CHAPTER 1

Lattice Models, Phase Transitions, and Critical Exponents

Lattice models lie at the heart of Statistical Mechanics, which aims to explain macroscopic behaviors in terms of their microscopic origins. They consist of simple degrees of freedom living on a lattice with local interactions. In this book, we consider lattice models which are families of random variables located on vertices that interact with their neighbors, in a translation-invariant manner.

Fundamental questions about lattice models concern the large-scale structures and behaviors that emerge when studied on larger and larger graphs. A natural setup that will be central in this book is to consider the discretization of a domain $\Omega \subset \mathbb{C}$ by a lattice Ω_δ of mesh size δ: the question of large-scale behavior then becomes that of describing the model on Ω_δ as $\delta \to 0$. It is within this framework that we will, in particular, consider *phase transitions*, i.e., sudden change of behavior as a function of the system parameters: for instance, in two dimensions, such models may either display discontinuous (also called first-order) or continuous (such as the second-order) phase transitions.[1]

In this chapter, we present several lattice models that exhibit particularly interesting behaviors at their critical points. For models with second-order phase transition, these behaviors are predicted by CFT, whereas it is not expected that tools from CFT provide any insight into discontinuous phase transitions.

1.1. Ising Model

The most celebrated lattice model is probably the Ising model [**86**]. This model lies at the center of a number of investigations; many lattice models that CFT describes can be seen as variants of it.

Consider a finite graph $\mathbb{G}$, with a set of vertices $\mathcal{V}$ and a set of non-oriented edges $\mathcal{E}$. The Ising model on $\mathbb{G}$ consists of ± 1-valued random variables σ_x, called *spins*, living on the vertices $v \in \mathcal{V}$, and interacting with their (graph) neighbors. The probability $\pi_\beta(\sigma)$ of a spin configuration $\sigma = (\sigma_x)_{x \in \mathcal{V}}$ is proportional to $e^{-\beta H[\sigma]}$, where $\beta > 0$ is the *inverse temperature* and the *energy* is $H[\sigma] = -\sum_{\{x,y\} \in \mathcal{E}} \sigma_x \sigma_y$:

$$\pi_\beta(\sigma) := \frac{1}{\mathcal{Z}_{\text{Ising}}(\beta)} \exp\Bigg(\beta \sum_{\{x,y\} \in \mathcal{E}} \sigma_x \sigma_y\Bigg).$$

[1] The classification of the order of the phase transition is given by the regularity of the thermodynamic quantities as functions of the model parameters.

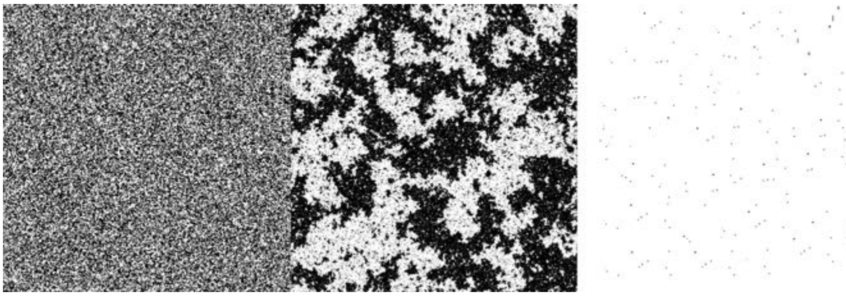

FIGURE 1.1.1. Typical configurations for the Ising model for different values of β with free boundary conditions: white pixels correspond to $+1$ spins, black pixels correspond to -1 spins. On the left figure, $\beta < \beta_c$ and the system is in a disordered configuration, on the right figure, $\beta > \beta_c$ and the spins tend to align and to take the same sign. In the center, a typical Ising configuration at criticality $\beta = \beta_c$.

In the definition above, $\mathcal{Z}_{\text{Ising}}(\beta) = \sum_\sigma \exp\Big(\beta \sum_{\{x,y\}\in\mathcal{E}} \sigma_x\sigma_y\Big)$ is the *partition function* of the Ising model that normalizes the measure π_β. Informally, the system favors lower-energy configurations and, as a result, the spin σ_x at a vertex x tends to align with the spins σ_y at neighboring vertices $y \sim x$; the strength of the alignment effect is controlled by β (see Figure 1.1.1).

EXERCISE 1.1.1. Consider the Ising model on a connected, finite graph $\mathbb{G}$ and the associated measure π_β on $\{+1,-1\}^{\mathcal{V}}$. Determine the behavior of π_β as $\beta \to 0$ and as $\beta \to +\infty$.

The two-dimensional Ising model is mostly studied on the square lattice, more precisely on the discretization $\Omega_\delta \coloneqq \Omega \cap \delta\mathbb{Z}^2$ of some planar bounded domain $\Omega \subset \mathbb{C}$ by a square grid of mesh size δ. This model has been the subject of almost a century of intense research and is now particularly well understood [**103**, **108**]. A remarkable property of this two-dimensional lattice model is the existence of a phase transition at the critical inverse temperature $\beta_c = \frac{1}{2}\ln\Big(\sqrt{2}+1\Big)$: depending on whether the parameter β is smaller or larger than β_c, we observe very different macroscopic behaviors as $\delta \to 0$. For instance, considering $+$ boundary conditions (i.e., forcing the spins on the boundary of Ω_δ to be $+1$), one obtains the following behavior as $\delta \to 0$:

- For $\beta < \beta_c$, the system is disordered: spins at large distances are essentially independent. In particular, for any a in the interior of Ω and any family of points a_δ in Ω_δ that converges to a, the magnetization $\mathbb{E}\left[\sigma_{a_\delta}\right]$ decays exponentially fast to zero as $\delta \to 0$.
- For $\beta > \beta_c$, a long-range alignment takes place: spins at arbitrary distance retain a uniformly positive correlation. In particular, for any a in the interior of Ω and any family of points a_δ in Ω_δ that converges to a, the magnetization $\mathbb{E}\left[\sigma_{a_\delta}\right]$ converges exponentially fast to $\mathcal{M}(\beta) > 0$ as $\delta \to 0$.

The limiting magnetization $\beta \mapsto \mathcal{M}(\beta)$ is continuous and exhibits a discontinuous derivative at β_c: the two-dimensional Ising model exhibits a continuous, second-order phase transition (see Figure 1.1.2).

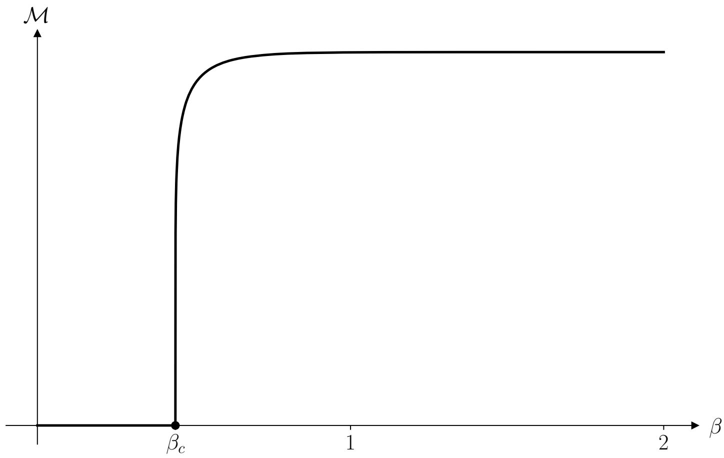

FIGURE 1.1.2. Magnetization $\beta \mapsto \mathcal{M}(\beta)$ for the critical Ising model: in the disordered phase, for $\beta < \beta_c = \frac{1}{2}\ln(1+\sqrt{2})$, the magnetization $\mathcal{M}(\beta) = 0$, whereas in the ordered phase, $\mathcal{M}(\beta) > 0$. At the critical point β_c, the function exhibits a discontinuous derivative and its behavior is given by $\mathcal{M}(\beta) \approx C(\beta - \beta_c)_+^{1/8}$.

In [**109**, **144**], it was proven that, in the so-called supercritical regime (defined by $\beta > \beta_c$), as we approach the critical inverse temperature, the limit

$$\lim_{\beta \searrow \beta_c} \frac{\mathcal{M}(\beta)}{(\beta - \beta_c)^{1/8}}$$

is finite and non-zero. What occurs at the phase transition point $\beta = \beta_c$ is the most interesting, and is in particular universal: the behavior at the critical point is indeed expected to be independent of many features of the model, e.g., of the choice of the lattice. For instance, if we had taken another lattice, such as the triangular lattice, though the value of β_c would change, the macroscopic picture we observe at the critical point would not be different from what we see at the critical point for the square lattice. Furthermore, the critical point sees the emergence of fascinating power-law behaviors; at $\beta = \beta_c$, as $\delta \to 0$:

- The magnetization $\mathbb{E}\left[\sigma_{a_\delta}\right]$ behaves like $d_{\sigma,\mathbb{Z}^2} C_\sigma(\Omega, a)\delta^{1/8}$, with $d_{\sigma,\mathbb{Z}^2}$ a lattice-specific constant, and C_σ a universal term.
- The normalized energy density $\mathbb{E}\left[\sigma_{a_\delta}\sigma_{a_\delta+\delta} - \frac{\sqrt{2}}{2}\right]$ behaves like $d_{\epsilon,\mathbb{Z}^2} C_\epsilon(\Omega, a)\delta$, with $d_{\epsilon,\mathbb{Z}^2}$ a lattice-specific constant, and C_ϵ a universal term.

Conformal Field Theory (CFT) methods predict both the value of the above critical exponents (i.e., the power-law exponents, 1/8 and 1 respectively) and the functions C_σ and C_ϵ (note however that CFT abstracts away the lattice-specific constants $d_{\sigma,\mathbb{Z}^2}$ and $d_{\epsilon,\mathbb{Z}^2}$).

1.2. Blume-Capel Model and Tricritical Ising Model

The so-called *Blume-Capel model* [**21**, **26**], is a classical variant of the Ising model. It consists of spins σ_x living on the vertices of a finite graph $\mathbb{G}$, taking values in $\{-1, 0, 1\}$. The probability of a spin configuration is proportional to $e^{-\beta H[\sigma] - \Delta V[\sigma]}$, where $\beta > 0$ and $H[\sigma] = -\sum_{\{x,y\}\in\mathcal{E}} \sigma_x \sigma_y$ are as in the Ising model, and where $\Delta \in \mathbb{R}$ and $V[\sigma] = \sum_{x\in\mathcal{V}} \sigma_x^2$ is the chemical potential:

$$\pi_{\beta,\Delta}(\sigma) := \frac{1}{\mathcal{Z}_{\mathrm{BC}}(\beta,\Delta)} \exp\left(\beta \sum_{\{x,y\}\in\mathcal{E}} \sigma_x \sigma_y - \Delta \sum_{x\in\mathcal{V}} \sigma_x^2\right).$$

Informally, it consists of spins σ_x that are either vacant (i.e., $\sigma_x = 0$), in which case they don't interact with their neighbors, or present (i.e., $\sigma_x = \pm 1$), in which case they behave as Ising spins; the strength of the alignment effect is modulated by $\beta > 0$, while the tendency for a spin to be present is controlled by $\Delta \in \mathbb{R}$. A competition exists between the interaction energy H and the potential V. The configurations that are most likely to occur are those that minimize $\beta H + \Delta V$ (see Figure 1.2.1).

While only a few mathematical results are available [**74**] on the two-dimensional Blume-Capel model, we have a very good heuristic picture for its behavior, thanks to simulations and physical arguments [**14**].

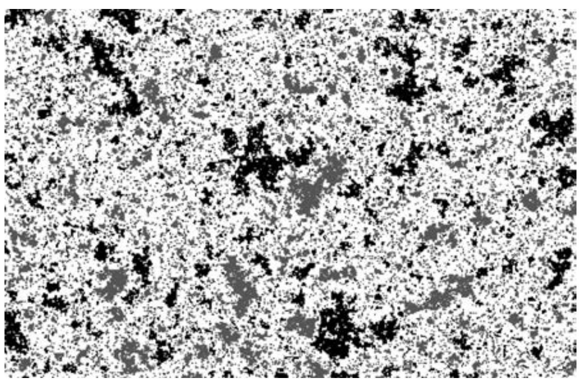

FIGURE 1.2.1. Instance of the Blume-Capel model near the tricritical point (black: $+1$ spins, gray: -1 spins).

Taking the same setup as for the Ising model, i.e., considering the model on $\Omega_\delta = \Omega \cap \delta\mathbb{Z}^2$ as $\delta \to 0$, with $+$ boundary conditions, we see (conjecturally) a critical value Δ_c and a critical line of the following nature $(\beta_c(\Delta), \Delta)$ (see Figure 1.2.2). As β varies, the model goes from a disordered phase for $\beta < \beta_c(\Delta)$ to an ordered phase for $\beta > \beta_c(\Delta)$ and:

- For fixed $\Delta > \Delta_c$, it does so in a discontinuous manner: the limiting magnetization is discontinuous at $\beta_c(\Delta)$.
- For fixed $\Delta < \Delta_c$, it does so in a continuous manner: it undergoes an Ising-type order-disorder transition at $\beta_c(\Delta)$ (notice that the $\Delta \to -\infty$ case is equivalent to the Ising model). At the critical point $\beta_c(\Delta)$, the critical exponents are expected to be the same.

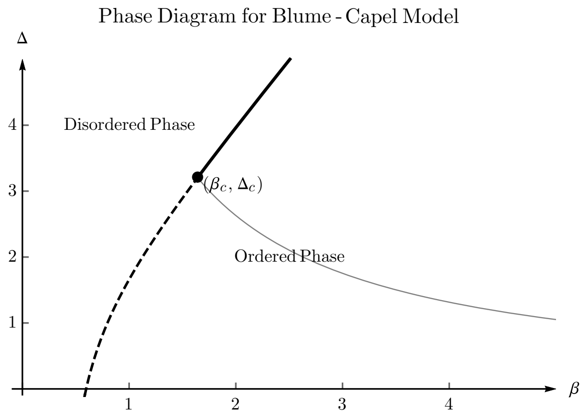

FIGURE 1.2.2. Qualitative (conjectural) picture of the phase diagram for the Blume-Capel model. For $\Delta > \Delta_c$ (thick line), there is a discontinuous (first-order) phase transition; for $\Delta < \Delta_c$ (dashed line), we see an Ising-like phase transition. The qualitative thin gray line divides the ordered phase into two regions: above the line (resp. below) for values of β and Δ sufficiently large, one observes an infinite cluster of vacancies (resp. of non-zero spins).Numerical simulations [**14**] position the tricritical point (β_c, Δ_c) around the point $(1.639, 3.222)$.

- At the $\Delta = \Delta_c$ point, it does so in a continuous manner: a new continuous phase transition, governed by a behavior different from the Ising one, takes place. The model at the point (β_c, Δ_c) where it occurs is called the *tricritical Ising model*.

At the tricritical point, it can be expected that an instance of the Blume-Capel model will consist, qualitatively, of a collection of conformally invariant fractal non-touching (and not self-touching) loops separating clusters of $\sigma = \pm 1$ spins from $\sigma = 0$ spins. Furthermore, each $\sigma = \pm 1$ cluster will have constant spin, chosen independently and with an equal probability between $\sigma = +1$ and $\sigma = -1$.

Even though the description of the tricritical point provided above is not yet understood rigorously, CFT methods give precise (conjectural) information about the tricritical Ising model phase transition at (β_c, Δ_c). In particular, it gives conjectural insight into the following; at $(\beta, \Delta) = (\beta_c, \Delta_c)$, as $\delta \to 0$:

- The magnetization $\mathbb{E}\left[\sigma_{a_\delta}\right]$ behaves like $d^{\mathrm{TIM}}_{\sigma,\mathbb{Z}^2} C^{\mathrm{TIM}}_{\sigma}(\Omega, a)\delta^{3/40}$, with $d^{\mathrm{TIM}}_{\sigma,\mathbb{Z}^2}$ a lattice-specific constant, and $C^{\mathrm{TIM}}_{\sigma}$ a universal term.
- The normalized energy density $\mathbb{E}\left[\sigma^2_{a_\delta}\right] - \frac{1}{2}$ behaves like $d^{\mathrm{TIM}}_{\epsilon,\mathbb{Z}^2} C^{\mathrm{TIM}}_{\epsilon}(\Omega, a)\delta^{1/5}$, with $d^{\mathrm{TIM}}_{\epsilon,\mathbb{Z}^2}$ a lattice-specific constant, and $C^{\mathrm{TIM}}_{\epsilon}$ a universal term.

Once again, CFT methods predict both the value of the above critical exponents ($3/40$ and $1/5$ respectively) and the values of $C^{\mathrm{TIM}}_{\sigma}$ and $C^{\mathrm{TIM}}_{\epsilon}$. It is quite remarkable that such precise information can be inferred about this lattice model, for which we only know a priori a purely qualitative (and conjectural) picture.

1.3. 3-Potts Model and Tricritical 3-Potts Model

A natural generalization of the Ising model is the 3-Potts model, where spins can take three (symmetrically equivalent) possible values. Representing these values by the three roots of unity $\{z \in \mathbb{C} \mid z^3 = 1\}$, the probability of a spin configuration is proportional to $e^{-\beta H[\sigma]}$, where $H[\sigma] = -\sum_{\{x,y\}\in\mathcal{E}} \Re\left(\sigma_x \overline{\sigma_y}\right)$:

$$\pi_\beta^{3\mathrm{P}}(\sigma) := \frac{1}{\mathcal{Z}_{3\mathrm{P}}(\beta)} \exp\left(\beta \sum_{\{x,y\}\in\mathcal{E}} \Re\left(\sigma_x \overline{\sigma_y}\right)\right).$$

Note that the Hamiltonian of the 3-Potts model can be replaced by $H[\sigma] = -\sum_{\{x,y\}\in\mathcal{E}} \delta_{\sigma_x,\sigma_y}$, where $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ if $a \neq b$. Informally, just as in the Ising model, the system tends to favor configurations where spins at adjacent vertices align; the strength of the alignment effect is controlled by β.

EXERCISE 1.3.1. Consider the measure $\tilde{\pi}_{\tilde{\beta}}^{3P}$ defined as follows for any $\tilde{\beta} \geq 0$:

$$\tilde{\pi}_{\tilde{\beta}}^{3\mathrm{P}}(\sigma) := \frac{1}{\mathcal{Z}'_{3\mathrm{P}}(\tilde{\beta})} \exp\left(\tilde{\beta} \sum_{\{x,y\}\in\mathcal{E}} \delta_{\sigma_x,\sigma_y}\right).$$

Determine the inverse temperature $\tilde{\beta}$ such that $\tilde{\pi}_{\tilde{\beta}}^{3\mathrm{P}} = \pi_\beta^{3\mathrm{P}}$.

The model on a square lattice exhibits a continuous phase transition at $\beta_c = \frac{2}{3}\ln(1+\sqrt{3})$ [**15**,**47**]. Consider the model on discretizations $\Omega_\delta = \Omega \cap \delta\mathbb{Z}^2$ of a domain $\Omega \subset \mathbb{C}$, with +1 boundary conditions: for any a in the interior of Ω, $\mathbb{E}\left[\sigma_{a_\delta}\right] \to \mathcal{M}(\beta)$ as $\delta \to 0$ where $\mathcal{M}(\beta) = 0$ for $\beta \leq \beta_c$ and the limit

$$\lim_{\beta \searrow \beta_c} \frac{\mathcal{M}(\beta)}{(\beta - \beta_c)^{1/9}}$$

is finite and non-zero [**2**,**13**,**107**,**114**].

Thanks to CFT, one can predict the following asymptotic behavior at $\beta = \beta_c$: when $\delta \to 0$, the magnetization $\mathbb{E}\left[\sigma_{a_\delta}\right]$ behaves like $d_{\sigma,\mathbb{Z}^2}^{3\mathrm{P}} C_\sigma^{3\mathrm{P}}(\Omega, a)\delta^{2/15}$. Once again, CFT methods predict both the value of the above critical exponent $2/15$ and the value of $C_\sigma^{3\mathrm{P}}(\Omega, a)$.

Allowing vacancies in the 3-Potts model, i.e., allowing spins to take the values $\{z^3 = 1\} \cup \{0\}$ and considering a probability measure proportional to $e^{-\beta H[\sigma] + \Delta V[\sigma]}$ where $\Delta \in \mathbb{R}$ and $V[\sigma] = \sum_{x \in V} |\sigma_x|$ leads to a picture analogous to that of the Blume-Capel model with the existence of a tricritical point. At the resulting tricritical point, the model is called the *tricritical 3-Potts model*. Again, CFT methods predict formulas for the limiting behavior of the model (e.g., $\mathbb{E}\left[\sigma_{a_\delta}\right]$ decays as $\delta^{2/21}$ as $\delta \to 0$).

1.4. Other Models Described by CFT

The lattice models mentioned above are only a few of the critical lattice models that can be described using CFT methods; the spectrum of such models is impressively vast and includes in particular:

- The Ising and 3-Potts model can be generalized to more states, by considering the *Q-Potts model* with spins taking values in $\{z : z^Q = 1\}$ and with energy $H[\sigma] = -\sum_{\{x,y\}\in\mathcal{E}} \delta_{\sigma_x,\sigma_y}$. The $Q = 4$ case can be seen as an Ashkin-Teller model [**115**], and can be described by a CFT. The Q-Potts models for $Q > 4$

exhibit a discontinuous phase transition, and are hence not described by any CFT [**15**, **45**].

- The *Ashkin-Teller models* (introduced in [**5**]) consist of two copies of the Ising model coupled via a four-spin interaction.
- The *Clock Potts model* [**39**, **50**] where the spins take values $\sigma_x \in \{z \in \mathbb{C}, z^Q = 1\}$ and with Hamiltonian $H[\sigma] = -\sum_{\{x,y\}\in\mathcal{E}} \Re\left(\sigma_x \overline{\sigma_y}\right)$, which is a another generalization of the 3-Potts model.[2]
- The vast family of *Restricted Solid-On-Solid models* (RSOS) defined by Andrews, Baxter, and Forrester [**3**, **82**].

Finally, it is worth mentioning that CFT techniques and ideas have also played relevant roles also on predicting probabilities of connection events for probabilistic models such as percolation. Unlike the Ising model, in *site percolation*, the ± 1 spins $(\sigma_x)_{x\in\mathcal{V}}$ living on the vertices of a graph $\mathbb{G}$, do not interact with each other, and they are all independent random variables identically distributed according to Bernoulli law with parameter p, i.e., $\mathbb{P}(\sigma_x = 1) = p$. Considering macroscopic observables such as *crossing probabilities* (i.e., probabilities that fixed points are linked by connected components of identical spin values), we can also see continuous phase transitions. There exists a critical value p_c [**75**] for which such observables become conformally invariant. This is in particular the content of the so-called *Cardy's formula* for crossing probabilities in a topological rectangle, which was initially derived using CFT methods [**28**, **60**, **95**, **126**] and later proven rigorously in [**127**].

1.5. Universality of Phase Transitions

A remarkable feature of the results and conjectures stated in Sections 1.1–1.3 is their universality: most of the specific details in the definitions of the above lattice models (such as the choice of the lattice) are irrelevant in the scaling limit description of the phase transition. As an example, a classical illustration of the universality phenomenon arises in the context of random walks made of independent identically distributed centered jumps: as long as the jumps have a finite variance, the large-scale behavior is essentially the same, and the scaling limit can be described by a universal object, the Brownian motion. Similarly, we expect a large class of models with ± 1-valued spins at each vertex of an arbitrary lattice, with short-range interactions and spin-flip symmetry to be described by the Ising universality class: while the values of the parameters where the phase transition occurs will typically depend on the lattice and the interaction range, the critical exponents and the rescaled correlations will essentially not.

As a result of the (mostly conjectural) universality, each of the models outlined above is representative of a universality class exhibiting the same large-scale behavior

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[2]This generalization of the 3-Potts model differs from the Q-Potts models discussed in Section 1.4. For instance, the 4-Clock Potts model is equivalent to two independent Ising models. Given two independant Ising models σ_1 and σ_2 with same inverse temperature, the model $\sigma(x) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}(\sigma_1(x) + i\sigma_2(x))$ defines a 4-Clock Potts model. Importantly, as outlined in Section 5.7.3, the scaling limit of this model is a CFT with a central charge $c = 1$. In contrast, the 4-Potts model is not equivalent to two independent Ising models. Its decription in terms of an Ashkin-Teller model includes a non-zero four-spin interaction term that couples the two Ising models. For a recent exploration of the Q-Clock Potts model when $Q > 4$, and its conjectural connection with the compactified boson theories, we refer to [**100**].

at criticality: it is expected that the critical behaviors discussed above occur in more general models, thereby greatly increasing the chance of seeing them in various contexts. By studying a specific model, such as the Ising model on the square lattice, one gains insight into a large universality class.

1.6. Outlook: Statistical Field Theory

Above in this chapter, we stated several claims about the phase transitions of some particular lattice models: at the critical point, remarkable power-law behaviors arise, and universal, exact results arise. This book aims to shed light on these predictions.

In Chapter 2, we will discuss how such phase transitions can be understood within a formalism at the intersection of Statistical Physics and Quantum Field Theory (QFT). The critical behaviors can be read from continuous universal objects that extract the relevant features of the models and which can be studied thanks to their conformal symmetry.