-z)dz.$$

Integrating with respect to x, we obtain the first part of the theorem. For any $f \in L_p$ and $\epsilon > 0$, we can find g such that g is continuous and $\|f - g\|_p \le \epsilon$.

$$\|S_N f - f\|_p \le \|S_N f - S_N g\|_p + \|S_N g - g\|_p + \|g - f\|_p \le \|S_N g - g\|_p + 2\epsilon$$

$\|S_N g - g\|_p \le \sup_x |S_N(g,x) - g(x)| \to 0$ as $N \to \infty$ and $\epsilon > 0$ is arbitrary. □

The behavior of $s_N(f,x)$ is more complicated. It is easy enough to observe that for $f \in C^2(\mathbf{T})$,

$$\lim_{N\to\infty} \sup_x |s_N(f,x) - f(x)| = 0.$$

The series converges uniformly, so $s_N(f,\cdot)$ has a uniform limit g. The Cesàro average $S_N(f,\cdot)$ has the same limit, which has just been shown to be f. Therefore $f = g$. The following theorem is again fairly easy.

THEOREM 1.5. *If $f \in L_1$ satisfies $|f(y) - f(x)| \le c|y-x|^\alpha$ at some x for some $\alpha > 0$ and $c < \infty$, then at that x,*

$$\lim_{N\to\infty} s_N(f,x) = f(x).$$

PROOF. We can assume without loss of generality that $x = 0$ and let $f(0) = a$. We need to show that

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{\mathbf{T}} f(y) \frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} dy = a,$$

or

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{\mathbf{T}} \frac{[f(y) - a]}{\sin\frac{y}{2}} [\sin(N+\frac{1}{2})y] dy = 0. \tag{1.13}$$

Because the function $g(y) = \frac{f(y)-a}{\sin\frac{y}{2}}$ with singularity only at 0 is dominated by $c|y|^{\alpha-1}$ for some $\alpha > 0$, it is integrable. (1.13) is a consequence of the Riemann-Lebesgue lemma, i.e., Theorem 1.1. □

Let us now assume that f is a function of bounded variation on $\mathbf{T}$ which has left and right limits a_l and a_r at 0. It is easy to check that

$$\left|\frac{1}{\sin(\frac{y}{2})} - \frac{1}{\frac{y}{2}}\right| \le C|y|.$$

It follows from the Riemann-Lebesgue lemma that

$$\lim_{\lambda\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sin\lambda y \left[\frac{1}{\sin(\frac{y}{2})} - \frac{1}{(\frac{y}{2})}\right] dy = 0.$$

By a change of variables, one can reduce the calculation of

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{\mathbf{T}} f(y) \frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}} dy$$

to calculating

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\lambda\pi}^{\lambda\pi} f(\frac{y}{\lambda})\frac{\sin y}{y}dy.$$

If we write

$$G(y)=\int_y^\infty \frac{\sin x}{x}dx,$$

then $G(\infty)=0$ and $G(0)=\frac{\pi}{2}$.

We have that

$$\begin{aligned} a_r(\lambda) &= \frac{1}{\pi}\int_0^{\lambda\pi} f(\frac{y}{\lambda})\frac{\sin y}{y}dy = -\frac{1}{\pi}\int_0^{\lambda\pi} f(\frac{y}{\lambda})dG(y) \\ &= \frac{1}{2}a_r + \frac{1}{\pi}\int_{0+}^{\lambda\pi} G(y)df(\frac{y}{\lambda}) = \frac{1}{2}a_r + \frac{1}{\pi}\int_{0+}^{\pi} G(\lambda y)df(y) \\ &\to \frac{1}{2}a_r \end{aligned}$$

by the bounded convergence theorem. Similarly,

$$b_r(\lambda)=\frac{1}{\pi}\int_{-\lambda\pi}^{0} f(\frac{y}{\lambda})\frac{\sin y}{y}dy \to \frac{1}{2}b_r.$$

This establishes the following.

THEOREM 1.6. *If f is a function of bounded variation on $\mathbf{T}$, then for every $x\in\mathbf{T}$,*

$$\lim_{N\to\infty} s_N(f,x)=\frac{f(x+0)+f(x-0)}{2}.$$

REMARK 1.7. On the other hand, the behavior of $s_N(f,x)$ for f in $C(\mathbf{T})$, the space of continuous functions on $\mathbf{T}$ or in $L_p[-\pi,\pi]$ for $1\le p<\infty$, is more complex.

For example, one can ask if $s_N(f,x)\to f(x)$ in L_p. Or how about convergence for almost all x? Let us define the linear operator

$$(T_N f)(x)=\int_{-\pi}^{\pi} f(x-y)\frac{\sin(N+\frac{1}{2})y}{\sin\frac{y}{2}}dy \tag{1.14}$$

on smooth functions f. It is more convenient to think of f as a periodic function of period 2π defined on $\mathbf{R}$. If $s_N(f,x)$ were to converge uniformly to f for every bounded continuous function, it would follow by the uniform boundedness principle that

$$\sup_x |(T_N f)(x)| \le C\sup_x |f(x)|$$

with a constant independent of f as well as N. Let us show that this is not possible. The best possible bound C_N for T_N is seen to be

$$C_N=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\sin(N+\frac{1}{2})y|}{|\sin\frac{y}{2}|}dy,$$

and because

$$\left|\frac{1}{\sin\frac{y}{2}}-\frac{2}{y}\right|$$

is integrable on $[-\pi,\pi]$, C_N differs from

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|2\sin(N+\frac{1}{2})y|}{|y|}dy = \frac{1}{\pi}\int_{-(N+\frac{1}{2})\pi}^{(N+\frac{1}{2})\pi}\frac{|\sin y|}{|y|}dy$$

by a uniformly bounded amount. The divergence of $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\sin y|}{|y|}dy$ implies that $C_N \to \infty$ as $N \to \infty$. By duality, this means that T_N is not uniformly bounded as an operator from $L_1[-\pi,\pi]$ into itself either. Again because of the uniform boundedness principle, one cannot expect $s_N(f,\cdot)$ to tend to $f(\cdot)$ in $L_1[-\pi,\pi]$ for every $f \in L_1[-\pi,\pi]$.

1.3. Special case $\mathbf{p = 2}$

When $p = 2$, we have a Hilbert space $L_2(\mathbf{T})$ with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbf{T}}\overline{f}\,g\,dx$$

and

$$\|f\|_2^2 = \frac{1}{2\pi}\int_{\mathbf{T}}|f(x)|^2\,dx.$$

We have taken the normalized Lebesgue measure $d\mu = \frac{dx}{2\pi}$ so that $\mu(\mathbf{T}) = 1$. The functions $\{e_n(\cdot) = e^{inx} : n \in \mathbf{Z}\}$ form a complete orthonormal basis, and the Fourier series is the expansion

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx},$$

which converges in $L_2(\mathbf{T})$ with a_n given by

$$a_n = \langle e_n, f\rangle = \frac{1}{2\pi}\int_{\mathbf{T}} e^{-inx} f(x)dx.$$

The Plancherel-Parseval identities state that

$$\sum_{n=-\infty}^{\infty}|a_n|^2 = \frac{1}{2\pi}\int_{\mathbf{T}}|f(x)|^2\,dx$$

and

$$\sum_{n=-\infty}^{\infty}\overline{a}_n b_n = \frac{1}{2\pi}\int_{\mathbf{T}}\overline{f(x)}g(x)dx.$$

1.4. Higher dimensions

If we have periodic functions $f(\mathbf{x}) = f(x_1,\ldots,x_d)$ of d variables with period 2π in every variable, then the Fourier transforms are defined on $\mathbf{Z}^d$. If $\mathbf{n} = (n_1,\ldots,n_d)$, then

$$a_{\mathbf{n}} = \frac{1}{(2\pi)^d}\int_{\mathbf{T}^d} e^{i\langle\mathbf{n},\mathbf{x}\rangle}d\mathbf{x}.$$

While most of the one-dimensional results carry over to d dimensions, one needs to be careful about the partial sums. Results that depend on the explicit form of the kernels k_N and K_N have to be reexamined. While partial sums over sets of the form $\bigcap_i\{|n_i| \le N\}$ or even $\bigcap_i\{|n_i| \le N_i\}$ can be handled, it is hard to analyze partial sums over sets of the form $\{\mathbf{n} : \sum_i n_i^2 \le N\}$. Decomposition of functions in $L_2(\mathbf{T}^d)$ into $\{e^{i<\mathbf{n},\mathbf{x}>}\}$ with $\mathbf{n} \in \mathbf{Z}^d$ goes through without difficulty.

1.5. Maximal inequality

We start with a useful covering lemma known as the Vitali covering lemma.

LEMMA 1.8. *Let $K \subset S$ be a compact subset of* $\mathbf{R}$ *and $\{I_\alpha\}$ be a collection of open intervals covering K. Then there is a finite subcollection $\{I_j\}$ such that*

(1) *$\{I_j\}$ are disjoint.*
(2) *The intervals $\{3I_j\}$ that have the same midpoints as $\{I_j\}$ but three times the length cover K.*

PROOF. We first choose a finite subcover. From the finite subcover, we pick the largest interval. In case of a tie, pick any of the competing ones. Then at every stage of the remaining intervals from our finite subcollection, we pick the largest one that is disjoint from the ones already picked. We stop when we cannot pick any more. The collection that we end up with is clearly disjoint and finite. Let $x \in K$. This is covered by one of the intervals J from our finite subcollection covering K. If J was picked, there is nothing to prove. If J was not picked, it must intersect some I_j that was picked. Otherwise J would be disjoint from all the I_j that were picked and would have been picked as well. Let us look at the first such interval and call it $\bar{I}$. J is disjoint from all the previously picked ones and J was passed over when we picked $\bar{I}$. Therefore in addition to intersecting $\bar{I}$, J is not longer than $\bar{I}$. Therefore $3\bar{I} \supset J \ni x$. □

The lemma is used in proving maximal inequalities. For instance, for the Hardy-Littlewood maximal function, we have the following result.

THEOREM 1.9. *Let $f \in L_1(\mathbf{T})$. Define*

$$M_f(x) = \sup_{0<r\le\pi} \frac{1}{2r}\int_{|y-x|<r} |f(y)|dy. \tag{1.15}$$

Then

$$\mu[x : M_f(x) > \ell] \le \frac{3\int |f(y)|dy}{\ell}. \tag{1.16}$$

PROOF. Let us denote by E_ℓ the set

$$E_\ell = \{x : M_f(x) > \ell\},$$

and let $K \subset E_\ell$ be an arbitrary compact set. For each $x \in K$, there is an interval I_x such that

$$\int_{I_x} |f(y)|dy \ge \ell\mu(I_x).$$

Clearly $\{I_x\}$ is a covering of K. By lemma, we get a finite disjoint subcollection $\{I_j\}$ such that $\{3I_j\}$ covers K. Adding them up, we get

$$\int_{\mathbf{T}} |f(y)|dy \ge \sum_j \int_{I_j} |f(y)|dy \ge \ell \sum_j \mu(I_j) = \frac{\ell}{3}\sum_j \mu(3I_j) \ge \frac{\ell}{3}\mu(K).$$

Since $K \subset E_\ell$ is arbitrary, we are done. □

There is no problem in replacing $\{x : |M_f(x)| > \ell\}$ by $\{x : |M_f(x)| \ge \ell\}$. Replace ℓ by $\ell - \epsilon$ and let $\epsilon \to 0$. This theorem can be used to prove the Lebesgue differentiability theorem.

THEOREM 1.10. *For any* $f \in L_1(\mathbf{T})$,

$$\lim_{h\to 0} \frac{1}{2h}\int_{|x-y|\le h} |f(y)-f(x)|dy = 0 \quad \textit{for a.e. } x. \tag{1.17}$$

PROOF. It is sufficient to prove that for any $\delta > 0$,

$$\mu[x : \limsup_{h\to 0} \frac{1}{2h}\int_{|x-y|\le h} |f(y)-f(x)|dy \ge \delta] = 0.$$

Given $\epsilon > 0$, we can write $f = f_1 + g$ with f_1 continuous and $||g||_1 \le \epsilon$, and

$$\begin{aligned}\mu[x : \limsup_{h\to 0} \frac{1}{2h}&\int_{|x-y|\le h} |f(y)-f(x)|dy \ge \delta]\\ &= \mu[x : \limsup_{h\to 0} \frac{1}{2h}\int_{|x-y|\le h} |g(y)-g(x)|dy \ge \delta]\\ &\le \mu[x : \sup_{h>0} \frac{1}{2h}\int_{|x-y|\le h} |g(y)-g(x)|dy \ge \delta]\\ &\le \frac{3||g||_1}{\delta} \le \frac{3\epsilon}{\delta}.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we are done. □

An easy consequence is the following corollary.

COROLLARY 1.11. *If* $f \in L_1[-\pi,\pi]$, *then* $\lim_{h\to 0} \frac{1}{2h}\int_{x-h}^{x+h} f(y)dy = f(x)$ *for a.e* x.

In other words, the maximal inequality is useful to prove almost sure convergence. Typically, almost sure convergence will be obvious for a dense set, and the maximal inequality will be used to interchange limits in the approximation.

Another summability method similar to the Fejér sum is the Poisson sum: For $0 \le \rho < 1$

$$S(\rho, x) = \sum_{n\in\mathbf{Z}} a_n \rho^{|n|} e^{inx},$$

and the kernel corresponding to it is the Poisson kernel

$$P(\rho, x) = \frac{1}{2\pi}\sum_n \rho^{|n|}e^{inx} = \frac{1}{2\pi}\frac{1-\rho^2}{(1-2\rho\cos x+\rho^2)} \tag{1.18}$$

so that

$$(P_\rho f)(x) = \int_{\mathbf{T}} f(y)P(\rho, x-y)dy.$$

It is left as an exercise to prove that for for $1 \le p < \infty$, every $f \in L_p$ $P_\rho f \to f$ in L_p as $\rho \to 1$. We will prove a maximal inequality for the Poisson sum so that, as a consequence, we will get the almost sure convergence of $P_\rho f$ to f for every f in L_1.

THEOREM 1.12. *For every* f *in* L_1,

$$\mu[x : \sup_{0\le\rho<1} |(P_\rho f)(x)| \ge \ell] \le \frac{C||f||_1}{\ell}. \tag{1.19}$$

The proof consists of estimating the Poisson maximal function in terms of the Hardy-Littlewood maximal function $M_f(x)$ defined in (1.15). We begin with some simple estimates for the Poisson kernel $P(\rho, x)$:

$$\begin{aligned} P(\rho, x) =& \frac{1}{2\pi} \frac{1-\rho^2}{(1-\rho)^2 + 2\rho(1-\cos x)} \le \frac{1}{2\pi} \frac{1-\rho^2}{(1-\rho)^2} \\ =& \frac{1}{2\pi} \frac{1+\rho}{1-\rho} \le \frac{1}{\pi} \frac{1}{1-\rho}. \end{aligned}$$

The problem therefore is only as $\rho \to 1$.

LEMMA 1.13. *For any symmetric function $\phi(x)$,*

$$\int_{-\pi}^{\pi} f(x)\phi(x)dx \le 2M_f(0) \int_0^{\pi} |x\phi'(x)|dx + 2\pi|\phi(\pi)||M_f(0)|,$$

where M_f is the Hardy-Littlewood maximal function.

PROOF.

$$\begin{aligned} &\Big|\int_{-\pi}^{\pi} f(x)\phi(x)dx\Big| \\ &= \Big|\int_0^{\pi} [f(x) + f(-x)]\phi(x)dx\Big| \\ &= \Big|\int_0^{\pi} \phi(x)\Big[\frac{d}{dx}\int_{-x}^{x} f(y)dy\Big]dx\Big| \\ &\le \Big|\int_0^{\pi} \phi'(x)\Big[\int_{-x}^{x} f(y)dy\Big]dz\Big| + \Big|\phi(\pi)\int_{-\pi}^{\pi} f(x)dx\Big| \\ &\le \int_0^{\pi} 2|x\phi'(x)|\frac{1}{2x}\Big[\int_{-x}^{x} |f(y)|dy\Big]dx + |\phi(\pi)|\int_{-\pi}^{\pi} |f(x)|dx \\ &\le 2M_f(0)\int_0^{\pi} |x\phi'(x)|dx + 2\pi|\phi(\pi)||M_f(0)| \end{aligned}$$

□

To estimate the maximal function for the Poisson kernel in terms of the Hardy-Littlewood maximal function $M_f(x)$, we check that, with $\phi(x) = P(\rho, x)$,

$$\begin{aligned} \Big|x\frac{d}{dx}P(\rho, x)\Big| &= \frac{1}{2\pi}\frac{1-\rho^2}{(1-2\rho\cos x + \rho^2)^2}2\rho|x\sin x| \\ &\le \frac{1}{\pi}\frac{(1-\rho)x^2}{(1-\rho)^4 + (1-\cos x)^2} \\ &\le C\frac{(1-\rho)x^2}{(1-\rho)^4 + x^4} \end{aligned}$$

and

$$\int_0^{\pi} \left|x\frac{d}{dx}P(\rho,x)\right| dx \le C\int_0^{\pi} \frac{(1-\rho)x^2}{(1-\rho)^4+x^4}dx$$
$$= \int_0^{\frac{\pi}{1-\rho}} \frac{x^2}{1+x^4}dx$$
$$\le \int_0^{\infty} \frac{x^2}{1+x^4}dx \le C_1$$

uniformly in ρ. Moreover,

$$P(\rho,\pi) = \frac{1}{2\pi}\,\frac{1-\rho^2}{(1-\rho)^2+2\rho(1-\cos\pi)} = \frac{1}{2\pi}\,\frac{1-\rho^2}{(1+\rho)^2} \le C.$$

There is nothing special about $\mathbf{T}$. We can carry out the argument on $\mathbf{R}^d$ or $\mathbf{T}^d$.

THEOREM 1.14. *If* $f \in L_1(\mathbf{R}^d)$ *and*

$$M_f(x) = \sup_{r>0} \frac{\int_{D(x,r)} |f(y)|dy}{|D(x,r)|}.$$

then

$$\mu[x : M_f(x) > \ell] \le \frac{3^d\|f\|_1}{\ell}.$$

Here, $D(x,r)$ *is the sphere with center* x *and radius* r*. It could also be the cube centered at* x*. All that matters is that if* $D(x,r)\cap D(y,s) \ne \emptyset$ *and* $s \le r$*, then* $D(x,3r) \supset D(y,s)$ *and* $|D(x,3r)| = 3^d D(x,r)$*.*

Proof is left as an exercise.

1.6. Exercises

(1) For $1 \le p < \infty$, if $f \in L_p$, prove the convergence of $P_\rho f \to f$ in L_p as $\rho \to 1$.
(2) Instead of the Fejér sum, if we use

$$(W_N f)(x) = \sum a_n w(N,n)e^{inx}$$

with $w(N,n) \to 1$ as $N \to \infty$, what simple additional conditions will ensure the convergence of $W_N f$ to f? In $L_1[\mathbf{T}]$ or $L_2[\mathbf{T}]$.
(3) What about $w(N,n) = e^{-\frac{|n|}{N}}$ or $w(N,n) = e^{-\frac{n^2}{N}}$?
(4) For a function $f \in L_1[\mathbf{T}]$, the harmonic extension to the interior of the circle is given by

$$U(f,r,\theta) = \frac{1}{2\pi}\int \frac{f(\alpha)}{(1-2r\cos(\theta-\alpha)+r^2)}d\alpha.$$

Show that for $f \in L_1[\mathbf{T}]$,

$$\lim_{r\to 1} U(f,r,\theta) = f(\theta)$$

for almost all $\theta \in \mathbf{T}$.
(5) Construct explicitly a continuous function f on $\mathbf{T}$ such that the Fourier series of f does not converge uniformly.

(6) The function $\frac{y}{2}$ is not well-defined on $\mathbf{T}$. If $x = \frac{y}{2}$ so is $x + \pi$. Why do the formulas in (1.8) and (1.11) make sense?
(7) Provide a proof for Theorem 1.14.