Preface

The developments of Statistical Physics and Quantum Field Theory (QFT) are among the most significant developments of twentieth-century science. The unification of these two subjects, sometimes called Statistical Field Theory (SFT), started to take place in the second half of the century.

Conformal Field Theory (CFT) is at the core of this unification, and has proven to be one of the most successful achievements in Mathematical Physics. CFT famously connects critical statistical mechanics models with conformally symmetric field theories, thus allowing for a very precise and beautiful description of the former and a concrete, applicable, and well-defined understanding of the latter.

The two-dimensional case is particularly remarkable, as the conformal symmetry leads to the emergence of the Virasoro algebra. The pioneering work of Belavin, Polyakov, and Zamolodchikov initiated a breakthrough that led to the classification of the so-called *Unitary Minimal Models of CFT*: a discrete series of field theories that describe the phase transitions of some of the most fundamental and classical lattice models of statistical mechanics.

This breakthrough was followed by an explosion of research, which gave an unprecedented level of understanding of statistical mechanics models in terms of beautiful mathematical structures; further, it has spearheaded numerous mathematical and physical developments, including representation theory (in particular the theory of vertex operator algebras), probability (most notably the theory of Schramm-Loewner Evolution [SLE]), and string theory. Despite CFT's tremendous success and impact, the very idea at the root of its development (the connection with critical lattice models) has largely remained conjectural: mathematical proofs of the spectacular predictions provided by CFT for lattice models were lacking, and the question of constructing precise bridges between the lattice models and the relevant field theories remained largely unaddressed, as well as the (related) question of constructing such field theories.

This book grew out of an attempt to bring clarity around the above questions. We lay down a path that brings us from simple lattice models to CFTs, by carefully building advanced structures on top of more elementary ones, and justifying each step in as convincing and precise a way as possible. Our approach is close in spirit to the original founding articles in the subject, but is considerably more detailed and precise, being notably enriched by a number of important mathematical developments in the last two decades. In particular, the study of the Ising model saw substantial progress, including proofs of most CFT-based conjectures and extensive connections with the field of stochastic processes. While the connection between other critical models and their corresponding CFT remains conjectural, the new results allow one to gain much concrete and convincing insight into the general picture. They provide a probabilistic understanding of CFT (through lattice models and random geometry), and pave the way for new avenues of research.

This book is devoted to explaining the connection between lattice models and CFT in light of recent progress in the fields. We present several two-dimensional lattice models (e.g., the Ising model, the tricritical Ising model, the 3-Potts model), their phase transitions, and outline their conjectural connection with CFT. Through these lattice models and their local fields, we introduce the fundamental ideas and results of two-dimensional CFTs, with a special emphasis on the so-called Unitary Minimal Models of CFT. This book delves into the delicate ideas that lead to the classification of these CFTs, discussing the assumptions on the lattice models whose scaling limits are described by CFTs. This allows us to provide a probabilistic rather than an axiomatic or algebraic definition of CFTs.

To do so, we expand on aspects of CFTs that are often overlooked in the literature, including a thorough study of *transformation laws* at both continuous and discrete levels, the geometric emergence of the holomorphic Stress Tensor, and closed-form expression of its correlation functions. Furthermore, we explain how the existence of a Hermitian transfer matrix at the discrete level leads to the unitarity condition for underlying CFTs. This unitarity property is crucial for understanding the central charges, scaling dimensions and correlations of local fields in the Unitary Minimal Models.

The book presents recent results showing the convergence of the correlations of numerous lattice scaling local fields of the Ising model to those found in the Ising CFT, which is the first Unitary Minimal Model. Using the complementary perspective of random geometry, we explain how this Ising CFT is related to the so-called Conformal Loop Ensemble (CLE). Based on the analysis performed in this book, along with specific insights from the Ising model study, we discuss how, conjecturally, the scaling limit of certain lattice models be described by specific Unitary Minimal Models. We also discuss the potential connection of these models with the CLE, offering readers a detailed and expanded view of CFTs and their relationship with random geometry and lattice models.

The book is structured into two parts: the first one serves as an introduction to the fundamental concepts and intuitions behind the relation between lattice models and CFT, while the second part delves deeply into this connection.

In the first three chapters, we address the core questions driving the investigation of CFT:

- Chapter 1 provides a brief overview of various two-dimensional lattice models and their phase transitions, highlighting notable CFT conjectures that aim to describe these phase transitions.
- Chapter 2 explores the notion of the scaling limit: critical lattice models conjecturally become QFTs. This chapter discusses CFT predictions regarding these QFTs. We use the Gaussian Free Field (GFF) as an illustration and discuss the QFT that underpins it, specifically the polynomial Free Boson Theory.
- Chapter 3 offers a broad perspective on the key themes and insights related to CFTs and their conformal symmetry, setting the stage for the rest of the book.

The second part investigates the principal ideas and insights of CFT that lead to the description of Unitary Minimal CFTs, using the lattice models introduced earlier as illustrations. Unlike most of the existing literature, this section places emphasis on explaining how all the objects in CFT should be (either conjecturally or provably) thought of within the context of the lattice models that the CFT aims to describe. More specifically:

- Chapters 4 and 5 are dedicated to the concept of SFT and their scaling lattice local fields, which are the lattice analogs of continuous local fields. These chapters explore the universality conjecture and its implications for the action of diffeomorphisms on critical parameters and lattice local fields—the so-called transformation laws. Emphasis is placed on the critical role of the Stress-Energy Tensor in CFT. Through the study of its poles when inserted near a local field (the Ward Identities), we show how the holomorphic modes of the Stress-Energy Tensor enable the construction of additional local fields from the so-called primary fields, providing a representation of the Virasoro algebra whose central charge depends on the CFT. These holomorphic modes allow also a better understanding of the transformation laws and the relation between 'algebraic' primary fields and 'geometric' primary fields.
- Chapters 6 and 7, delve into the key implication of a Hermitian Transfer Matrix in the underlying statistical model, namely the unitarity condition in both the CFT and the Virasoro Algebra representations. This condition leads to a precise description of the Virasoro algebra's representations encountered in the scaling limit of lattice models, allowing for a precise characterization of the central charge, left/right-scaling dimensions, and null fields that appear in such unitary CFT.
- Chapter 8 synthesizes our understanding to describe the conjectural scaling limits of the models introduced in the introductory part. Importantly, we show that the existence of null fields in Unitary Minimal Models lead to differential equations that correlations of primary fields have to satisfy. We provide a detailed description of the relation between the Ising model, a cornerstone of statistical mechanics, and its scaling limit, which corresponds to the first Unitary Minimal CFT. In this setting, the scaling lattice local fields are rigorously known to converge, and the CFT structures can be revealed at the lattice level. We also describe its link with the geometric point of view on CFT, obtained by considering a family of random curves, the CLEs. Drawing from the Ising model analysis, we provide intuition on the conjectural relationship between various lattice models and their scaling limit, as well as their conjectural connection with the CLE framework.

This book is aimed at graduate students and researchers in both mathematics and physics (assuming typical undergraduate introductions to probability and to complex analysis as prerequisites), in particular in the field of statistical mechanics, probability, and QFT. It should especially appeal to those looking for concrete and precise explanations of the links between lattice models and field theories. While this book grew out of a desire to fill a perceived gap in the mathematical literature, it can be expected to be equally interesting for mathematicians and physicists: a special effort is made to ensure that rigor and precision don't come at the expense of concreteness and physical relevance, and that the amount of detail does not make the text deviate from the spirit and goals of the founding articles. In particular, this book aims at providing specific insight at places often not very much detailed in the physics literature, and at building a conceptual framework where each mathematical definition comes in naturally.

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