1.4 Function Composition Using Crackers and Cheese

Julie Barnes, Western Carolina University

Concepts Taught: function composition

Activity Overview

When students are introduced to function composition, it often has no meaning to them. Without any meaning, students may compose the wrong direction, turn composition into multiplication, or make any number of other mistakes. However, most students already have a very clear understanding of both the concept of a sandwich and how to make one. In this activity, sandwich making is broken down into two steps, each introduced as a function; students then use crackers and cheese to represent a variety of different compositions of the two functions. By changing the order of the sandwich-making steps, and relating this to function notation, students have an opportunity to make that connection between the notation for function composition and its meaning.

Supplies Needed (per group)	Class Time Required	Group Size
1 individual snack pack of crackers and cheese* Small paper plate	10 minutes	2-4 students

*The snack packs come in boxes and are convenient to use in class because they are easy to distribute and are the right size (roughly 5 or 6 crackers) for a group. It is also possible to use a large box of crackers and anything spreadable like cheese, jelly, or jam; for this to work, have a station where students can pick up the crackers, a plastic knife, and a spoonful of cheese, jelly, etc., on their plates for use during the activity.

Running the Activity

Provide each group with a paper plate, a snack pack of cheese and crackers, and a handout. Have students draw an "x" in the middle of the plate, as seen in Figure 1.5, and define the following two functions that will be used in this activity.

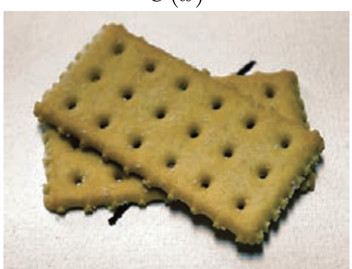
$C(x)$ is the function of placing a cracker on x.
$S(x)$ is the function of spreading one teaspoon of cheese on x.

It is helpful to have a brief discussion about the role of x as a placeholder and not just the mark on their plate. Ask questions such as, "What would C(chair) or C(hand) represent?"

Figure 1.5: Starting by drawing an "X" on a paper plate.

$C(x)$

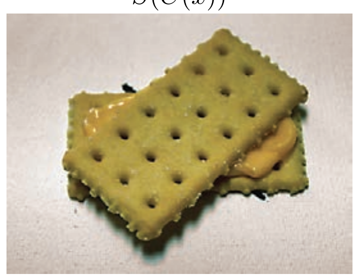
$S(x)$



$C(C(x))$

$S(C(x))$

$S(C(C(x)))$



$C(S(C(x)))$

Figure 1.6: Photographs of constructions from the handout.

or "Would you want to create S(chair)? Why or why not?" Once everyone is comfortable with the notation, have them work through the problems on the handout. The handout asks students to explore a variety of possible compositions of $C(x)$ and $S(x)$ by creating each composition with their crackers and cheese. While students are working, circulate through the room to provide hints and answer questions. See Figure 1.6 for examples of the kinds of function compositions students will create.

Once the students have built their cracker creations, have a class discussion about the activity. What did they notice about the order of steps? How does this relate to the symbols describing what they did? How is composition different from basic addition? If they do not see a difference, ask them what $C(x) + S(x)$ would look like. It would have to involve two plates; one would have a cracker and one would have cheese spread on x. This is clearly different from what they just created.

Suggestions and Pitfalls

This is an easy activity to implement but it could generate crumbs; it is a good idea to bring paper towels or baby wipes. Also, be aware that students may have food allergies and some might not want to touch crackers or cheese for this reason. For a large class, you may want to do this as a class demonstration.

2.5 Graphing Functions from Derivative Information Using Bendable Sticks

Julie Barnes, Western Carolina University

Concepts Taught: derivatives, critical points, increasing and decreasing, inflection points, concavity, graphs

Activity Overview

With the amount of technology available, students often expect calculators or computers to do all their graphing for them. However, this could keep them from making valuable connections between the derivatives of a function and the shape of that function's graph. In this activity, students use information about derivatives to graph functions without being provided with equations; this forces them to consider those connections between derivatives and graphs. Students are also able to experience the function properties of the first and second derivatives by physically creating a curve instead of just sketching it on paper.

Supplies Needed (per group)	Class Time Required	Group Size
3 bendable sticks, preferably in different colors	20-30 minutes	2-3 students

*Two brands of easy-to-use bendable sticks that work well are Wikki Stix® and Bendaroos® which can be found online and sometimes in craft stores. An added bonus to using these is that they will adhere to paper and yet can be removed much like sticky notes can be removed; therefore, graphs will stay in place while students are working. Pipe cleaners can also be used, but they don't stick to the paper nicely.

Running the Activity

Have each group of students begin by sketching a coordinate system on their paper for $-2 < x < 2$ and $-2 < y < 2$, with roughly $1''$ per unit. The only point that absolutely needs to be marked on the graph is $(1, 0)$. Provide each group with three different colored bendable sticks and a handout. Have students work through the handout in groups.

In the first problem, students are given a list of conditions about the first and second derivative of a function h and are asked to use a bendable stick to create a graph that meets the given criteria as seen in Figures 2.5a and 2.5d. Then they need to use a second bendable stick to graph the first derivative of h (Figures 2.5b and 2.5e) and a third to graph the second derivative of h (Figures 2.5c and 2.5f). Since the sticks can be moved and bent, students are able to work with different criteria in any order, possibly bending the stick enough to satisfy increasing and decreasing features and then bending further to obtain the correct

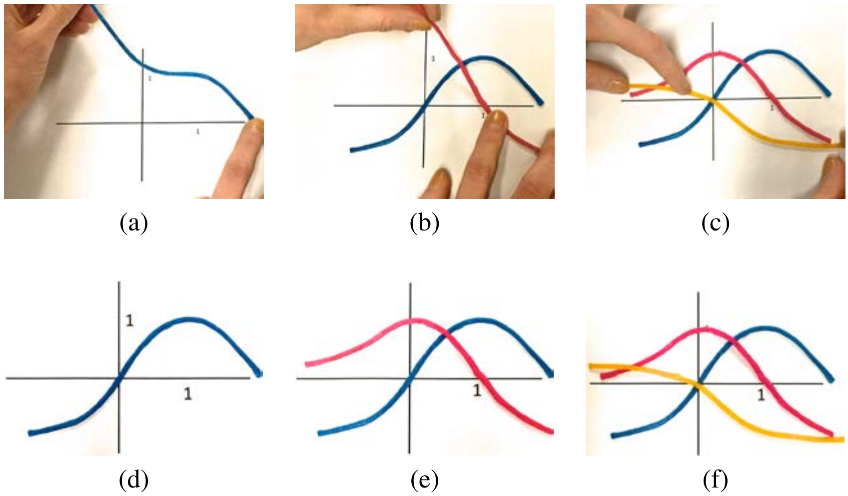


Figure 2.5: (a)-(c) Students using Wikki Stix to create graphs for h, h', and h'' as described in Problem 1 of the handout. (d) A possible graph for h. (e) A graph of h' with h. (f) A graph of h'' along with h' and h.

concavity. This means students can keep modifying their work until it meets all criteria, and they do not need to erase. Once students have a function and its derivatives graphed, they are asked to compare the derivatives to the information used to create the function in the first place.

The second problem asks students to graph any function of their choosing. Sometimes this catches them off guard because they are used to being told exactly what to graph instead of just being told to make sure it passes the vertical line test. Once they have a graph, they are asked to use a second bendable stick to graph a new function that has the same derivative. For students who are struggling, you can mention that the easiest way to do this is to bend the second stick on top of the first one, and then pick it up and move it up or down the y-axis. Make sure students understand why this works.

While the students work, circle around the room to answer questions and check their work. Note that because of the nature of the Wikki Stix, it is easy do things like start where the derivative is zero, and then bend the curve accordingly.

Suggestions and Pitfalls

This activity also works well with feather boas. In Figure 2.6 we see students working on Problem 2 from the handout. Feather boas take a little more setup time and space, as seen in Activity 1.6, Graphing Piecewise Functions with Feather Boas, but students tend to work more as a team with the larger objects.

Figure 2.6: Students using feather boas to graph two functions that have the same derivative, as in Problem 2 from the handout.

3.7 Fun with Infinite Series

Jennifer Hutchison, Cedarville University

Concepts Taught: geometric series, series convergence, modeling

Activity Overview

Series convergence and divergence tend to be counterintuitive concepts for many students. Adding up infinitely many numbers to obtain a finite sum, but only sometimes, can seem like sophistry, and that makes sense: the dividing line between converging and diverging is very fine. Geometric series are an excellent example of this. They are especially useful for examples since there is an easy formula to calculate the sum of a convergent geometric series. This activity includes three concrete examples of infinite processes that illustrate both convergent and divergent series: cutting paper in half, the Koch snowflake curve, and a bouncing ball. A class could do one, two, or all three of these activities, with varying levels of instructor involvement. For example, an instructor could set up handouts and supplies for all three on separate tables as stations and students would choose what interests them. All three activities are designed to be self-guided, but it is helpful to have a whole-class debriefing at the end about what students learned or observed, in order to draw out the desired illustration of convergence and divergence.

Supplies Needed	Class Time Required	Group Size
Fun with Paper Scrap paper (1-3 pieces per student) Scissors (1 pair per student)	20-25 minutes	1-4 students
Fun with Fractals Straight edge (1 per student) Pencil and eraser (1 per student)	30-40 minutes	2-4 students
Fun with Gravity Meter stick (1 per group) Bouncy ball (1 per group) Calculators (at least one per group) Stopwatches (optional)	45 minutes	3-5 students

Running the Activity

For each activity, hand out the handouts and supplies, or set up stations containing them. Labeling each table by activity can be helpful if students get to choose their activity. Students should follow the instructions on each handout and answer the questions, discussing with their group as they go. The handout is designed to guide students through the activity, and it is usually helpful to observe what students are doing and ask helpful questions about their progress and thoughts. After students have had time to fully analyze the situation (or run out of time), discuss what they observed and calculated.

Fun with Paper

The handout directs students to cut a piece of $8.50'' \times 11''$ paper into a square, then cut the square into successive halves using two methods, each of which is illustrated on the handout. The handout then has students explore what would happen if they were to continue each process indefinitely. After calculation, students will find that in either case, there would be a finite area of paper in infinitely many pieces, giving a useful concrete illustration of a convergent series. In the first method, the length of paper cut is infinitely long (illustrating a divergent series), while in the second, they would cut a finite distance (illustrating a convergent series). For this activity, one interesting point is that the methods of cutting are ostensibly similar but result in quite different outcomes (finite versus infinite cutting lengths).

Fun with Fractals

After the students draw several iterations of the Koch snowflake, they should begin figuring out lengths and areas as they answer the questions on the handout. They should find that while the added amount of perimeter and area each decrease to zero as the number of iterations tends to infinity, the perimeter tends to infinity while the area tends to a finite amount. This provides both an example of the failure of the converse of the nth term test for series divergence, and somewhat concrete examples of convergent and divergent series. In particular, the students are forced to confront an object with an infinite perimeter and a finite area. This should cause them to be taken aback and to realize that there are situations in which our intuition needs to be verified by mathematics, since most students would consider this impossible.

Fun with Gravity

Prepare a location in which students can drop bouncy balls on a good rebounding surface (such as a tabletop or desktop). Each group will drop the ball several times, take measurements of the rebound height using the meter stick, then model the distance traveled by an ideal ball that rebounds to exactly the same percent of its height each time and continues bouncing in this way forever. They will then apply the projectile equation to analyze the length of time this will take. They should find that the distance and time are both finite (e.g., that their series converge).

For this activity, in addition to the benefit of seeing a concrete example of a convergent series, there is an especially good use of detailed thinking when modeling. For instance, one can discuss the role of the assumption about rebound height, and discuss with students what would happen if they picked a different percent. (Hopefully, they will observe that only a percent less than 100% would be reasonable, and would still result in a convergent series.) Also, it is interesting to discuss how to set up the series given that the ball travels only down for the first distance and both up and down for every subsequent bounce; students could either pull the first trip down out of the series and then reset the initial height, or write the series as though the ball goes both up and down the first time and then subtract off the first distance. You will probably have to lead them into considering how to account for this in their series model.

Suggestions and Pitfalls

Fun with Paper

After several minutes of paper-cutting, it is helpful to remind the students that there is math to be done. Often students begin a competition as to who can cut the smallest pieces of paper; I usually allow this as long as they get to the math in a reasonable amount of time. One of the goals is to have fun, since series is usually a fairly intense chapter. For the mathematical analysis, one can use the actual dimensions of the paper, but it is easier to declare the paper squares 1 unit by 1 unit.

Fun with Fractals

Students nearly always carry a student ID, and this can make a useful straight edge for the level of accuracy needed. It is more difficult to complete this activity with a pen since then one cannot erase the middle line segments before constructing the triangles, so strongly encourage the use of pencils and erasers. Also, it might be helpful to have extra copies of the triangle. In addition, it is helpful to check that students are finding and replacing all line segments with the "mountain" shape; usually they miss the remaining segments of the original triangle.

Fun with Gravity

Students may take a while to get used to estimating rebound height. Help any who are particularly disturbed by inaccuracy to accept the idea of estimating by discussing the nature of mathematical models and perhaps suggesting that they include a statement about what level of accuracy they might be obtaining. For the questions on the time elapsed, students may decide they would rather take data and find an average or pattern that way, instead of using the projectile equation that is given on the handout. It will probably be helpful to push them to collect sufficient data to draw good conclusions in this case. If you wish to include this as part of the activity, you could provide stopwatches.

If you have more ready access to yardsticks than meter sticks, change the units (3 ft rather than 1 m) and the gravitational constant ($s(t) = h - 16t^2$ instead of $s(t) = h - 4.9t^2$).

4.2 Building Functions of Two Variables with Cookies

Jessica Libertini, Virginia Military Institute

Concepts Taught: Concepts taught: multivariable functions, partial derivatives, Riemann sums, volume approximations, contour lines

Activity Overview

When first learning about functions of one variable, students are often asked to evaluate a function at several points, plot those points, and sketch the graph to help them understand the shape and behavior of that function. As we move to functions of two variables, paper sketches become problematic, requiring us to look at cross-sections or develop strong perspective drawing skills. While software packages such as Mathematica allow students to visualize functions of two variables, they cannot replace the intuition students develop by plotting points in space and generating their own graphs of multivariable functions. In this activity, students use cookies to generate three dimensional graphs of functions of two variables. This activity also provides a useful foundation for a variety of topics such as partial derivatives, multivariable integration, and contour lines as explained in the Extensions Section on page 117.

Supplies Needed (per group)	Class Time Required	Group Size
One large package of cookies* Cookie-sized coordinate system**	20-30 minutes for activity 10 minutes per extension	3-5 students

*Each group will need between 20-60 stackable cookies or crackers to make their functions, so if using packages that have less than 60 cookies, encourage groups to share. Alternatively, this activity can be done using coins or other stackable objects, as shown in Figure 4.5.
**Based on the size of the cookies, you will need to make a handout for each group with an appropriately sized coordinate system on it; in other words, the unit length of the grid is the diameter (or the major axis, for noncircular cookies) of a cookie. Since each group will be graphing a different function over a different domain, it is best to have the students label the axes and origin.

Figure 4.3: Three dimensional graph of a function of two variables using cookies.

Running the Activity

Divide the class into groups of 3-5 students. You may wish to have them rearrange their desks so they can share a work space. Provide each group with a box or bag of cookies, a cookie-sized coordinate system, and a copy of the handout. Assign each group a domain and a function of two variables whose value is positive over the prescribed domain. Each group of students should then calculate, for each pair of coordinates on the grid, the number of cookies they need to stack in order to build a three dimensional representation of their function.

Problems 1 and 2 on the handout guide groups through the creation of a three dimensional cookie-graph of their function, as shown in Figure 4.3. Problems 3 and 4 address the limitations of using cookies to plot functions. Problem 5 asks students to discuss what other groups' graphs might look like based on equations. Lastly, Problem 6 has them visit other groups to see how the cookie graphs compare with their answers to Problem 5. Be sure to allow time for them to do the gallery walk to see the work of their classmates.

Suggestions and Pitfalls

In order to see a nice portion of the graph while minimizing time, take advantage of symmetry. For the sample functions given in the handout, it is best to make a 5×5 grid centered about the origin. OREO® cookies are approximately $1.75''$ in diameter, so you can fit four across a sheet of paper, but not quite five. Generic cookies are often a little smaller, allowing you to fit a row of five cookies with boxes of $1.7''$ or smaller.

Be sure that you've calculated the number of cookies required to graph each function and that each group has access to enough cookies to complete their graph. Sandwich cookies, such as OREO cookies, give students the opportunity to be creative in representing non-integer values as seen in Figure 4.4a. You may wish to have them present their graph side by side with a computer-based representation, such as a three dimensional plot or a contour plot, as shown in Figure 4.4.

(a) (b)

Figure 4.4: Three dimensional cookie graphs displayed with corresponding Mathematica contour plots.

Figure 4.5: Three dimensional graph of a function of two variables using coins.

Extensions

This activity can be used simply to introduce graphs of functions of two variables; however, it can also be used to support several multivariable calculus concepts, as described below.

Partial Derivatives and Directional Derivatives. Due to the inherently discrete nature of the cookie graph, students can approximate directional derivatives and partial derivatives by comparing the heights of neighboring stacks of cookies. For example, you might ask each group to use their cookie graph to approximate the partial derivative at $(0, 1)$ in the y-direction.

Volume Approximations. When making a graph in three-space, it would be wonderful if we could hang points, or cookies, in midair; however, in this exercise, we represent the height of the function at a point by building a stack of cookies. The lower cookies, while not actually part of the graph of the function, literally play a supporting role (holding up the upper cookie); they also represent an approximation of the volume beneath the surface. Therefore have your students approximate the volume captured between their surface and the xy-plane by totaling the number of cookies used in their graph.

Contour Plots and Contour Lines. For this add-on activity, you should have markers or crayons available. Have your students replace each stack of cookies with a color-coded marking, e.g., all blocks that contained stacks of 3 cookies get colored red while all blocks containing stacks of two cookies get colored orange, etc. By having your students connect similarly colored blocks, they can explore contour lines. Ask your students what it might mean for two adjacent blocks to have the same color.

5.1 Crowdsourcing to Create Slope Fields

Karen Bliss, Virginia Military Institute
Jessica Libertini, Virginia Military Institute

Concepts Taught: differential equations, slope fields

Activity Overview

A slope field is a powerful tool to visualize the family of solutions of a first-order differential equation. Typically, students use a computer to generate a slope field to avoid the repetitive and tedious task of generating one by hand. However, if students are never asked to create a slope field by hand, they may struggle with understanding its relationship to the differential equation.

In this activity, the class as a whole creates the slope field for each of three differential equations. Students each calculate the slopes for just a few points by hand, and these answers are used collectively to generate the slope field. Hence, students get an opportunity to actually compute the slope at just a handful of points, reinforcing the concept of a slope field, but through the crowdsourcing, also get to see a good visualization without the tedium and repetition required for an individual student to generate an entire slope field.

Supplies Needed	Class Time Required	Group Size
Thin strips of colored paper* Painter's tape axes** A few rolls of painter's tape or masking tape	40 minutes	Whole class

*You will need three different colors of paper, such as construction paper or card stock. For each color, cut 81 strips, each approximately $4'' \times 0.5''$.
**For each slope field, use 4 ft lengths of painter's tape each for the x-axis and y-axis. Choose a space where your class will be able to gather around to view the slope field for discussions at the end of the activity, such as a wall or a tiled floor. Write directly on the tape with a marker to indicate the grid scaling from -2 to 2 in increments of 0.5 in both x and y, each with an approximate scale of 1 ft per unit. If working on a tiled floor, you can use the floor grid to help you maintain even spacing; otherwise, you may wish to use a ruler to help maintain regular spacing. The axes can be made by the instructor while students are performing their slope calculations, or they can be prepared before class.

Running the Activity

After giving your students a brief overview of how slope fields are generated, tell them that they will be making three slope fields by hand as a class. Let them know that, since the calculation process is straightforward and repetitive, they will be working collectively to generate their slope fields, rather than each person doing all the calculations.

Each slope field will be generated on a grid from -2 to 2 with 1/2-unit spacing, meaning that there are 81 points in each grid. Assign these points to individual students in your class; this can be done using a projector or by passing out cards marked with the points. Once the students have their assigned points, provide them the handout that includes the differential equations for each of the three slope fields. Tell your students which color coordinates with each of the three slope fields, and have them record that information on the handout. Next, have your students calculate the slopes for each of the three slope fields at their assigned points and record their answers on the handout.

The handout also provides guidance on how to contribute to the construction of the slope field. In order to facilitate the construction phase, set up a central station with several rolls of painter's tape and a pile of paper strips of each color. Once students have completed their calculations, they should tape an appropriately colored strip of paper, angled to match their calculated slope, at each of their assigned points on the three coordinate systems, resulting in slope fields such as those in Figure 5.1.

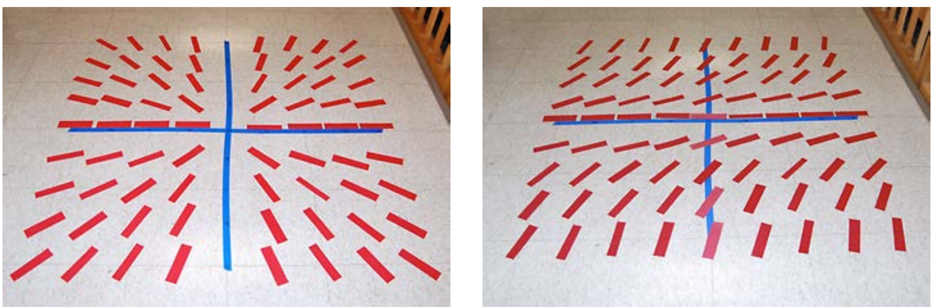


Figure 5.1: Slope fields for differential equations (A) and (C) from the handout.

After all students finish taping their paper strips onto the coordinate systems, have the students gather around the resulting slope fields. Before starting a formal discussion relating to the topics on the handout, it is beneficial to ask the class to assess and correct the slope fields; a significantly incorrect slope will stand out. Once the class is happy with each of the slope fields, lead the class in a discussion addressing the points that you would like to emphasize which could include the utility of slope fields and the types of information that can be gathered from them, families of solutions, particular solutions, equilibria, and the value of technology in generating slope fields.

Suggestions and Pitfalls

Some students work faster than others in computing the slopes, so look for ways to keep them productively contributing to the activity, such as asking these students to help with computing the slopes for the points not assigned (in case the number of points does not divide evenly into the number of students in the class), to serve as the official photographers so that the class can have a record of their work, and to facilitate the construction process as access to tape can be a limiting factor.

If time permits, you can extend the discussion portion and have students use tape or yarn to mark out particular solutions that result from different initial conditions. Also, if you wish to discuss the classification of equilibrium values, differential equation (C) offers an opportunity to see a semi-stable equilibrium value.

Clean up goes a lot faster if you ask the students to all help with removing the paper strips and tape.

6.7 Discovering Catalan Numbers Using M&M's®

Ann N. Trenk, Wellesley College

Concepts Taught: Catalan numbers, combinatorics

Activity Overview

The Catalan numbers ($c_0 = 1$, $c_1 = 1$, $c_2 = 2$, $c_3 = 5$, $c_4 = 14$, $c_5 = 42$, etc.) are a fascinating sequence of numbers that arise in different contexts, and as a consequence have been rediscovered many times. One way to define c_n is the number of sequences of n red and n green M&M's so that for each k, among the first k M&M's in the sequence, there are never more greens than reds. This is a complicated counting question and students in combinatorics often have an easier time understanding such a question if they first list some sample elements in the set they are trying to count as well as some elements that fall outside this set. This activity provides such an opportunity.

There is a lovely recursive formula for the Catalan numbers ($c_n = c_0 c_{n-1} + c_1 c_{n-2} + \cdots + c_{n-2} c_1 + c_{n-1} c_0$) as well as an explicit formula $c_n = \frac{1}{n+1}\binom{2n}{n}$. The handout following this activity leads students to conjecture the explicit formula. The activity also can be helpful when presenting a proof of the recursive formula.

Supplies Needed	Class Time Required	Group Size
Red and green M&M's 1 container per student* Yarn**	10-15 minutes	Whole class

*Cupcake liners work well, but anything small that can hold six M&M's, such as a plastic bag or a bowl, is fine.
**The yarn needs to be long enough so that when the students are standing shoulder to shoulder, there is yarn in front of all of them. This yarn marks the edge of a mud pit. You can have one student hold the end of the yarn and another unravel it so it is stretched out in front of the whole class.

Running the Activity

For each student, prepare a container with three red and three green M&M's as seen in Figure 6.10. Have the students line up, shoulder to shoulder, holding their M&M's and facing an imaginary mud pit (Figure 6.11a). Ask the students to close their eyes and choose a piece of candy from their bowl. Once they have chosen the candy, they can open their eyes, look at its color, and eat or discard it. Have those students who chose a red candy take

Figure 6.10: M&M's placed in cupcake liners before starting the activity.

a step backwards away from the mud pit; have those students who chose a green candy take a step forward into the mud pit and kneel down to symbolize that they are stuck in the mud. Once a student is in the mud pit, he or she is done taking steps, but can continue to pick and eat M&M's with the rest of the class. The situation at this point is illustrated in Figure 6.11b.

The activity continues in the same way. Again the students close their eyes, pick a candy, then open their eyes and look at its color. If it is green they take a step forward, and if it is red they take a step back. All steps are the same size. After eating two pieces of candy, students who chose GG or GR will be kneeling down in the mud pit, students who chose RG will be back at the edge and those who chose RR will be two steps back from the edge. The activity ends when the students have picked all six of the M&M's. Take note of how many are kneeling down and how many are safe. Students should observe that those standing are back in their starting position at the edge of the mud pit.

After the activity, the students are prepared to work through the handout which guides them to conjecture the explicit formula $c_n = \frac{1}{n+1}\binom{2n}{n}$. The activity is also helpful if the instructor chooses to give a combinatorial proof of the recursive formula $c_n = c_0c_{n-1} + c_1c_{n-2} + \cdots + c_{n-2}c_1 + c_{n-1}c_0$. In that proof, the term $c_{k-1}c_{n-k}$ counts the number of safe sequences in which the first return to the edge of the mud pit is after eating exactly k red and k green candies, and this can be described by placing a safety line one step back from the edge of the mud pit.

Suggestions and Pitfalls

If you have a large class, or a small classroom, you can do the activity twice with half the class participating each time and the other half observing. If the weather is nice, you could bring the class outside and use the edge of a walkway as the edge of the mud pit.

(a)

(b)

Figure 6.11: (a) Students at the beginning of the activity. (b) After eating some M&M's.

M&M's® Activity – Class Handout

Clifford stands at the edge of a mud pit, facing the edge and holding a bag containing n red and n green M&M's. He draws out the candies one at a time and eats them. If he draws a red one, he takes a step back. If he draws a green one, he takes a step forward. All steps have the same size.

1. How many different arrangements of n red and n green M&M's are there?

Call a sequence of n red and n green M&M's *safe* if it leads to Clifford remaining at the edge of the mud pit. For example, when $n = 4$, the sequence $RRRGGGRG$ is safe, but $RGGRRRGG$ is not.

2. For each of the specific cases $n = 1, 2, 3$, list all safe sequences.

3. For each of the specific cases $n = 1, 2, 3$, calculate the probability that Clifford does **not** go over the edge. *Note that the probability a sequence is safe is* $\frac{\textit{\# safe sequences}}{\textit{total \# sequences}}$.

4. Based on these cases, develop a conjecture for the probability that he stays safe in the general case.

5. Use your conjecture from (4) and your answer to (1) to develop a conjecture for the number of safe sequences in the general case.