1

Tic-Tac-Whoa!

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"It is a happy talent to know how to play."
– Ralph Waldo Emerson

Remember when you figured out the strategy for Tic-Tac-Toe as a child? Perhaps there was joy in realizing you would never lose to a sibling again, or maybe there was pride in knowing that your parents would no longer have to 'let' you win. Even though people may have stopped playing it with you, discovering the strategy for Tic-Tac-Toe probably came with a strong sense of accomplishment. This activity aims to rekindle this sense of accomplishment by revisiting a cherished childhood game and connecting it to a new one.

This chapter describes an in-class activity that offers students a glimpse into game theory while facilitating engagement in structured mathematical thought, cultivating interest in deeper mathematical questions, and (anecdotally) strengthening their sense of self-efficacy. Students first analyze Tic-Tac-Toe and then a second game, which they discover is isomorphic to Tic-Tac-Toe.

1.1 Background

Context. Northern Kentucky University is a public, regional, primarily undergraduate institution serving over 14,000 students. As part of the general education requirement, all undergraduate degree-seeking students are required to complete a quantitative inquiry course, all but one of which are offered by the Department of Mathematics and Statistics. One of these classes is entitled Mathematics for the Liberal Arts, which enrolls a large number of students majoring in world languages, the fine arts, and elementary education. Sections of the course typically enroll around 30 students. In developing a course plan, it is vital to engage students in the course material. As such, small-group inquiry-based-learning activities that frequently involved games or puzzles are frequently used to support week-long modules on different mathematical topics.

This activity is one of my favorites from the course for multiple reasons. The primary reason is its utility in facilitating students' engagement in structured mathematical thought, cultivating their interest in deeper mathematical questions, and (anecdotally) strengthening their sense of self-efficacy. The activity's low floor and high ceiling enable students of all abilities and backgrounds both to enter with sufficient prior knowledge to be successful, and also to have the opportunity to engage with mathematical topics typically not introduced until upper level mathematics courses. Of significant note is the near universal student engagement for the full class session, which is a common challenge for this course.

My ultimate goal when teaching this course is to ensure that every student has at least one positive mathematical experience. To avoid spoiling students' positive experiences, I do not assess students' retention of the mathematical ideas involved in this activity. However, there would certainly be ways to incorporate this activity into assessments relating to topics commonly incorporated into this type of course (e.g. symmetry, magic squares).

Tic-Tac-Toe. For completeness, this section discusses the rules for Tic-Tac-Toe (aka noughts and crosses). Tic-Tac-Toe is a two-player game played on a 3×3 grid, commonly depicted using the pound sign (aka hashtag, aka octothorpe). Players alternate turns marking a square, typically by using X's and O's. The first player who successfully has three marks in a horizontal, vertical, or diagonal row is the winner. When played optimally, Tic-Tac-Toe is known to end in a tie. Figure 1.1 depicts a sample game.

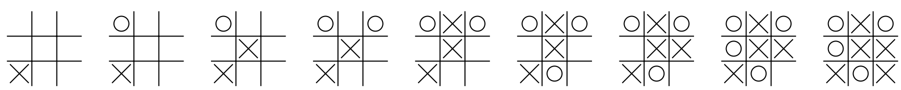

Figure 1.1. A sample game of Tic-Tac-Toe where play progresses from left to right.

1.2 Implementation

The described activity fits nicely into a single, 50-minute session and can naturally extend to fill a 75-minute session. The questions described in this chapter are summarized in a worksheet that can be found at the book's webpage as a Supplementary Material (`https://bookstore.ams.org/clrm-65/`).

The proposed activity can also be included as part of a week-long game theory module that covers various combinatorial games.

The Hook. Another reason this activity is one of my favorites is the opportunity to start class by predicting the outcome of a game of Tic-Tac-Toe. Don't believe I can do it? Let's play! I'll go first by putting an X in the center of the Tic-Tac-Toe board and then continue as follows:

- If your first move is in a corner, I will always put my next X in the square immediately clockwise from your last move.

- If your first move is on a side, I will always put my next X in the square immediately counterclockwise from your last move.

Ready? Go!

Done? Good work. Does the end state of Figure 1.1 look familiar? It does from where I'm sitting.

Okay, so the prediction only works up to rotation. As long as your students don't see your hand-written prediction, it has the appearance of an exact prediction if you can reveal it by holding it up with the proper rotation.

The activity. For the active exploration, students work in pairs to explore two "different" games and discover that the games are, in fact, isomorphic. The conversational activity itself is well-suited for a 50-minute class meeting, but can be easily extended for longer sessions. While the activity can be conducted using nothing more than a pencil and paper, using one deck of playing cards for every eight students allows for more physical movement and a clearer explanation of rules for the second game.

Part 1: Tic-Tac-Toe. After the hook, class begins with an announcement that we will be analyzing Tic-Tac-Toe from a mathematical perspective. Students are given some time to refresh their memory by playing a few games. As students rediscover strategies, they are prompted to discuss their strategy and how successful it is. Some students are convinced that they have a strategy that will always win. If they truly mean 'always,' I view it as an opportunity to discuss the logic behind counterexamples, and then provide one to their claim with a quick game. The fact that Tic-Tac-Toe should always end in a tie typically arises somewhat quickly, albeit not as quickly as I expected the first time through.

At this point, there is an opportunity to provide students with perspective regarding the general education curriculum. When students are asked *how* they know Tic-Tac-Toe ends in a tie, I have always received a response analogous to "I played a lot of games and that's how they all ended." Often the wording of students' responses allows for pointed questions about the precision of their language and the specific meanings of words. For example, students' use of the word *all* often comes with exceptions when one of the players made a mistake. To conclude this brief interlude, I describe the students' approach as scientific and announce that we will now analyze Tic-Tac-Toe strategies using a structured mathematical approach.

Students are prompted with the following questions and given time to explore them.

(1) What is one way to measure the "strength" of a square?

(2) Under your measure, what is the strength of each square?

(3) Are there groups of squares with the same strength? If so, can you come up with an idea as to why?

As intended, students consistently decide to measure a square's strength (and do so accurately) by the number of possible winning rows, columns, and diagonals it is in. As shown in Figure 1.2, squares can be grouped by strength into corners, sides, and the center. Identifying rotational symmetry as the reason behind groupings of squares with the same strength sometimes takes some very directed prompting.

Part 2: Three-to-Fifteen. Having analyzed a familiar game, I inform students that we are going to practice this structured mathematical thought with a new game and

3	2	3
2	4	2
3	2	3

Figure 1.2. The strength of squares in Tic-Tac-Toe.

explain how the game works. In three-to-fifteen, nine playing cards – ace through 9 of a suit – are placed face-up on a table between two players. Players take turns picking one card from the table and adding it to their hand. Similar to the card game Rummy, the winner is the first player who is able to lay down (from their hand) three cards whose sum is 15; cards count as their face value with ace as 1. In the absence of playing cards, students can play by writing 1-9 on a piece of paper and indicating which cards they have picked using, for example, single vs double underlines. Figure 1.3 shows a potential game.

Player 1										**Player 2**
	1	2	3	4	5	6	7	8	9	
4	1	2	3		5	6	7	8	9	
	1	2	3		5	6	7		9	8
4 6	1	2	3		5		7		9	
	1	2	3				7		9	5 8
2 4 6	1		3				7		9	
	1		3				7			5 8 9
2 4 6 7	1		3							

Figure 1.3. A sample game of three-to-fifteen where play progresses from top to bottom. Player selections move a card from the middle to their respective columns. The game ends with Player 1 laying down $2 + 6 + 7 = 15$.

Students are given time to play a few games. The most common rule clarification I make is that the game does not end after six turns (when both players hold three cards); play continues until either a player can lay down a winning set, or all cards have been picked up. I have also had to clarify that putting down two cards face up (whose sum is 15) with another card face down does not count as three cards that sum to 15.

Students are then prompted with the following (intentionally similar) questions:

(4) What is one way to measure the "strength" of a card?

(5) Under your measure, what is the strength of each card?

(6) Are there groups of cards with the same strength? If so, can you come up with an idea as to why?

Students have generally been quick to measure strength by the number of possible winning sums a card is in, but benefit from a guided enumeration of the eight possible sums of three. In Figure 1.4, the possible sums are ordered below by largest number in sum.

$9+5+1 = 9+4+2 = 8+6+1 = 8+5+2 = 8+4+3 = 7+6+2 = 7+5+3 = 6+5+4$

Card	A	2	3	4	5	6	7	8	9
Strength	2	3	2	3	4	3	2	3	2

Figure 1.4. The eight winning subsets of cards, along with each card's strength.

Part 3: Making the Connection. For every class I come prepared to prompt students to compare their answers to Questions 2,3 with Questions 5,6 in an attempt to recognize the similar structure to these games. However, every class has noticed the similarities before being prompted to look for them. Instead, I tell them that if they believe their claim that, for example, the second game is really just Tic-Tac-Toe in disguise, then they should prove it. For the sake of time, I add that if their claim is true, then they should be able to put every card onto a Tic-Tac-Toe board so that every row, column, and diagonal sum to 15 (see Figure 1.5).

8	3	4
1	5	9
6	7	2

Figure 1.5. A magic square depicting the isomorphism between games.

Extensions. There are a variety of ways in which this activity can be extended to include higher level topics. Alternatively, this activity could be condensed and used in an upper level course as a fun introduction to topics.

A Hint of Research. For an opportunity for exploration and imagination, inform students that mathematical research builds upon established truths. Then ask "Since we know how Tic-Tac-Toe works, what could be next?" Prompt students to think of a way to alter the game and then explore the impact of the change. As an example, what if the player who gets three in a row loses?

Exploring Symmetries. To make connections with high school curriculum, expand upon rotational symmetry of the empty Tic-Tac-Toe board and include reflective symmetry. I like to ask students if those symmetries hold throughout the game and give them an opportunity to explore when certain symmetries hold. In an upper level course, one could transition into a conversation about the dihedral group of order eight.

One of Many Combinatorial Games. Do your students want to analyze more games? The game Chomp offers an opportunity to discuss a strategy stealing argument. The game MasterMind provides a setting for detailed analysis and information processing while pursuing a strategy; there are 8-move strategies that are somewhat straightforward to analyze. The open-source textbook *Discovering the Art of Mathematics: Games and Puzzles* provides explorations for 1-, 2-, and 3-pile games; 1-pile is accessible for open exploration by changing the number of stones a player is allowed to pick on each turn [**1**].

Magic Squares. Make bigger magic squares and explore the magic constant (i.e. row sum) for an $n \times n$ magic square: $n(n^2+1)/2$.

Isomorphisms. Have more advanced students describe the isomorphism in a more formal manner.

Counting Non-Isomorphic Boards. While counting the number of non-isomorphic Tic-Tac-Toe boards may be more of an undergraduate project, the question could be used to motivate coverage of Burnside's Lemma and the Redfield-Pólya Theorem.

Predicting the Outcome. Let your students try to figure out how to force the outcome of a game of Tic-Tac-Toe. Careful enumeration of all possible games and a few hours should be sufficient when students know the end configuration to aim for. If you allow reflection, it is possible to force the same configuration as the second player with X starting in the center.

1.3 Conclusion

Remember how the outcome of our game of Tic-Tac-Toe was predictable up to rotation, but did not give you any real choice after your first move? If I changed my third move, you get a choice and the outcome is still predictable up to rotation. If this is not magic enough to hook your students into the lesson, Martin Gardner's *Mathematics Magic and Mystery* describes a card trick that builds a magic square through a game of Tic-Tac-Toe [**2**]. Have fun!

About the author. Axel Brandt is passionate about exposing people to advanced mathematical concepts in fun and approachable ways. His mathematical training in graph theory provides a library of topics for both interactions with K-12 students and also research with undergraduates.

References

[1] J. Fleron, V. Ecke, P. Hotchkiss, and C. von Renesse, *Discovering the art of mathematics* (2008), available at `http://artofmathematics.org/`.

[2] Martin Gardner, *Mathematics, Magic and Mystery*, Dover Publications, Inc., New York, 1956. MR0082445