2

PSP Summaries: The Collection at a Glance

The following overviews briefly describe the primary sources and content goals of the PSPs in this collection. Except where otherwise noted, these projects are fully independent of each other, and thus can be used in any order or combination within a given course. They also assume no prior knowledge on the part of students of the topics that they treat, in keeping with their intended use as a first introduction to those topics. Information about any prerequisite knowledge assumed by a particular project is included in the "Notes to Instructors" section of that PSP. Shorter projects that can be implemented in 1–2 class periods are designated with an asterisk.

2.1 Real Analysis

- **Chapter 4. Why Be So Critical? Nineteenth-Century Mathematics and the Origins of Analysis,*** by Janet Heine Barnett

  In the early nineteenth century, mathematicians began to express concerns about the relation of calculus (analysis) to geometry, as well as the status of calculus (analysis) more generally. The language, techniques, and theorems that developed as the result of the critical perspective adopted in response to these concerns are precisely those which students encounter in an introductory analysis course, but without the context that motivated nineteenth-century mathematicians. This 1–2 day project employs excerpts from texts written by Niels Abel (1802–1829), Bernhard Bolzano (1781–1848), Augustin-Louis Cauchy (1789–1857), and Richard

Dedekind (1831–1916), as a means to introduce students to that larger context in order to motivate and support development of the more rigorous and critical view required of students for success in an analysis course.

- **Chapter 5. Investigations into Bolzano's Bounded Set Theorem,** by David K. Ruch

  Bernhard Bolzano (1781–1848) was among the first mathematicians to rigorously analyze the completeness property of the real numbers. This project investigates his formulation of a version of the least upper bound property from a paper written by Bolzano in 1817. Students read his proof of a theorem on this property, a proof that inspired Karl Weierstrass (1815–1897) decades later in his proof of what is now known as the Bolzano-Weierstrass Theorem. Although this PSP is based on the same source as the PSP "Bolzano on Continuity and the Intermediate Value Theorem" in Chapter 8 of this collection, there is no substantive overlap between the two projects.

- **Chapter 6. Stitching Dedekind Cuts to Construct the Real Numbers,** by Michael P. Saclolo

  As a fledgling mathematics professor, Richard Dedekind (1831–1916) was unsatisfied with the lack of foundational rigor with which differential calculus was taught, and, in particular, with the way the set of real numbers and its properties were developed and used to prove the most fundamental theorems of calculus. His efforts to rectify this situation resulted in his 1872 monograph *Continuity and Irrational Numbers*. This project guides students through his construction of the real numbers as Dedekind cuts: pairs of subsets of the set of rational numbers that represent a real number. The properties of the real numbers emerge out of corresponding properties of the rationals. Project tasks ask the students to interpret, scrutinize, and reflect on the source text, and challenge them to fill in details that Dedekind decided to leave out.

- **Chapter 7. Investigations into d'Alembert's Definition of Limit,*** by David K. Ruch

  The modern definition of a limit evolved over many decades. One of the earliest attempts at a precise definition is credited to Jean le Rond d'Alembert (1717–1783). This 2–3 day project investigates the definition of limit, beginning with d'Alembert's formulation and a modern introductory calculus textbook definition. D'Alembert's definition is completely verbal, and project tasks lead students through examples that guide them towards a translation of his definition written in modern notation and quantifiers. Students are also asked to find examples illustrating the differences between the modern and d'Alembert definitions. The project closes with tasks that prompt students to investigate two limit properties stated by d'Alembert, including their modern proofs.

- **Chapter 8. Bolzano on Continuity and the Intermediate Value Theorem,** by David K. Ruch

  Students read and work through the proof of the Intermediate Value Theorem given by Bernhard Bolzano (1781–1848) in an 1817 paper in which he gave a definition of continuity and formulated a version of the least upper bound property of the real numbers. Although this PSP is based on the same source as the PSP "Investigations into Bolzano's Bounded Set Theorem" in Chapter 5 of this collection, there is no substantive overlap between the two projects.

- **Chapter 9. Understanding Compactness: Early Work, Uniform Continuity to the Heine-Borel Theorem,** by Naveen Somasunderam

  Like many concepts in topology, the modern notions of compactness emerged out of the work of early nineteenth-century analysts and evolved over many decades. This project focuses on the open cover definition of compactness and its connection to the Uniform Continuity Theorem, which states that continuous functions on closed bounded intervals are uniformly continuous. After examining an early proof of that theorem given by Gustav Lejeune Dirichlet (1805–1859), the core of the project is based on works by Emile Borel (1871–1956) and Henri Lebesgue (1875–1941) in which the idea encapsulated in today's open cover definition of compactness was first studied. Lebesgue used that idea to give a "nice proof" of the Uniform Continuity Theorem, which students are asked to compare to Dirichlet's earlier proof. The final section of the project introduces the fully general modern statement of the Heine-Borel Theorem for $\mathbb{R}$ and prompts students to complete its proof via a series of tasks.

- **Chapter 10. An Introduction to a Rigorous Definition of Derivative,** by David K. Ruch

  Augustin-Louis Cauchy (1789–1857) is generally credited with being among the first to define and use the derivative in a near-modern fashion. This project introduces the derivative with some historical background from Isaac Newton (1643–1727), George Berkeley (1685–1783), and Guillaume L'Hôpital (1661–1704). Students then read from Cauchy's 1823 calculus text, written while he was teaching at the École Polytechnique, and explore relevant examples and basic properties of the derivative based on his definition.

- **Chapter 11. Rigorous Debates over Debatable Rigor: Monster Functions in Introductory Analysis,** by Janet Heine Barnett

  Based in part on correspondence between Gaston Darboux (1842–1917) and Guillaume Hoüel (1823–1886) in which they discussed issues related to rigor and proof, this project fosters students' ability to read and critique proofs in modern analysis, thereby enhancing their understanding of current standards of proof and rigor in mathematics more generally. Project tasks also prompt students to refine their intuitions about continuity, differentiability, and the relationship between them, and, in an optional section, introduces them to the concept of uniform differentiability. The project closes with an examination of Darboux's proof of the theorem in analysis that now bears his name: every derivative has the intermediate value property.

- **Chapter 12. The Mean Value Theorem,** by David K. Ruch

  The Mean Value Theorem has come to be recognized as a fundamental result in a modern theory of the differential calculus. In this project, students read from the 1823 efforts of Cauchy to rigorously prove this theorem for a function with continuous derivative. Later in the project, students explore a very different approach that was developed some forty years after Cauchy's proof, by mathematicians Joseph Serret (1819–1885) and Pierre Ossian Bonnet (1819–1892).

- **Chapter 13. Euler's Rediscovery of** ***e*****,*** by David K. Ruch

  The constant *e* appears periodically in the history of mathematics. In this 1–2 day project, students read about *e* and logarithms from Leonhard Euler's (1707–1783) 1748 book *Introductio in Analysin Infinitorum* (*Introduction to the Analysis of the Infinite*) and use his ideas to justify the modern definition: $e = \lim_{j\to\infty}(1 + 1/j)^j$.

- **Chapter 14. Abel and Cauchy on a Rigorous Approach to Infinite Series,** by David K. Ruch

  Infinite series were of fundamental importance in the development of the calculus. Questions of rigor and convergence were of secondary importance early on, but things began to change in the early 1800s. When Niels Abel (1802–1829) visited in Paris in 1826, he became aware of certain paradoxes concerning infinite series and wanted big changes. In this project, students read from the 1821 *Cours d'analyse* (*Course on analysis*), in which Augustin-Louis Cauchy (1789–1857) carefully defined infinite series and proved some properties. They then read from the 1826 paper in which Abel attempted to correct a flawed series convergence theorem from Cauchy's book.

- **Chapter 15. The Definite Integrals of Cauchy and Riemann,** by David K. Ruch

  Attempts to rigorously define the definite integral began in the early 1800s. In this project, students read from an 1823 study of the definite integral for continuous functions by Augustin-Louis Cauchy (1789–1857). They then read from the 1854 paper in which Bernard Riemann (1826–1866) developed a more general concept of the definite integral that could be applied to functions with infinitely many discontinuities.

- **Chapter 16. Lebesgue on the Development of the Integral Concept,*** by Janet Heine Barnett

  The primary goal of this 2–3 day project is to consolidate students' understanding of the Riemann integral (and its relative strengths and weaknesses) by contrasting it with the Lebesgue integral, as described by Henri Lebesgue (1875–1941) himself in a relatively non-technical 1926 paper. The project's second goal is to introduce the important concept of the Lebesgue integral, which is rarely discussed in an undergraduate course on analysis. Lebesgue's overview of the evolution of the integral concept further exposes students to the ways in which mathematicians hone various tools of their trade (e.g., definitions, theorems).

2.2 Topology

- **Chapter 17. The Cantor Set before Cantor,*** by Nicholas A. Scoville

  A special construction used in both analysis and topology today is known as the Cantor set. Georg Cantor (1845–1918) used this set in a paper in the 1880s. Yet a variation of this set appeared as early as 1874, in the paper "On the Integration of Discontinuous Functions" by the Irish mathematician Henry John Stephen Smith (1826–1883). Through excerpts from Smith's paper, this 2-day project explores the concept of nowhere dense sets in general, and the Cantor set in particular. **Also suitable for use in Real Analysis courses.**

- **Chapter 18. Topology from Analysis,*** by Nicholas A. Scoville

  Topology is often described as having no notion of distance, but a notion of nearness. How can such a thing be possible? Isn't this just a distinction without a difference? In this project, students discover the notion of "nearness without distance" by studying the work of Georg Cantor (1845–1918) on a problem involving Fourier series. In particular, they see that it is the relationship of points to each other, and not their distances per se, that was essential to Cantor's efforts. In this way, this 2-day project leads students to see the roots of topology organically springing from analysis. **Also suitable for use in a Real Analysis courses.**

- **Chapter 19. Nearness without Distance,** by Nicholas A. Scoville

  Motivated by a question of uniqueness of a Fourier expansion, Georg Cantor (1845–1918) developed a theory of nearness based on the notion of limit points over several papers written over a decade, beginning in 1872. Emile Borel (1871–1956) then took Cantor's ideas and began to apply them to a more general setting. Finally, Felix Hausdorff (1868–1842) developed a coherent theory of topology in his famous 1914 book *Grundzüge der Mengenlehre* (*Fundamentals of Set Theory*). This project follows the development of topology, starting with the question in analysis initially posed by Cantor, into a theory of nearness of points in the works of Borel and Hausdorff. Although two of the subsections based on Cantor's work in this project overlap with the material developed in the shorter project "Topology from Analysis" in Chapter 18 of this collection, this full-length PSP develops Cantor's work and the notion of "nearness without distance" more generally and in much greater detail.

- **Chapter 20. Connectedness: Its Evolution and Applications,** by Nicholas A. Scoville

  Connectedness has become a fundamental concept in modern topology. The concept seems clear enough: a space is connected if it is a "single piece." Yet today's definition of connectedness is a classic example of a definition that evolved over decades. The need to define the concept of "connected" is first seen in an 1883 work in which Georg Cantor (1845–1918) gave a rigorous definition of a continuum. After its inception by Cantor, definitions of connectedness were given by others, including Arthur Schoenflies (1853–1928) and Nels Lennes (1874–1951). Following this, connectedness was studied for its own sake by Bronisław Knaster (1893–1980) and Kazimierz Kuratowski (1896–1980). This project traces the development of the concept of connectedness through the works of these five mathematicians, proving many fundamental properties of connectedness along the way. Instructors who wish to study connectedness via a PSP should choose between either this full-length project or the 2-day project "Connecting Connectedness" in Chapter 21 of this collection.

- **Chapter 21. Connecting Connectedness,*** by Nicholas A. Scoville

  Focusing on the same concept that is developed in the full-length PSP "Connectedness: Its Evolution and Applications" in Chapter 20 of this collection, this project briefly traces the evolution of the definition of connectedness. It begins with the definition given by Georg Cantor (1845–1918), journeys through the works of Camille Jordan (1838–1922) and Arthur Schoenflies (1853–1928), and culminates with the modern definition given by Nels Lennes (1874–1951). Instructors who wish to study connectedness via a PSP should choose between either the full-length PSP or this more abbreviated 2-day PSP.

- **Chapter 22. From Sets to Metric Spaces to Topological Spaces,*** by Nicholas A. Scoville

  One of the significant contributions that Felix Hausdorff (1868–1942) made in his influential 1914 book *Grundzüge der Mengenlehre* (*Fundamentals of Set Theory*) was to clearly lay out the differences and similarities between sets, metric spaces, and topological spaces. From his work, it is easy to see how metric and topological spaces are built upon sets as a foundation, as well as what is added to that foundation in order to obtain metric and topological spaces. This 2-day project follows Hausdorff as he built topology "from the ground up" with sets as his starting point.

- **Chapter 23. The Closure Operation as the Foundation of Topology,*** by Nicholas A. Scoville

  Today's axioms for a topology are well–known: closure under unions of open sets, closure under finite intersections of open sets, and the entire space and empty set are open. However, in the early twentieth century, multiple axiomatic systems were proposed as equivalent options for defining a topology. One such system, based on the closure operation, was the subject of the doctoral thesis of Kazimierz Kuratowski (1896–1980). In this 2-day project, students work through a proof that today's axioms for a topology are equivalent to Kuratowski's closure axioms by studying excerpts from both Kuratowski and Felix Hausdorff (1868–1942).

- **Chapter 24. A Compact Introduction to a Generalized Extreme Value Theorem,*** by Nicholas A. Scoville

  In a short paper published two years prior to his doctoral thesis, Maurice Fréchet (1878–1973) gave a simple generalization of what we today call the Extreme Value Theorem: continuous real-valued functions attain a maximum and a minimum on a closed bounded interval. Developing this generalization was a matter of coming up with "the right" definitions to make things work. In this 2-day project, students work through Fréchet's entire 1.5-page paper to arrive at an Extreme Value Theorem for a more general type of topological space: those which, to use Fréchet's newly-coined term, are compact.

2.3 Complex Variables

- **Chapter 25. The Logarithm of** -1**,*** by Dominic Klyve

  Understanding the behavior of multi-valued functions can be a difficult hurdle to overcome in the early study of complex analysis. Many eighteenth-century mathematicians also found this difficult. This 1–2 day project looks at excerpts from letters in the correspondence between Leonhard Euler (1707–1783) and Jean le Rond d'Alembert (1717–1783) in which they argued about the value of $\log(-1)$. This argument not only set the stage for the rise of complex analysis, but helped to end a longstanding friendship.

- **Chapter 26. Riemann's Development of the Cauchy-Riemann Equations**, by David K. Ruch

  This project examines the Cauchy-Riemann equations (CRE) and some of their consequences from the perspective of Bernard Riemann (1826–1866), using excerpts from his 1851 *Inauguraldissertation*. Students work through Riemann's argument that the differentiability of a complex function $w = u(x, y) + iv(x, y)$ of a complex variable $z = x + iy$ is equivalent to satisfying the CRE. Riemann also introduced Laplace's equation for the u and v components of w, which allows students to explore some basic ideas on harmonic functions. Riemann's approach with differentials works nicely at an intuitive level and motivates the standard modern proof that the CRE follow from differentiability. In the final section of the project, students are introduced to the modern definition of derivative and revisit the CRE in that context.

- **Chapter 27. Gauss and Cauchy on Complex Integration,** by David K. Ruch

  This project begins with a short excerpt on the meaning of definite complex integrals and a claim about their path independence taken from an 1811 letter written by Carl Friedrich Gauss (1777–1855). Students then work through the development of a definite complex integral given by Augustin-Louis Cauchy (1789–1857) in 1825, culminating in a a parametrized version allowing for evaluation of these integrals. They then apply Cauchy's parametric form to illustrate Gauss's ideas on path independence for certain complex integrals.

4

Why Be So Critical? Nineteenth-Century Mathematics and the Origins of Analysis

Janet Heine Barnett, Colorado State University Pueblo

One striking feature of nineteenth-century mathematics, as contrasted with that of previous eras, is the higher degree of rigor and precision demanded by its practitioners. This tendency was especially noticeable in *analysis*, a field of mathematics that essentially began with the development of calculus by Gottfried Leibniz (1646–1716) and Isaac Newton (1642–1727) in the mid-seventeenth century. Unlike the calculus studied in an undergraduate course today, however, the calculus of Newton, Leibniz, and their immediate followers focused entirely on the study of geometric *curves*, using algebra (or "analysis") as an aid in their work. This situation changed dramatically in the eighteenth century when the focus of calculus shifted instead to the study of *functions*, a change due largely to the influence of the Swiss mathematician and physicist Leonhard Euler (1707–1783). In the hands of Euler and his contemporaries, functions became a powerful problem solving and modelling tool in physics, astronomy, and related mathematical fields such as differential equations and the calculus of variations. Why then, after nearly 200 years of success in the development and application of calculus techniques, did nineteenth-century mathematicians feel the need to bring a more critical perspective to the study of calculus? This project explores this question through selected excerpts from the writings of the nineteenth-century mathematicians who led the initiative to raise the level of rigor in the field of analysis.

4.1 The Problem with Analysis: Bolzano, Cauchy, and Dedekind

To begin to get a feel for what mathematicians felt was wrong with the state of analysis at the start of the nineteenth century, we will read excerpts from three well-known analysts of the time: Bernard Bolzano (1781–1848), Augustin-Louis Cauchy (1789–1857) and Richard Dedekind (1831–1916). In these excerpts, these mathematicians expressed their concerns about the relation of calculus (analysis) to geometry, and also about the state of calculus (analysis) in general. As you read what they each had to say, consider how their concerns seem to be the same or different. The project tasks that follow these excerpts will then ask you about these comparisons, and also direct your attention towards certain specific aspects of the excerpts.

Bernard Bolzano, 1817[1]

In the study of equations there are two propositions for which it could be said until recently that a fully correct proof is unknown. One is the proposition: *Between any two values of the unknown quantity that give opposite [sign] results, there must always lie at least one real root of the equation.* The other states: *Every algebraic rational entire function*[2] *of a variable can be decomposed into real factors of the first or second degree.* — After unsuccessful attempts by *d'Alembert, Euler, de Foncenex, La Grange, La Place, Klügel* among others, *Mr. Gauss* finally provided us last year with a couple of proofs of the latter theorem that should no longer leave anything to be desired. Indeed as early as the year 1799 this excellent scholar provided us a proof of this theorem, though it still had a mistake as he admitted himself, in that he based the purely analytic truth on a geometric consideration. His two newest proofs are indeed completely free of this mistake, as the trigonometric functions that appear in the last one can and should be understood in a purely analytic sense.

The other proposition that we mentioned above does not belong to those which, until now, have occupied the reflection of scholars in an excellent way.

[1]The translation of this excerpt from *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reele Wurzel der Gleichung liege* (*A purely analytic proof of the theorem, that between any two values that give opposite* [*sign*] *results, there lies at least one real root of the equation*) was prepared by Michael P. Saclolo, St. Edward's University, 2023.

[2]Today, we would simply say "polynomial function" in place of "algebraic rational entire function."

In the meantime we find that mathematicians of high esteem are tackling this theorem and have already attempted several ways of proving it. To anyone who wants to be convinced of this, compare the various presentations of this theorem that, for instance, *Kästner, Clairaut, Lacroix, Metternich, Klügel, La Grange, Rösling*, and many others have given.

A thorough examination of these proof methods very readily shows that none of them could be considered sufficient. The most common proof method relies upon a truth borrowed from geometry: Namely, *that each continuous line of a simple curve, whose ordinates are first positive then negative (or vice versa), must necessarily cut the abscissa axis at a point somewhere between those ordinates*. There are no objections whatsoever to both the correctness of and the evidence for this geometric statement. But it is also just as obvious that it is an intolerable violation of good method to want to derive the truths of pure (or general) mathematics (i.e., Arithmetic,[3] Algebra, or Analysis) from considerations that belong to only an applied (or special) part of it, namely to Geometry. ...

Augustin-Louis Cauchy, 1821[4]

As for the methods [in this text], I have sought to give them all the rigor that is demanded in geometry, in such a way as never to refer to reasons drawn from the generality of algebra. ... ... One should also note that [reasons drawn from the generality of algebra] tend to cause an indefinite validity to be attributed to the algebraic formulae, even though, in reality, the majority of these formulae hold only under certain conditions, and for certain values of the variables which they contain. By determining these conditions and values, and by fixing precisely the meaning of the notations of which I make use, I remove any uncertainty; ...

Augustin-Louis Cauchy, 1823[5]

My principal aim has been to reconcile rigor, which I took as a law in my *Cours d'analyse*, with the simplicity that results from the direct consideration of infinitesimals. For this reason, I believed I should reject the expansion of functions by infinite series whenever the series obtained was divergent; and I found myself forced to defer Taylor's formula until the integral calculus, [since] this formula can not be accepted as general except when the series

[3]As was not uncommon in the nineteenth century, Bolzano's use of the word "Arithmetic" here referred to the mathematical discipline that is today called "number theory."

[4]The translation of this excerpt from *Cours d'analyse* (*Course on analysis*) was prepared by the project author.

[5]The translation of this excerpt from *Résumé des leçons sur le calcul infinitésimal* (*Summary of lessons on the infinitesimal calculus*) was prepared by the project author.

it represents is reduced to a finite number of terms, and completed with [a remainder given by] a definite integral. I am aware that [Lagrange] used the formula in question as the basis of his theory of derivative functions. However, despite the respect commanded by such a high authority, most geometers[6] now recognize the uncertainty of results to which one can be led by the use of divergent series; and we add further that, in some cases, Taylor's theorem seems to furnish the expansion of a function by a convergent series, even though the sum of that series is essentially different from the given function.

Richard Dedekind, 1872[7]

My attention was first directed toward the considerations which form the subject of this pamphlet in the autumn of 1858. As professor in the Polytechnic School in Zürich I found myself for the first time obliged to lecture upon the elements of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic.[8] In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. Even now such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question until I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis.

Task 1 In what way do the concerns of these three mathematicians about the relation of calculus (analysis) to geometry, and about the state of calculus (analysis) in general, seem to be the same or different?

---

[6] The meaning of the word "geometer" also changed over time; in Cauchy's time, this word referred to any mathematician (and not just someone who worked in geometry).

[7] The translation of this excerpt from *Stetigkeit und irrationale Zahlen* (*Continuity of Irrational Numbers*) is taken from [Dedekind, 1901].

[8] Unlike Bolzano's use of the word "arithmetic" to mean "number theory," Dedekind's use of the expression "scientific foundation for arithmetic" was related to the set of real numbers and its underlying structure.

Task 2 This task looks at some of the mathematical results mentioned by Bolzano, Cauchy, and Dedekind.

(a) Note that Bolzano discussed two specific theorems and Dedekind discussed one specific theorem. Identify each of these three theorems.
(b) Note that Cauchy made reference to the Taylor formula and related results. Look back to see what he had to say, and briefly describe his concerns.
(c) Which of the results in parts (a) and (b) are familiar to you? For each that is, try to state it in "modern" terms, or give its "modern name."
(d) Which of the results in parts (a) and (b), if any, do you believe to be true and why (or why not)?

4.2 Niels Abel: "Hold your laughter, friends!"

In this section, we will examine an excerpt from a letter written by young Norwegian mathematician Niels Abel (1802–1829) to his high school teacher, Bernt Michael Holmboe, on January 26, 1826. Abel is often remembered for his celebrated impossibility proof in the theory of equations in which he proved that a "quintic formula" for the general fifth degree polynomial equation (akin to the quadratic formula for second degree polynomial equations) does not exist — a proof that marked an important step in the mathematical quest for algebraic solutions to polynomial equations which began with the development of Babylonian procedures for solving quadratic equations in 1700 BCE. Abel is equally well known for his work in analysis, and especially the theory of elliptic functions. In his letter to Holmboe, written during a study-abroad trip to Paris and Berlin, Abel described some of his concerns about the state of analysis in general, and about the use of infinite series in particular. The **letter itself (in English translation[9]) appears after Tasks 3–6**; read through these tasks first in order to have them in mind while you read Abel's letter; then complete your responses to Tasks 3–6 below after you've finished reading the letter.

Task 3 Find at least two references in Abel's letter to infinite series as an important concept or issue in mathematics. To what degree do the concerns that Cauchy expressed about series agree with Abel's view of series?

[9]The translations of excerpts from Abel's letter in this project were taken from [Bottazzini, 1986, pp. 87–89], with minor changes made by the project author based on the original text of the letter (pp. 15–18 of Norwegian section) and its French translation (pp. 15–19 of French section) in [Holst et al., 1902]. The English translations in [Bekken, 2003] and French translations in [Abel, 1839, pp. 266–268] were also consulted.

Task 4 What was it that Abel thought was "extremely surprising" about the state of mathematics at the time? Be specific here! Do you agree with his reaction to that state of affairs? Explain.

Task 5 Towards the end of the excerpt, Abel remarked that a series of the following form can converge for "x smaller than 1," but diverge for $x = 1$:

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

(a) Provide an example in which this occurs, specifying both the series (by giving values for the coefficients $a_0, a_1, \ldots$) and the function $\phi(x)$ to which that series converges for "x smaller than 1." (This doesn't really take much work, so don't make it harder than it is!)

(b) Notice that Abel went on to speculate that an even worse situation might occur. Namely, he proposed the possibility that a series $\phi(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ might be convergent for "x smaller than 1" *and* convergent for $x = 1$, but in such a way that $\lim_{x \to 1} \phi(x)$ is not equal to $\phi(1)$. What mathematical concept is involved here? That is, if such a function ϕ did in fact exist, what function property would ϕ be lacking?

Task 6 Consider the following series discussed by Abel at the end of the excerpt:

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \textit{etc.}$$

(a) Describe how this series is different from a power series.

(b) Complete Abel's argument about the numerical aspects of this series by determining what is absurd about this formula for $x = \pi$.

(c) Next complete Abel's comments about the derivative of this series by differentiating the formula term by term in order to show what can go wrong when one "performs every kind of operation on infinite series, as if they were finite." Be sure to say what is wrong with the derivative results!

Heinrik Abel, 1826, Letter to Holmboe

Another problem that has greatly concerned me is the summation of the series

$$\cos mx + m\cos(m-2)x + \frac{m(m-1)}{2}\cos(m-4)x + \cdots.$$

If m is a positive integer, the sum of this series, as you know, is $(2\cos x)^m$, but if m is not an integer, this is no longer the case unless x is smaller than $\pi/2$.

There is no other problem that has occupied mathematicians as much as this in the recent past. Poisson, Poinsot, Plana, Crelle, and many others have sought to resolve it, and Poinsot is the first who has found an exact sum, but his reasoning is completely false, and no one as yet has been able to find out why. Happily, I have succeeded with complete rigor. A memoir about this will appear in the *Journal*, and I will soon send it to France to appear in Gergonne's *Annales de Mathematiques*.

> *[There follows a discussion, omitted here, of some results that Abel had found concerning the above series.]*

Divergent series are on the whole devilish and it is a shame to base the slightest demonstration on them. You can get whatever you want when you use them, and they are what has produced so many failures and paradoxes. Can one think of anything more horrible than to say that

$$0 = 1 - 2^n + 3^n - 4^n + etc.$$

where n is a positive integer? *Risum teneatis amici!*[10] My eyes have at last opened up in a striking way because, except for cases of the most extreme simplicity, for example geometric series, there is hardly anywhere in the whole of mathematics a single infinite series whose sum is determined in a rigorous manner. In other words, that which is the most important in mathematics is without foundation. Most things are correct, this is true; and this is extremely surprising. I strive to find the reason for this. An exceedingly interesting subject.

I do not think you can show me many propositions where infinite series appear, where I cannot make fundamental objections against their demonstration. Do it, and I will answer you. Even the binomial formula is not yet rigorously demonstrated.

> *[There follows a discussion, omitted here, about the Binomial Series, about which Abel had derived certain results.]*

In order to show by a general example (*sit venia verbo*)[11] how poorly one can reason and how it is necessary to be prudent, I will choose the following example:

Let

$$a_0 + a_1 + a_2 + a_3 + a_4 + \text{etc.}$$

[10] Latin for "Hold your laughter, friends!"

[11] Latin for "pardon the expression."

be any infinite series. You know that a very common manner of finding the sum is to take the sum of

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + etc.$$

and then to let $x = 1$ in the result. This is probably correct, but it seems to me that we cannot accept it without demonstration, because if we prove that

$$\phi(x) = a_0 + a_1x + a_2x^2 + \cdots$$

for all values of x smaller than 1, it does not follow that one can say the same for $x = 1$. It is very possible that the series $a_0 + a_1x + a_2x^2 + \cdots$ approaches a completely different value than $a_0 + a_1 + a_2 + \cdots$ as x approaches 1. This is clear in the general case where the series $a_0 + a_1 + a_2 + \cdots$ is divergent, because then it does not have any sum. I have demonstrated that this is correct when the series is convergent. The following example shows how one can err. One can rigorously demonstrate that

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - etc.$$

for all values of x smaller than π. It seems that consequently the same formula must be true for $x = \pi$; but this will give [an absurdity] ...

... ...

One performs every kind of operation on infinite series, as if they were finite, but is it permissible? I do not think so. Where has it been demonstrated that one can obtain the derivative of an infinite series by taking the derivative of each term? It is easy to cite examples where this is not right, for example:

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - etc.$$

4.3 Conclusion

The concerns expressed by Abel, Bolzano, Cauchy, and Dedekind in the excerpts we have read in this project were emblematic of the state of analysis at the turn of the nineteenth century. Ultimately, mathematicians of that century responded to this set of concerns by moving to the requirement of *formal proof* as a way to certify knowledge via the *rigorous use of inequalities* intended to capture the notion of two real numbers “being close” that underlies the limit concept. Other factors that influenced this direction included new teaching and research situations, such as the École Polytechnique in Paris, that required mathematicians to think carefully about their ideas in order to explain them to others. Today, this nineteenth-century response remains at the core of the study and practice of real analysis. The final task in this project takes one last look at the motivations of those who led the way in formulating this response, as they expressed it in their own words.

Task 7 Look back at the excerpts from the works of Abel, Bolzano, Cauchy, and Dedekind that we have read in this project. What questions or comments would you address to these mathematicians about aspects of their concerns that are not addressed in the earlier tasks? (Write at least one question and at least one comment, please!)

Bibliography

Abel, N. H. (1839). *Oeuvres complètes de N. H. Abel, mathématicien, avec des notes et développements* (*Collected works of N. H. Abel, mathematician, with notes and annotations*), volume 2. C. Gröndahl, Christiania. Edited by B. Holmboe.

Bekken, O. (2003). The Lack of Rigour in Analysis: From Abel's Letters and Notebooks. In Bekken, O. and Mosvold, R., editors, *Study the Masters: The Abel-Fauvel Conference*, pages 9–21. NCM Göteborgs Universitet, Göteburg.

Bolzano, B. (1817). *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege* (*A purely analytic proof of the theorem, that between any two values that give opposite* [*sign*] *results, there lies at least one real root of the equation*). Gottlieb Haase, Prague.

Bottazzini, U. (1986). *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass*. Springer-Verlag, New York.

Cauchy, A.-L. (1821). *Cours d'analyse de l'École Royale Polytechnique* (*Course on analysis of the Royal Polytechnic School*). De Bure, Paris.

Cauchy, A.-L. (1823). *Résumé des leçons données à L'École Royale Polytechnique sur le calcul infinitésimal* (*Summary of lessons given on the infinitesimal calculus at the Royal Polytechnic School*). De Bure, Paris.

Dedekind, R. (1872). *Stetigkeit und irrationale Zahlen* (*Continuity and irrational numbers*). F. Vieweg und Sohn, Braunschweig.

Dedekind, R. (1901). *Essays on the Theory of Numbers*. Open Court, Chicago. Includes English translation by W. W. Beman of [Dedekind, 1872].

Holst, E., Stømer, C., and Sylow, L., editors (1902). *Niels Henrik Abel: Memorial publié a l'occasion du centenaire de sa naissance* (*Memorial published on the occassion of the centary of his birth*). Jacob Dybwad, Kristiana. Includes full text of Abel's correspondence in the original Norwegian and in French translation.

Notes to Instructors

PSP Content: Topics and Goals. This PSP is designed for use in an introductory course in real analysis. It has also been used in history of mathematics courses and capstone seminars for mathematics majors. Its goal is to provide context for the use of rigorous proofs and precise ϵ-inequalities that developed out of concerns about the state of analysis that first arose in the nineteenth century, but which remain defining characteristics of today's analysis. Both these tools of the current trade (i.e., rigorous proof, precise inequalities) offer challenges to students of introductory real analysis, who have typically encountered calculus only as a procedural and applied discipline up to this point in their mathematical studies. By offering a glimpse into the problems that motivated nineteenth-century mathematicians to shift towards a more formal and abstract study of the concepts underlying these procedures and applications, the readings in this PSP provide students with a context for making a similar shift in their own understanding of these concepts. Completing this PSP early in the course can also provide students and instructors with a basis for reflection on and discussion of current standards of proof and rigor throughout the course.

Student Prerequisites. The project assumes that students are familiar with fundamental concepts from a first-year calculus course, including basic results about limits and power series. However, no prior study of analysis or experience with formal proof writing is needed.

PSP Design and Task Commentary. This project consists of two main parts. In Section 1, brief excerpts from works by Bolzano, Cauchy, and Dedekind paint a general picture of the issues that motivated nineteenth-century mathematicians to attempt to infuse greater rigor into the study of analysis. Section 2 then examines a letter written by Abel in which he discussed concerns about infinite series in particular. Tasks 5 and 6 in the Abel section are the most technical parts of the project, but are still reasonably straightforward to complete. (Setting all coefficients equal to 1 in the series in Task 5(a) yields, for instance, a geometric series with ratio x.) Nevertheless, these two tasks can seem baffling to students who have not studied infinite series recently. Reassuring them that they should not make these tasks overly complicated can be helpful, as can some well-timed Calculus 2 reminders.

Note that none of the excerpts or tasks in this project describe how the study of analysis changed as a result of the concerns expressed by Abel, Bolzano, Cauchy, and Dedekind. Rather, the quotes from these mathematicians used in this project simply lay out the worries of the day. This is intentional, in that those changes (e.g., use of ϵ–δ inequalities, the arithmetization of analysis, increased rigor and precision in definitions and proofs) are precisely what students will encounter (and wrestle with!) throughout their introductory real analysis course. The "Summary Discussion Notes" in a later section of these Notes provide some additional details that instructors may find useful in helping to make the connection between the issues raised by Abel, Bolzano, Cauchy, and Dedekind in the excerpts in this PSP, and the ways in which they and others responded to these issues helped to shape analysis in the nineteenth century.

Sample Implementation Schedule (based on a 50-minute class period). The following sample schedule, based on the implementation strategy described in Chapter 1 of this collection, offers several options to help instructors tailor that mode of implementation to their course goals and available class time. Depending on the exact combination of individual/small-group/whole-class work, this method of implementation requires 1.5–2 class days.

- **Preparation for Day 1.** Read project introduction and all of Section 1; prepare answers to Tasks 1–2 for class discussion. Also read introduction to Section 2 and full Abel excerpt in that section; prepare answers to Tasks 3–4 for class discussion.
- **Day 1.**
  - *Optional*: Mini-lecture by instructor (∼10 minutes) to provide overview of pre-nineteenth-century calculus themes (based on table in first bullet of the "Summary Discussion Notes" below); this could instead be presented as part of the closing discussion on Day 2.
  - Small-group discussion of Tasks 1–2 (∼20 minutes), followed by whole-class summarizing discussion of Section 1, segueing into Section 2 by soliciting students' general comments and reactions to Abel's letter (∼10 minutes).
  - Whole-class discussion of Tasks 3–4 (∼10 minutes); alternatively, students could discuss these tasks in small groups (with more time allotted for this).
  - Time permitting, begin individual or small-group work on Task 5.
- **Preparation for Day 2.** Prepare answers to Tasks 5–6 for class discussion.
- **Day 2.**
  - Small-group discussion of Tasks 5–6 (15–20 minutes).
  - Whole-class discussion (15–30 minutes) of Section 2 (including answers to Tasks 5–6 as desired) and the PSP in general (including comments on the nineteenth-century response to the set of concerns raised in the PSP, per the final bullet of the "Summary Discussion Notes" below).
- **Homework.** A complete formal write-up of Tasks 2, 5, 6, and 7, to be due at a later date (e.g., one week after completion of the in-class work).

Alternative Classroom Implementation Strategy. The author typically uses this PSP as the basis of a whole-class discussion held at the beginning of the second week of her introductory real analysis course. On the first day of class, students are provided with a copy of the project that leaves blank space below each task where they can write their final responses and assigned to read the entire PSP and respond (in writing) to all tasks therein prior to class discussion. This allows students about a week to complete the project, which provides sufficient time for the careful advance reading that is needed for a high-quality in-class discussion. As they work on the project during that week, students are encouraged to discuss the readings and PSP tasks with each other or with the instructor (outside of class time) before the assigned due date (with the sole provision that their final written responses must be their own). While there is no prohibition against using additional resources to complete the PSP (e.g., a calculus text), it is important to assure students that there is no need to do any historical research in order to complete it.

On the assignment due date, a whole-class discussion of the reading is conducted by the instructor, with student responses to various PSP tasks elicited during that discussion. An instructor-prepared handout containing solutions to select tasks (especially Task 2) can be helpful during this discussion. The completed written work is typically collected at the close of that class period; however, the discussion could also be conducted after the instructor has collected and read students' written PSP work. The author does evaluate students' individual written work for a grade. That evaluation and the associated grade are based primarily on completeness, but also takes into account both presentation (e.g., use of complete sentences) and accuracy (particularly with regard to the mathematical details in Tasks 2, 5, 6).

A set of "Summary Discussion Notes" is offered below as a possible structure for a whole-class discussion of the PSP. Although some type of summarizing discussion is highly recommended, that discussion need not adhere to the notes provided here.

Summary Discussion Notes

- Overview of pre-nineteenth-century calculus themes

Century	**Content Focus** *What objects should we study?*	**Primary justification of "correctness"** *How do we know results are "true"?*
17th	Calculus of CURVES (using algebra as a tool)	Methods produced results that matched previously known results (obtained from geometry)
18th	Calculus of FUNCTIONS (with physics as primary motivation)	Methods produced correct predictions (in physics)

- Overview of the situation at the end of eighteenth / start of nineteenth century
  With suggested prompts for whole-class discussion given in italics.

  I. Increasing mistrust of geometric intuition as evidence for analytic statements, general frustration with use of non-analytic "proofs" to verify analytic "truths"
     Underlaying concern: Is it valid to borrow "truths" from one domain (e.g., geometry, physics) to justify truths in another (e.g., mathematics)?
     Ask for evidence of this in the assigned reading.

  II. Concern that existing algebraic proof methods lack adequate rigor
      Ask for evidence of this in assigned reading; two subthemes to elicit here.
      – Euclid had long been a model of rigor; desire to bring back something like an axiomatic approach as a foundation for certain knowledge
      – Algebra allows too much generality (e.g., unrestricted); this makes it too easy to assume that properties (e.g., continuity, rationality) that hold at all "lower" values will also hold in the limit
        Elicit or mention Abel power series example here.

  III. Use of power series (in particular) lacks firm foundation
       Ask for evidence of this in assigned reading; two specific points to elicit:
       – Current view on $\sum_{n=1}^{\infty} x^n$ (converges if $|x| < 1$, diverges otherwise)
         Discuss Abel's use of the phrase "x less than 1,*" which he meant as equivalent to current view (although today we would write "*$|x| < 1$*").*
       – Abel's claim that it's possible for $\phi(x) = \sum_{n=1}^{\infty} a_n x^n$ to converge for $|x| \leq 1$ with $\lim_{x\to 1} \phi(x) \neq \phi(1)$
         Ask students for their answers to Tasks 4 and 5 here.

  IV. General concerns about foundations: If we don't base calculus on power series, what do we use instead?
      – Some possibilities (and early proponents of each):
        * Fluxions (Newton)
        * Infinitesimals (Leibniz)
        * **Limits** (d'Alembert) ← **The "winner"!**
      – Nineteenth-century mathematicians' chosen option of "limit" raised yet another new question: What is a limit really??

- Ultimate nineteenth-century response to this set of concerns:

  Require $\underbrace{\textit{formal proofs}}_{\text{as way to certify knowledge}}$ via $\underbrace{\textit{rigorous}\text{ use of }\textit{inequalities}}_{\text{as way to talk about "being close"}}$

  – This response, which forms basis of work done throughout an introductory real analysis course, is often described as "the arithmetization of analysis"

- ***An optional historical aside related to item III in discussion list:*** The use of series and power series itself was not new in the nineteenth century!
  - Power series were around well before the development of calculus, but were considered part of "precalculus" (at least through the eighteenth century) in the sense that understanding power series was considered a prerequisite to the study of calculus
  - Newton (and others) used power series extensively as infinite polynomials that are easy to integrate and differentiate
  - An infinite series result[12] from the eighteenth century: $1-1+1-1+\cdots = \frac{1}{2}$
    - ∗ A first "proof":

$$(1-1)+(1-1)+\cdots = 0$$
$$1-(1-1)+(1-1)+\cdots = 1$$
$$\text{Series value is the average: } \tfrac{0+1}{2} = \tfrac{1}{2}$$

    - ∗ A second "proof" (endorsed by Euler, among others):

$$\sum_{n=1}^{\infty}(-1)^n = \frac{1}{1-(-1)} = \frac{1}{2}$$

- ***An optional historical aside related to nineteenth-century French mathematicians:*** Commenting on his experience during a visit to Paris, Abel wrote the following to Holmboe on October 24, 1826.

> Legendre is an extremely amiable man, but unfortunately "as old as stones." Cauchy is mad and there is no way to get anywhere with him, although at present he is the [only] mathematician who knows how to treat mathematics. His works are excellent, but he writes in a very confused manner. In the beginning, I understood almost nothing that he wrote, now that's going better. ... Cauchy is extremely Catholic and bigoted. A very strange thing for a mathematician. ... Poisson is a small man with a nice little belly. He carries himself with dignity. Likewise Fourier. Lacroix is terribly bald and remarkably old. ... Otherwise I do not like the French as much as the German: the French are extremely reserved with foreigners. It is very difficult to make their close acquaintance. And I dare not count on doing so.[13]

[12]For more about this and other divergent series in the eighteenth century, see the June 2006 MAA On-line column *How Euler Did It* by Ed Sandifer, available at `http://eulerarchive.maa.org/hedi/HEDI-2006-06.pdf`.

[13]The translation of this excerpt was prepared by the project author based on original text of the letter (pp. 41–42 of Norwegian section) and its French translation (p. 45 of French section) in [Holst et al., 1902].

8

Bolzano on Continuity and the Intermediate Value Theorem

David K. Ruch, Metropolitan State University of Denver

The foundations of calculus were not yet on firm ground in the early 1800s. Mathematicians such as Joseph-Louis Lagrange (1736–1813) made efforts to put limits and derivatives on a firmer logical foundation, but were not entirely successful.

Bernard Bolzano (1781–1848) was one of the great success stories of the foundations of analysis. He was a theologian with interests in mathematics and a contemporary of Carl Friedrich Gauss (1777–1855) and Augustin-Louis Cauchy (1789–1857), but was not well known in mathematical circles. Despite his mathematical isolation in Prague, Bolzano was able to read works by Lagrange and others, and published work of his own.

This project investigates results from his important pamphlet *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege*[1] [Bolzano, 1817]. We will read excerpts from this paper related to two major theorems that Bolzano studied in it. The first of these is the main theorem in Bolzano's Section 12, where he stated and proved a property of bounded sets. The second major theorem, which appears in Section 15 of Bolzano's pamphlet, concerns continuous functions, and some version of this result is found in nearly every introductory calculus text. Naturally Bolzano's

[1]The title of Bolzano's pamphlet translates into English as *A purely analytic proof of the theorem, that between any two values that give opposite [sign] results, there lies at least one real root of the equation.*

concept of continuity is vital for understanding both of these theorems, so we will first study it.

8.1 Bolzano's Definition of Continuity

Bolzano was very interested in logic, and he was dissatisfied with many contemporary attempts to prove theorems using methods he found inappropriate. Here are some excerpts from the preface of his 1817 work in which he described his concerns.[2] As you read, remember that when Bolzano wrote his pamphlet, there were not yet precise and universally agreed upon definitions of limit or continuity.

> In the study of equations there are two propositions for which it could be said until recently that a fully correct proof is unknown. One is the proposition: *Between any two values of the unknown quantity that give opposite [sign] results, there must always lie at least one real root of the equation*. ... ...
>
> In the meantime, we find that mathematicians of high esteem are tackling this proposition and have already attempted several ways of proving it. To anyone who wants to be convinced of this, compare the various presentations of this proposition that, for instance, *Kästner, Clairaut, Lacroix, Metternich, Klügel, La Grange, Rösling*, and many others have given.
>
> A thorough examination of these proof methods very readily shows that none of them could be considered sufficient.
>
> I. The most common proof method relies upon a truth borrowed from geometry: Namely, *that each continuous line of a simple curve, whose ordinates are first positive then negative (or vice versa), must necessarily cut the x-axis*[3] *at a point somewhere between those ordinates*. There are no objections whatsoever to both the correctness of and the evidence for this geometric statement. But it is also just as obvious that it is an intolerable violation of good method to want to derive the truths of pure (or general) mathematics (i.e., Arithmetic,[4] Algebra, or Analysis) from considerations that belong to only an applied (or special) part of it, namely to Geometry.

[2]All translations of Bolzano excerpts in this project were prepared by Michael P. Saclolo, St. Edward's University, 2023, with minor changes made by the project author for readability. The translations [Russ, 1980] and [Russ, 2004, pp. 251–278] were also consulted by the project author.

[3]Bolzano actually used the German word "Abscissenlinie" (literally, "abscissae-line") here. Russ points out that "Abscissenlinie" suggests a geometric measuring line in contrast to the more abstract "x-axis" [Russ, 2004, p.254n].

[4]As was common in the nineteenth century, Bolzano's use of the word "Arithmetic" here referred to the mathematical discipline that is today called "number theory."

Task 1 Rewrite Bolzano's preface proposition ("between any two values ... at least one real root of the equation") in your own words with modern terminology. Sketch a diagram illustrating the proposition. What theorem from a first-year calculus course does this remind you of?

Task 2 Do you agree with Bolzano's philosophical criticism of using geometry to try to prove his preface proposition ("between any two values ... at least one real root of the equation")? Explain why or why not. Start by restating Bolzano's criticism in your own words.

Later in his preface, Bolzano asserted a "proper" definition of continuity and gave an interesting example as a footnote. As you read his definition in the next excerpt, think about whether you agree with it, and how you could rewrite it with modern language.

To explain it properly, we understand the expression that a function $f(x)$ varies according to the laws of continuity for all values x that lie within or beyond certain bounds,† to mean that if x is any such value, the difference $f(x+\omega)-f(x)$ can be made smaller than any given quantity whenever ω can be taken to be as small as desired.

† *Bolzano's footnote:* There are functions that are continuously variable for all values of their argument,[5] e.g., $\alpha x + \beta x$. But there are also others that vary according to the laws of continuity only within or beyond certain limiting values of their argument. For instance, $x + \sqrt{(1-x)(2-x)}$ varies continuously only for all values x that are $< +1$ or $> +2$, and not for those values that lie between $+1$ and $+2$.

Task 3 For a function $f : \mathbb{R} \to \mathbb{R}$, rephrase Bolzano's definition of continuity at x using modern ϵ–δ terminology and appropriate quantifiers.

Task 4 Use your definition in Task 3 to give a modern ϵ–δ proof of the continuity of the function $f(x) = 3x + 47$ at $x = 2$.

---

[5] *Translation note:* Bolzano used the German word "Wurzel" here, which is generally translated as "root." Given the context of the footnote, however, it is clear that he was referring to what we would today call the independent variable, or argument, of a function.

Task 5 Consider the function Bolzano discussed in his footnote.

(a) Sketch a graph of this function on the interval $[0, 3]$.
(b) Based on the preface proposition that Bolzano was discussing, why is this an interesting example?
(c) How could you adjust the function to make it better fit the issues surrounding the preface proposition?

Task 6 Adjust your continuity definition in Task 3 to include the notion of domain, so it applies to functions defined on an interval I within $\mathbb{R}$. Do you think Bolzano's footnote function should be continuous at $x = 1$ and at $x = 2$? Give an intuitive justification.

Task 7 Suppose a function h is continuous for all x in $[0, 4]$ and $h(3) = 6$. Show that there is a $\delta > 0$ for which $h(x) \geq 5$ for all $x \in (3 - \delta, 3 + \delta)$.

Task 8 Define $g(x) = 3-5x^2$ with domain $I = [4, 7]$. Show that g is continuous at an arbitrary $\alpha \in I$ using your continuity definition.

Bonus. For Task 8, change the domain of g to be $\mathbb{R}$. Show that g is continuous at an arbitrary $\alpha \in \mathbb{R}$. You may need to adjust your proof from Task 8.

Task 9 We define a function to be continuous on an interval if it is continuous at each point in the interval. Suppose that functions f and g are both continuous on an interval I. Prove that $f - 47g$ is also continuous on I, using your continuity definition.

For the next two tasks, the following properties of the sine and cosine functions for $a, b \in \mathbb{R}$ will be useful:

$$\sin a - \sin b = 2\sin\left((a-b)/2\right)\cos\left((a+b)/2\right), \quad |\sin a| \leq |a|$$

$$\cos a - \cos b = 2\sin\left((b-a)/2\right)\sin\left((a+b)/2\right), \quad |\sin a| \leq 1, \quad |\cos a| \leq 1$$

Task 10 Prove that $\sin x$ is continuous on $\mathbb{R}$.

Task 11 Prove that $\cos x$ is continuous on $\mathbb{R}$.

Task 12 Define $S(x) = x \sin(1/x)$ for $x \neq 0$. Find a value for $S(0)$ so that S will be continuous at $x = 0$. Prove your assertion.

In Section 3 of this project, we will return to Bolzano's proposition about equation roots that you examined in Task 1, and work through his proof of a related result. This material will involve continuous functions, but we will set continuity aside for now to study another important theorem from Bolzano's pamphlet.

8.2 Bolzano's Bounded Set Theorem

In this section we will leave continuity and study an important theorem from Bolzano about what we would today call bounded sets. The theory of sets had not been developed during Bolzano's era, so he did not use the same set-theoretic language we might expect in a modern discussion of his ideas. As you read the next excerpt from Bolzano's pamphlet, think about how you could translate his ideas into set terminology.

§11

Prelude. In the studies of applied mathematics the case happens on occasion that one learns about a [nonnegative][6] variable quantity x, all of whose values less than a certain u have a particular property M, without at the same time learning the property no longer applies to those values greater than u. In such cases there can exist perhaps some u' that is $> u$, for which all values x below it have property M, in much the same way that applies to u. Indeed, this property can perhaps apply to all x without exception. If on the other hand one learns that M does not apply to all x at all, then from the combination of these two statements one will now be justified to conclude that there is a certain value U that is the largest of which it is true that all smaller x have property M. The following theorem demonstrates this.

§12

Theorem. If property M does not apply to all values of a [nonnegative] variable quantity x, but rather to all such that are less than a certain u, then there is always a quantity U that is the largest among them for which it can be asserted that all smaller x have property M.

Let's look at some examples of this concept that Bolzano was discussing.

Task 13 Let M be the property "$x^2 < 3$" applied to the set $\{x \in \mathbb{R} : x \geq 0\}$.

(a) Find rational numbers u, u' such that the property M applies to all smaller nonnegative values for this example and $u < u'$. (The values of u, u' are not unique.) What is the value of "the quantity U that is the largest among them for which it can be asserted that all smaller x have property M" for this example?

[6]Bolzano intended to discuss only $x \geq 0$ in this note and his Section 12 theorem statement. The term "nonnegative" has been included in this project for clarity.

(b) Let S_M be the set of ω values for which all ω' values satisfying the inequality $0 \leq \omega' < \omega$ possess property M. Sketch S_M on a ω number line. Are the theorem hypotheses met for this property M?

(c) Does U possess property M?

Task 14 Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x$, and let $\alpha \in \mathbb{R}$ be arbitrary. Let M be the property "$f(\alpha + \omega) \leq f(\alpha) + 2$" applied to the set $\{\omega \in \mathbb{R} : \omega \geq 0\}$.

(a) Find rational numbers u, u' such that the property M applies to all smaller nonnegative values for this example and $u < u'$. Are these values unique? What is the value of the greatest such quantity U for this example?

(b) Let S_M be the set of ω values for which all ω' values satisfying the inequality $0 \leq \omega' < \omega$ possess property M. Sketch S_M on a ω number line. Are the theorem hypotheses met for this property M?

(c) Does U possess property M?

Task 15 Rewrite Bolzano's theorem from his Section 12 using modern terminology and set notation.

We will refer to the theorem you stated in Task 15 as **Bolzano's Bounded Set Theorem**. In Section 12 of his pamphlet, Bolzano went on to give a proof of his Bounded Set Theorem based on a Cauchy sequence-like convergence assumption for infinite series. While the proof is correct given that assumption, we will omit it for this project.[7] Instead, we next look at Section 15 of Bolzano's pamphlet, to see how he used both his definition of continuity and his Bounded Set Theorem to prove his main result on the solution of certain equations involving continuous functions.

8.3 An Application of Bolzano's Bounded Set Theorem

We are now ready to work through Bolzano's main result, given in the excerpt below. He broke his proof into three parts, and we will pause after each part to do tasks that will help unpack his proof and rephrase it with modern language.[8]

[7] You can explore the details of that proof in the related project "Investigations into Bolzano's Bounded Set Theorem" in Chapter 5 of this collection.

[8] Throughout his Section 15 theorem statement and proof below, Bolzano wrote fx where we would write $f(x)$, and similarly deleted argument parentheses for other functions and variables. The translator has inserted these parentheses to reduce distractions for the modern reader.

§15

Theorem. If two functions of x, $f(x)$ and $\varphi(x)$, vary according to the laws of continuity either for all values of x or for all those that lie between α and β, and if further, $f(\alpha) < \varphi(\alpha)$ and $f(\beta) > \varphi(\beta)$, then each time there is a certain value x lying between α and β for which $f(x) = \varphi(x)$.

Proof. I.1. First assume that both α and β are positive and that (because it is of no consequence) β is the larger of the two, and so let $\beta = \alpha + i$, where i indicates some positive value. As $f(\alpha) < \varphi(\alpha)$, so is $f(\alpha+\omega) < \varphi(\alpha+\omega)$, where ω indicates a positive value that can be as small as one likes. For since $f(x)$ and $\varphi(x)$ vary continuously for all x lying between α and β, and $\alpha+\omega$ lies between α and β as long as it can be taken that $\omega < i$, then $f(\alpha+\omega) - f(\alpha)$ and $\varphi(\alpha+\omega) - \varphi(\alpha)$ can become as small as one wants them to be, if ω is taken to be small enough. Therefore, if Ω and Ω' also denote quantities that can be as small as one wants, we can set $f(\alpha+\omega) - f(\alpha) = \Omega$ and $\varphi(\alpha+\omega) - \varphi(\alpha) = \Omega'$. Therefore,

$$\varphi(\alpha+\omega) - f(\alpha+\omega) = \varphi(\alpha) - f(\alpha) + \Omega' - \Omega.$$

But by assumption $\varphi(\alpha) - f(\alpha)$ equals some positive constant A. Thus,

$$\varphi(\alpha+\omega) - f(\alpha+\omega) = A + \Omega' - \Omega,$$

which remains positive when one lets Ω and Ω' be small enough, i.e., when ω is given a very small value and further for all smaller values. Thus we can claim that the two functions $f(\alpha+\omega)$ and $\varphi(\alpha+\omega)$ are in a relationship of a smaller value to a larger one relative to each other for all ω that are smaller than a particular one. Denote this property of the variable ω by M; then we can say that all ω that are smaller than a particular one have property M. Nevertheless, that this property M does not apply to all values of ω is clear, in particular not to the value $\omega = i$, because according to the assumption $f(\alpha+i) = f(\beta)$ is no longer $<$ but rather $> \varphi(\alpha+i) = \varphi(\beta)$. From this, according to Theorem §12,[9] there must be a value U that is the largest among them for which it can be claimed that all ω that are $<$ U possess property M.

Task 16 Sketch a diagram with graphs of f and φ that illustrates the theorem statement and label α, β, and A. For an arbitrary ω possessing property M, label Ω' and Ω. Also draw an ω number line and label key values i, U, and the set of values ω possessing property M.

[9] This is the theorem that you re-wrote in modern terminology in Task 15.

Task 17 Bolzano stated that ω, Ω, and Ω' can be made "as small as one wants." Explain the dependencies between these quantities. Use ϵ–δ terminology to clarify what is going on.

Task 18 Rewrite Bolzano's claim in the first two sentences of I.1 using modern terminology and call this Lemma 1.

Task 19 Convert Bolzano's argument in the first part of I.1 into a proof of Lemma 1 with your modern definition of continuity.[10]

Task 20 Rewrite with symbols Bolzano's definition of property M in I.1. Then rephrase his statement that "all ω that are smaller than a particular one have the property M" using set notation, and name this set S_M.

Task 21 Consider $f(x) = 4 + (x-2)(x-4)(x-6)$, $\varphi(x) = 4$ with $\alpha = 1$ and $\beta = 7$. Informally find the set S_M and the value of U for this example.

Task 22 We can summarize the results of Part I.1 of the Bolzano's proof by stating some facts about U. First, Bolzano showed that the hypotheses of his §12 theorem are met, so that the quantity U exists. What additional facts does the conclusion of his §12 theorem tell us about U? In particular, explain why U must be positive.

We now proceed to Bolzano's Part I.2 of his proof.

2. And this U must lie between 0 and i. First it cannot be $= i$, as this would mean that each $f(\alpha+\omega) < \varphi(\alpha+\omega)$ whenever $\omega < i$, however close it may come to the value i. But in the same way that we just proved that the result $f(\alpha+\omega) < \varphi(\alpha+\omega)$ follows from the assumption $f(\alpha) < \varphi(\alpha)$, as long as ω is taken to be small enough, so does it unfold that the result $f(\alpha+i-\omega) > \varphi(\alpha+i-\omega)$ follows from the assumption $f(\alpha+i) > \varphi(\alpha+i)$, as long as ω is taken to be small enough. So it is not true that the two functions $f(x)$ and $\varphi(x)$ are in a relationship of a smaller value to a larger one for all values of x that are $< \alpha + i$. Secondly, even less so can it be that $\mathsf{U} > i$, because otherwise i is also one of the values of ω that are $< \mathsf{U}$, and therefore it must be that $f(\alpha+i) < \varphi(\alpha+i)$ as well, which the assumption of the theorem downright contradicts. Thus, U surely lies between 0 and i as it is positive, and consequently, $\alpha + \mathsf{U}$ lies between α and β.

[10] Bolzano's argument for why this lemma is true ends with the sentence "Thus we can claim that ...."

Task 23 Rewrite Bolzano's claim that "the result $f(\alpha + i - \omega) > \varphi(\alpha + i - \omega)$ follows from the assumption $f(\alpha + i) > \varphi(\alpha + i)$, as long as ω is taken to be small enough" using modern terminology and call this Lemma 2. Also provide a proof using your modern definition of continuity.

Task 24 We can summarize this part of Bolzano's plan as the claims "$0 < \mathsf{U} < i$" (so that $\alpha < \alpha + \mathsf{U} < \beta$), followed by his proof of the claim that $\mathsf{U} < i$. Rewrite his proof using your own words and modern terms, referencing the set S_M and Bolzano's Bounded Set Theorem from Task 15.

We now read Part I.3 of Bolzano's proof.

3. Now begs the question of how do the functions $f(x)$ and $\varphi(x)$ compare to each other for the value $x = \alpha + U$. First of all, it cannot be that $f(\alpha+U) < \varphi(\alpha + U)$, since this would also give $f(\alpha + U + \omega) < \varphi(\alpha + U + \omega)$, if one takes ω to be small enough; and consequently $\alpha + U$ would not be the largest value for which it can be claimed that all values x below it have property M. Secondly, neither can it be that $f(\alpha+U) > \varphi(\alpha+U)$, because this would also give $f(\alpha + U - \omega) > \varphi(\alpha + U - \omega)$, as long as ω is taken to be small enough; and so it would go against the supposition that property M is not true for all x below $\alpha + U$. Thus, there is nothing else left but $f(\alpha + U) = \varphi(\alpha + U)$; and consequently it has been proven that there is a value x between α and β, namely $\alpha + U$, for which $f(x) = \varphi(x)$.

Task 25 Adjust your Lemmas 1 and 2 to give modern justifications of the first two claims in this excerpt.

Task 26 What property of the real numbers justifies the statement "there is nothing else left but $f(\alpha + U) = \varphi(\alpha + U)$"?

Task 27 At the beginning of part I.1 of the proof, Bolzano made the assumption "that α and β are both positive." Can you find a place in the proof where he used this assumption?

Bolzano continued in his Section 15 to address the cases α and β are both negative, one is zero, and they are of opposite sign. We will omit these proofs, as they are not terribly enlightening.

Now that we have completed our journey with Bolzano through his proof, let's return to his preface proposition that you examined in Task 1.

Task 28 Consider Bolzano's proposition from his preface that: "between any two values of the unknown quantity that give opposite [sign] results, there must always lie at least one real root of the equation."

(a) In Task 1, you restated this proposition using modern terminology. Look back at your answer to that task with Bolzano's §15 theorem in mind. Do the hypotheses of his theorem apply to your restatement? If not, modify your restatement of the proposition as needed so that they do.

(b) Use Bolzano's theorem to prove your restated version of Bolzano's preface proposition.

Task 29 Use Bolzano's theorem to prove the following result from a standard introductory calculus text, which is typically referred to as the "Intermediate Value Theorem."

> Consider an interval $I = [a, b]$ in the real numbers $\mathbb{R}$ and a continuous function $f : I \to \mathbb{R}$. If $f(a) < L < f(b)$ then there is a $c \in (a, b)$ such that $f(c) = L$.

8.4 Conclusion

As the final task of the last section illustrates, Bolzano's §15 theorem is a generalized version of the Intermediate Value Theorem that is featured in a standard first-year calculus textbook. Today, various versions of the Intermediate Value Theorem serve as powerful tools in numerical analysis and other areas of mathematics. The Bounded Set Theorem that Bolzano used to prove his version of the Intermediate Value Theorem was a highly original idea, and is closely linked to what are nowadays called the least upper bound and greatest lower bound existence properties of the real numbers. If you have studied these completeness properties, you might enjoy the following task.

Task 30 Let S be a nonempty subset of $\mathbb{R}$ such that $s > 47$ for all $s \in S$. Use Bolzano's Bounded Set Theorem to prove that S has a greatest lower bound. *Caution*: S is a subset of the interval $(47, \infty)$, but don't assume $S = (47, \infty)$.

It is interesting to note that Augustin-Louis Cauchy, a key player in building the theory of calculus, also proved a version of Bolzano's preface proposition, probably a few years later than Bolzano.[11] Although Cauchy used a very different method of

[11] Bolzano's proof was published in 1817, but wasn't widely known when it first appeared. Cauchy's proof appeared in a note on the numerical solution of equations that was included at the end of his textbook *Cours d'analyse* (*Course on analysis*), published in 1821.

proof,[12] his approach also depended on the completeness property of the real numbers. Given the strong influence of the approach to reshaping calculus which was adopted by Bolzano, Cauchy, and their contemporaries in the nineteenth century, it is no coincidence then that some version of the completeness property continues to lie at the foundation of the proofs of the Intermediate Value Theorem that are found in today's real analysis textbooks.

Bibliography

Bolzano, B. (1817). *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege* (*A purely analytic proof of the theorem, that between any two values that give opposite* [*sign*] *results, there lies at least one real root of the equation*). Gottlieb Haase, Prague.

Grabiner, J. V. (2005). *The Origins of Cauchy's Rigorous Calculus*. Dover, Mineola NY. Originally published by MIT Press, Cambridge MA, 1981.

Jahnke, H. N., editor (2003). *History of Analysis*. American Mathematical Society, Providence RI.

Lützen, J. (2003). The Foundation of Analysis in the 19th Century. In Jahnke, H. N., editor, *A History of Analysis*, pages 155–196. American Mathematical Society, Providence RI.

Russ, S. B. (1980). A Translation of Bolzano's Paper on the Intermediate Value Theorem. *Historia Mathematica*, 5:156–185.

Russ, S. B. (2004). *The Mathematical Works of Bernard Bolzano*. Oxford University Press, New York.

Notes to Instructors

PSP Content: Topics and Goals. This PSP is designed to introduce continuity and the Intermediate Value Theorem (IVT) in an introductory course in real analysis. Its specific content goals are to:

(1) Develop a modern continuity definition with quantifiers based on Bolzano's definition.

[12] To learn how Cauchy's approach to the Intermediate Value Theorem compares to that of Bolzano (without actually reading Cauchy himself!), see the overviews of Cauchy's proof found in [Lützen, 2003, pp. 175–176] and [Grabiner, 2005, pp. 69–74].

(2) Develop facility with the modern continuity definition by applying it to various functions.

(3) Analyze Bolzano's Bounded Set Property and rewrite it in modern terminology.

(4) Modernize Bolzano's proof of his Intermediate Value Theorem.

(5) Apply Bolzano's Intermediate Value Theorem to obtain two other formulations of the Intermediate Value Theorem.

Student Prerequisites. This project is written with the assumption that students have done a rigorous study of quantifiers and limits of real-valued functions. It also assumes that students have seen the least upper bound property for bounded sets of real numbers, but the project could be used to introduce this concept, with instructor supplements.

PSP Design and Task Commentary. Section 1 of this project contains excerpts from Bolzano introducing his version of the IVT and his definition of continuity. Most of the tasks in this section focus on developing the definition of continuity for a function defined on an interval, the appropriate setting for a discussion of the IVT. Getting a correct definition of continuity in Task 3 is crucial before going much further; a class discussion of Task 3 and the next problem can be helpful after students work on them for awhile or in groups.

Bolzano's choice of $x + \sqrt{(1-x)(2-x)}$ in his footnote is mildly perplexing, as it does not change signs in its domain $[1, 2]$. Indeed, the first set of students using the project were rather critical of Bolzano's footnote function. They inspired Task 5. In Bolzano's defense, he discussed the function $x + \sqrt{(x-2)(x+1)}$, which lacks a root between -1 and 2, later in his very lengthy preface.

Task 7 foreshadows a crucial result in the next section, namely writing a modern ϵ–δ proof of Bolzano's assertion: "As $f(\alpha) < \varphi(\alpha)$, so is $f(\alpha + \omega) < \varphi(\alpha + \omega)$, where ω indicates a positive value that can be as small as one likes." This is difficult for some students, and they may need a hint/reminder that THEY get to choose ϵ if f is known to be continuous.

The final group of exercises in Section 1, Tasks 8–12, are standard problems to sharpen skills in working with continuity. Instructors may sample the set for classroom examples or homework problems. However, they are not needed for the flow of Bolzano's discussion.

Section 2 is written with the assumption that students have seen the least upper bound property for bounded sets of real numbers. The theorem in Bolzano's Section 12 basically asserts this property for a special class of bounded sets, but in a form students (and the PSP author!) have not seen before. It is a bit tricky to unravel and put into modern set notation. The first two tasks should help ease this process for students. Task 14 should help them with Bolzano's Section 15, where the theorem is applied.

Section 3 focuses on Bolzano's proof of his IVT. Part I.1 of this proof is crucial and contains some subtleties; it is worth taking time to make sure students understand it. In addition the symbols Ω', Ω may cause confusion for some students. It is also not completely clear from Bolzano's writing whether the set of all ω with the property M in this part of his proof should be the set $\{\omega : f(\alpha + \omega') < \varphi(\alpha + \omega') \text{ for all } \omega' \in [0, \omega)\}$ or the set $\{\omega : f(\alpha + \omega) < \varphi(\alpha + \omega)\}$. Task 21 illustrates the difference. An opportunity for discussion and careful reading![13]

Bolzano actually began the proof of the Section 15 theorem with the following:

> We must remember that in this theorem, the values of the functions $f(x)$ and $\varphi(x)$ are to be compared with each other merely according to their absolute values, i.e., without regard to sign, or as if they were not quantities of opposite signs at all. But it is dependent upon the signs of α and β.

Bolzano then split his proof into several cases I–IV, beginning with case $0 < \alpha < \beta$. Here is S. B. Russ's explanation [Russ, 1980, p. 183]:

> Bolzano always used inequality signs to apply only to the magnitude of quantities and not to their position on a number line. This was common practice at the time as there was still no standard symbol for modulus. Thus in Bolzano's usage $x > -1$ meant the range we should now describe as $x < -1$. For example, in Section 2 he stated that the general term of a geometric progression ae^r becomes "ever larger when $e > \pm 1$" and "ever smaller when $e < \pm 1$." And in Section 15.IV he described the range of values of x between α and β when α is negative and β positive as "all values of x which if negative are $< \alpha$, and if positive are $< \beta$."

Bolzano's opening comments and his proof cases II–IV have thus been excluded from the project because they seem likely to cause considerable confusion with little payoff to most students of analysis. Instructors could bring up these issues in a class discussion of Task 27.

The statement and proof of Lemma 2 in the task set following Part I.2 of the IVT proof in Section 15 of Bolzano's pamphlet may seem a bit repetitive. However, it should clarify things for some students and serve as proof writing practice, especially if Lemma 1 is done in class.

[13]Note that in either case, the value of U produced by Bolzano's §12 theorem is the least upper bound of the first of these two options; that is, the set $S_M = \{\omega : f(\alpha + \omega') < \varphi(\alpha + \omega') \text{ for all } \omega' \in [0, \omega)\}$.

Sample Implementation Schedule (based on a 50-minute class period). Full implementation of the project can be accomplished in 4 class days as follows.

For Class 1, students read through the introductory material and the beginning of Section 1, and do Tasks 1–3 before the first class. After discussing their results at the beginning of Class 1, students work on and discuss Tasks 4 and 7. Homework practice with the definition of continuity could be some subset of Tasks 5, 6, 8–12. However, none of these are essential for the following material.

As preparation for Class 2, students read the first Bolzano excerpt in Section 2 and do Task 13. After discussing their results at the beginning of Class 2, students work on and discuss Tasks 14 and 15. Students then begin Section 3 by reading the first part of Bolzano's IVT proof and doing Task 16, which is essentially drawing diagrams for the proof.

As preparation for Class 3, students do Tasks 17, 18. After discussing their results at the beginning of Class 3, students work on and discuss Tasks 19–22.

As preparation for Class 4, students read Bolzano's Part I.2 proof excerpt and do Task 23. After discussing their results at the beginning of Class 4, students work on and discuss Tasks 24–30. Some of these tasks can be given as homework, as time permits.

Recommendations for Further Reading. The translations [Russ, 1980] and [Russ, 2004] include interesting background on Bolzano as well as commentary on some of the subtleties of Bolzano's work. The articles in [Jahnke, 2003] give some perspective on other works in analysis during Bolzano's era.

21

Connecting Connectedness

Nicholas A. Scoville, Ursinus College

Connectedness has become a fundamental concept in modern topology. The concept seems clear enough — a space is connected if it is a "single piece." Yet the definition of connectedness we use today is not what was originally written down. As we will see, today's version of connectedness is a classic example of a definition that evolved over decades. The first definition of connectedness was given by Georg Cantor (1845–1918). Cantor is best known for his work in set theory. His work in set theory, however, began with questions concerning Fourier series in an 1872 paper [Cantor, 1872]. In his study of Fourier series, Cantor was interested in finding conditions for when a function has a unique Fourier expansion. This study compelled him to define for the first time some purely topological concepts, including the concepts of a point-set, a neighborhood, and a derived set. Cantor's early topological investigations were the precursors to a series of six papers published between 1879–1883 that were themselves part of his work on set theory. It is with his fifth paper in this series that we begin this project on connectedness. Given this history, one can trace not only the origins of modern set theory back to Cantor, but also the origins of modern point-set topology.

21.1 Cantor: A Continuum

We begin our investigation into connectedness with the paper "Ueber unendliche, lineare Punktmannichfaltigkeiten, [Teil] 5" ("On infinite, linear manifolds of sets, Part 5") [Cantor, 1883]. Cantor's reason for writing this paper, however, does not appear to have been a consideration of connectedness. Cantor wrote:[1]

The point of this examination is to establish a precise criterion — but one that is simultaneously as generally applicable as possible – for designating P as a continuum.

Let's put ourselves into Cantor's shoes. What comes to mind when you think of a continuum? A straight line? A closed disk? An open disk? The irrational numbers? Definitions in mathematics tend to be driven by our examples; that is, what do all the examples of objects that you think of as a continuum have in common? Furthermore, what do all the examples that you think of as violating a continuum have in common? The goal is to abstract away the particulars and find the common properties that a continuum satisfies. As the quote above indicates, this is what Cantor desired to do in this paper.

Task 1 Consider a straight line, a closed disk, an open disk, and the irrational numbers. Which of these do you consider a continuum? Why?

One concept that Cantor had at his disposal was that of the derived set.

For the sake of brevity I call a given finite or infinite number of ... points on the line a *point-set*. If a point-set is given in a finite interval, a second point-set is generally given along with it, and with the latter a third, etc., which are essential to understanding the first set. In order to define these point-sets, we must begin with the concept of a *limit point of a set*. I define a "**limit point of a point-set** P" to be a point of the line situated in such a way that each neighborhood of it contains *infinitely* many points of P, and it may happen that the point itself belongs to the set. By a "**neighborhood of a point**" I mean any interval that has the point *in its interior*. ... Every point of the line is now in a definite relation to a given set P, either being a limit point of P or not, and thereby along with the point-set P the set of limit points of P is *conceptually* given, a set which I wish to denote by P' and call the **first derived point-set** of P. Unless the point-set P' contains only a finite number of points, it also has a derived set P'', which I call the **second derived point-set** of P. By υ such transitions one obtains the concept of

[1] All translations of Cantor, Jordan, and Schoenflies excerpts in this project were prepared by the project author.

the υ**th derived set** $P^{(\upsilon)}$ of P. ... [If] P is such that the derivation process produces no change:

$$P = P^{(1)}$$

and therefore

$$P = P^{(\gamma)}$$

[then] such sets P I call **perfect** point-sets.

Task 2 Using Cantor's definition of limit point and derived set, determine which of the following sets are perfect in $\mathbb{R}$.

(a) $P = [0, 1]$
(b) $\{(a, b), a < b \in \mathbb{R}\}$
(c) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
(d) $\left\{\frac{1}{n} : n = 1, 2, 3, \dots\right\}$
(e) $\mathbb{Q}$
(f) $\mathbb{R}$
(g) $[\mathbb{Q} \cap (0, 1)] \cup [-4, 2)$

You may have noticed that, for the above examples, each set that you would intuitively think of as constituting a continuum is a perfect set, and each one that you would not think of as a continuum is not perfect. This raises the question: should a perfect set be defined as a continuum?

Task 3 Is the set $[0, 1] \cup [2, 3]$ a perfect set? Should it be a continuum?

As the example above illustrates, being perfect might be a necessary condition for a set to be a continuum, but it is not sufficient. The missing piece that Cantor added to the condition of being perfect in order to complete his definition of a continuum was that the set also be connected. Cantor wrote:

A [closed and bounded] point-set T is **connected** if for every two of its points t and t', and arbitrary given positive number ϵ, there always exists a finite number of points $t_1, t_2, \dots, t_n$ of T such that the distances $tt_1, t_1t_2, \dots, t_nt'$ are smaller than ϵ.

Task 4 Rewrite this definition using modern notation (and possibly terminology), and use that definition to prove that $[0, 1] \cup [2, 3]$ is not connected. Then prove that $[0, 1]$ is connected.

Task 5 According to Cantor's definition, what can be said in terms of connectedness about the set $[0, 1] - \left\{\frac{1}{2}\right\}$?

21.2 Jordan: Distance between Sets

Cantor's definition of connectedness appealed to a notion of distance, which leads to a setting that is not as general as that which one encounters in topology today. This approach to the definition was nevertheless taken up by French mathematician Camille Jordan (1838–1922) in his *Cours d'analyse* (*Course on analysis*) [Jordan, 1893].

Let E, E' be two [closed] sets with no point in common. The distance between the various points p in E to the various points p' in E' form a set of nonnegative numbers. It is therefore bounded below, and it admits a minimum Δ, positive or 0, we call the **distance** between the sets E, E'. If the distance is greater than 0, we say that the sets E, E' are **separated**.

Task 6 Using Jordan's definition, compute the distance between two closed, disjoint disks in $\mathbb{R}^2$.

Jordan immediately made the following statement and provided a proof.

If two bounded and closed sets E, E' which have no points in common have distance Δ, then they will contain at least a couple of points whose mutual distance is precisely Δ ... .

Task 7 Show that this is not necessarily true if the sets in question are not closed.

We are finally ready for Jordan's definition. Jordan wrote as follows:

We say that a closed and bounded set E is a **component** if it cannot decompose into several separated sets.

Task 8 Using Jordan's definition of a component, prove that a closed and bounded set E in $\mathbb{R}^n$ is connected if and only if E itself is a component.

21.3 Schoenflies: No Distance Required

We have noted that both Cantor and Jordan defined connectedness by appealing to a distance. The German mathematician Arthur Schoenflies (1853–1928), who is best known for the Jordan-Schoenflies theorem, instead gave a purely set-theoretic definition of connectedness which did not appeal to the concept of distance. The definition that he gave in his paper "Beiträge zur Theorie der Punktmengen I" ("Contributions to the theory of point-sets I") [Schoenflies, 1904], was simple and elegant.

> A perfect set T is called **connected** if it can not be decomposed into subsets, each of which is perfect.

Task 9 Rewrite Schoenflies's definition of connected with modern terminology and notation.

Task 10 Does Schoenflies's definition give us any better indication of whether or not the set $[0,1] - \left\{\frac{1}{2}\right\}$ is connected? Why or why not?

Although Schoenflies did other things with this definition, our main purpose here is to note that he abstracted away any need for distance, yet kept the definition very much in the spirit of Jordan and Cantor, as the following task illustrates.

Task 11 Prove that if a set satisfies the Jordan definition, then it must satisfy the Schoenflies definition. Why is the converse not necessarily true?

21.4 Lennes: The Modern Definition

Finally, we turn to the work of Nels Johann Lennes (1874–1951). A Norwegian-born mathematician, Lennes earned his Ph.D. at the University of Chicago and lived out most of his mathematics career at Montana State University. Lennes was aware of the mathematical thought on connectedness up to this point. In his paper "Curves in Non-Metrical Analysis Situs with an Application in the Calculus of Variations," [Lennes, 1911], he wrote:

> It is apparent that in a geometry possessing linear order and continuity curves and limit curves exist independently of metric properties. ... Schoenflies uses metric hypotheses in the proof of practically every important theorem dealing with curves and the regions defined by them ... his treatment makes full use of metric properties.

Lennes then gave his own definition of limit point.

A point ℓ is a **limit point** of a set of points P if there are points of P other than ℓ within every [neighborhood] of which ℓ is an interior point.

Task 12 Compare Cantor's definition of limit point with that of Lennes. Are they equivalent? If so, prove it. If not, give a counterexample.

Lennes was interested in the Jordan Curve Theorem, one of the most important and difficult theorems of late-nineteenth and early-twentieth century mathematics. This theorem states:

Jordan Curve Theorem. Let J be a closed curve in $\mathbb{R}^2$ which does not self-intersect. Then $\mathbb{R}^2 - J$ is disconnected with exactly two open, connected components.

Although easy to state and intuitively obvious, a rigorous and satisfying proof of this fact eluded mathematicians for many years. In order to attempt a rigorous proof, Lennes provided a careful and precise definition of connectedness.

A set of points is a **connected set** if at least one of any two complementary subsets contains a limit point of points in the other set.

Lennes's definition turns out to be equivalent to the modern definition. To substantiate our claim, here is a definition from a modern classic book on point-set topology [Kelley, 1975, p. 53].

Definition. A topological space is **connected** if and only if X is not the union of two non-void separated, open subsets.

Task 13 Show that this definition and the one given by Lennes are equivalent. Then use it determine whether or not the set $[0,1] - \left\{\frac{1}{2}\right\}$ is connected.

21.5 Conclusion

We have seen how the definition of connectedness, starting with Cantor, has evolved into the modern definition. For Cantor, connectedness was somewhat of a side note: connectedness needed to be defined in order to properly understand the concept of a continuum. Jordan thought this definition of connected interesting enough to take it up in his own work, and studied it as a concept in its own right. Schoenflies then realized that there was no need to appeal to a notion of "distance" to give a coherent definition of connected. Finally, Lennes tweaked Schoenflies's definition sufficiently to obtain the one that is used by mathematicians today. So remember this the next time you see a definition in a textbook. The crisp, clean, and pithy definition may have taken some of the world's greatest mathematicians years to arrive at!

Bibliography

Cantor, G. (1872). Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen (On the extension of a theorem from the theory of trigonometric series). *Mathematische Annalen*, 5(1):123–132.

Cantor, G. (1883). Über unendliche, lineare Punktmannigfaltigkeiten, [Teil] 5 (On infinite linear manifolds of points, Part 5). *Mathematische Annalen*, 21:545–586.

Jordan, C. (1893). *Cours d'analyse de L'École Polytechnique* (*Course on analysis of the Polytechic School*), volume I. Gauthier-Villars, Paris, 2nd edition.

Kelley, J. L. (1975). *General Topology*. Springer-Verlag, New York. Graduate Texts in Mathematics, No. 27. Reprint of the 1955 edition, Van Nostrand, Toronto.

Lennes, N. J. (1911). Curves in Non-Metrical Analysis Situs with an Application in the Calculus of Variations. *American Journal of Mathematics*, 33(1–4):287–326.

Schoenflies, A. (1904). Beiträge zur Theorie der Punktmengen I (Contributions to the theory of point-sets I). *Mathematische Annalen*, 58:195–238.

Notes to Instructors

PSP Content: Topics and Goals. This PSP is intended for use in an introductory course in topology. While it is designed in part to introduce students to the concept of connectedness, it also serves a broader two-fold purpose. First, it is meant to show where definitions come from. This is seen in Cantor's attempt to study a continuum. It is made most explicit in Task 3 where the student should see that there are perfect point-sets that one ought not consider a continuum. The second purpose is to show how a definition can change over time. Because Cantor was only working with Euclidean space, it made sense for connectedness to be defined in the setting of Euclidean space. As more general spaces started to be explored, more general definitions were called for, some which made sense in these more general settings (like connectedness) while others did not.

Both points (where definitions come from and how definitions change) are just two aspects of the mathematical discovery process that is often hidden from students. As such, it is important to emphasize that this project is showing that there is something much greater to mathematics than applying formulas. It is the hope that this project will help students to see (or even consider for possibly the first time) that mathematics is a human endeavour with both struggles and unclear solutions.

Student Prerequisites. This project does assume some background familiarity with point-set topology. In particular, the students should have been exposed to the concept of derived sets. Otherwise, the definition of a perfect set in Section 1 comes seemingly out of left field.

PSP Design and Task Commentary. As noted above, the general purpose of this project is two-fold. Its first goal of showing where definitions come from is mostly achieved in this PSP in the section on Cantor, in two different ways. The first way is by illustrating the process through which mathematicians make precise an intuitive idea. The discussion on a continuum is meant to imitate the working out of a definition. We all have in our minds paradigmatic examples of sets that we do and don't consider to be continua. Let us take all those examples, abstract away the particulars, and be left with what they all have in common; this is then the essence of what it means to be a continuum. A good way to accomplish this in class is to ask the students to give examples of what they think are and are not continua, and to write their responses in two columns on the board. This should generate a discussion of some of the properties that students do and don't consider to be essential for a set to be a continuum. There may be students for whom nothing comes to mind when they hear "continuum." That is okay. But by the end of this task, the class as a whole should have a somewhat unified, even if still vague, idea of what "continuum" means.

The next step in this process is then attempting to precisely formulate the meaning of a continuum in a definition. In order to tease this out, students are asked to recall the definition of a perfect point-set. After a little reflection, it seems that our intuitive idea of what ought and ought not constitute a continuum coincides exactly with that of a perfect set. (In fact, I have had classes where those sets which the class considered a continuum were precisely the perfect point-sets, while sets which were not considered a continuum were precisely those which are not perfect.) Task 3, however, serves as an example to show why equating the two concepts is not appropriate. This leads into the second way in which the project shows where definitions come from, by considering the question: what property needs to be added to "perfect" in order to exclude examples like the one in Task 3 from being considered a continuum? There is then a need to define this additional property, and the concept that seems to work well is that of being "connected." Students are then asked to wrestle with Cantor's original formulation of the definition of connectedness.

The section on Jordan begins to address the second project goal of showing how a definition can evolve, either in its verbal formulation or in its point of view. While Cantor defined what it means for a set to be connected, Jordan added the viewpoint of "separation" as a way to look at connectedness. That is, now we have a positive definition of this concept (connected) as well as a negative concept (separation). The concept of separation can furthermore be quantified in the sense that if a separation exists, we can sometimes assign a number to it (e.g., Task 6).

The Schoenflies section is brief and meant to be a bit ambiguous. What did he mean by "decomposed into"? A partition? Only two sets? A finite number of sets? Students will wrestle with these question and justify answers that make the most sense relative to what Schoenflies was trying to do in Task 10. However, it should be noted that Schoenflies gave a definition that is not equivalent to those given by Cantor and Jordan, since there is no appeal to a metric in Schoenflies's definition.

The final definition due to Lennes is easily motivated with a word about the Jordan Curve Theorem, and it is satisfying for the students to see that this is the first time that we see the current definition being used.

There is a running example throughout the PSP of determining whether or not the set $[0,1]-\left\{\frac{1}{2}\right\}$ is connected or not. For both Cantor and Jordan, the concept of connectedness only applied to closed and bounded sets, so that this question would not even make sense to them, a seemingly major drawback of the definition. Depending on how they interpret Schoenflies, students may or may not find the set connected according to his definition. However, it should once again be satisfying to see this simple example, unclear for many years, now easily shown to be disconnected using the Lennes definition.

Sample Implementation Schedule (based on a 50-minute class period).

The following outline provides a schedule for implementing this PSP in 2 class days.

Day 1

- **In class discussion (10 minutes).** Before handing out the project, write the question "what is a continuum?" on the board. Ask the class for examples of objects (usually mathematical but not necessarily) that are a continuum. List these in one column on the board and at some point, begin a second column of "non-examples" or things that are not a continuum. You may find some disagreement among students, and you can put these examples in the middle. If they are not brought up, suggest a closed interval, an open interval, and a disconnected interval as sets for the class to consider and place in a column. You may wish to save this list or return to it after the class has seen Cantor's definition of a continuum. Then you can see which of the objects fit Cantor's definition and which do not.

- **Working in groups (15 minutes).** Hand out the project and have students work in small groups up to and including Task 2. Note that Task 1 simply reiterates the in-class discussion so students have a chance to say anything else about a continuum in their groups that they didn't have a chance to say in class.
- **Debrief (5 minutes):** After students work in groups, the next 5 minutes can be spent in a whole-class discussion, regrouping and making sure everyone is comfortable with limit points and derived sets.
- **Working in groups (10 minutes).** Students can spend the next 10 minutes working on Tasks 3–4. The goal here is for students to see that more is needed on top of perfect in order to be a continuum.
- **Debrief and Task 5 (10 minutes).** After a brief discussion about what was done in groups, end class by working on Task 5 together. Ask if the answer seems appropriate.
- **Homework.** Write up solutions to all tasks completed in-class and begin to read and write up tasks in Section 2 for homework.

Day 2

- **In class discussion (10 minutes).** Begin by reviewing what was done the previous day and move into a discussion with the class of the main ideas in Section 2. Ask if there are any questions.
- **Working in groups (20 minutes).** Have students continue working in their groups to both discuss Section 2 tasks as well as begin working through Section 3.
- **Debrief (5 minutes).** Have the class reconvene to discuss what they just worked on. Emphasize that Schoenflies has abstracted away the metric and is now working in a general topological space.
- **Working in groups (15 minutes).** Students can spend the last minutes working on Section 4. Anything they do not finish can be done for homework.

25
The Logarithm of -1

Dominic Klyve,Central Washington University

Logarithms were introduced to the world by John Napier (1550–1617) in 1614, in a work entitled *A description of the wonderful table of logarithms* (or, in the Latin original, *Mirifici logarithmorum canonis descriptio*), as a bridge between an arithmetic sequence and a geometric one. Four hundred years later, logarithms are ubiquitous in mathematics and science, and it is difficult to imagine a world without them.

The eighteenth century was in a time of transition in the story of logarithms (as it was in so many fields of mathematics). The elementary properties of logarithms were well understood, but many questions about them remained. This project will focus on one of these questions — namely, how to extend the domain of the logarithm function to negative numbers.

Recall that $\log_b(x) = a$ is equivalent to $b^a = x$, so, for example, $\log_{10}(1000) = 3$. Given any function, it is convenient for its domain to be as large as possible, and for mathematical reasons (the applications would come many years later), eighteenth-century mathematicians wished to find a way to do this for the logarithm function. As it turns out, deciding the best way to do so was not easy, and would lead to a serious disagreement in the mathematical community.

Task 1 Why is it difficult to find the logarithm of a negative number? For example, why can't you easily find a value for $\log_{10}(-100)$?

25.1 Functions with Several Values?

You have likely seen a formal definition of a function in the past. Although there are small differences between texts, the definition usually looks something like this:

> **Definition.** Let X and Y be sets. A **function** $f : X \to Y$ defined on X is a rule that assigns to each element $x \in X$ an element $y \in Y$, in which case we write $y = f(x)$.

Task 2 Check the definition of function in a current textbook. Does it differ in any significant way from the definition above? If so, how?

Task 3 A very important property of functions seems to be that for each x, $f(x)$ has only one value. (This is related to the "vertical line test" that you have seen in calculus.) Can you think of any "function" (possibly expanding or modifying the definition a bit) for which $f(x)$ has more than one value? Try to give at least one example — or more if you can!

25.2 Extending the Domain of the Logarithm

One of the strongest voices in the effort to expand the scope of the logarithm function was Leonhard Euler (1707–1783), the leading mathematician and physicist of the eighteenth century. He had been thinking about logarithms as early as 1727, in correspondence with his mentor Johann Bernoulli (1667–1748). Bernoulli had written in the past that $\log(-x) = \log(x)$ (so, in particular, $\log(-1) = 0$). Euler was unconvinced, though he recognized that there were arguments in favor of this position. Indeed, in a letter to Bernoulli dated December 10, 1728, Euler suggested one such argument himself.[1]

> If $\log(xx) = z$, it will be that $\frac{1}{2}z = \log(\sqrt{x})$, but $\sqrt{xx}$ is both $-x$ as $+x$, therefore $\frac{1}{2}z$ is $\log(x)$ and $\log(-x)$. But it could be objected that, if xx has two logarithms, it should be judged to have infinitely many.

Task 4 Explain Euler's argument in your own words.

Task 5 Does Euler's argument above convince you that $\log(-x) = \log(x)$? Why or why not?

[1] All translations of excerpts from Euler's correspondence and other primary sources in this project were prepared by the project author.

Task 6 What argument do you think Euler may have had in mind when he stated that if the logarithm function has two values, then "it should be judged to have infinitely many"? Write down as many ideas as you can come up with for this. Do you agree with the claim at this point? Why or why not?

25.3 The Beginning of a Conflict

By the end of 1746, Euler was much more mature, both as a mathematician in general and in his understanding of logarithms. Indeed, he had come to strongly suspect that the values of logarithms of negative values would involve imaginary and complex numbers, although he had yet to publish these ideas. On December 29 of that year, Euler wrote a letter to Jean le Rond d'Alembert (1717–1783), a French philosopher and thinker who had just submitted a paper to the Berlin *Mémoires* (a mathematics journal edited by Euler) in which he tried to prove the Fundamental Theorem of Algebra. In his paper, d'Alembert wrote that for any positive number x, $\log(-x) = \log(x)$. Euler wrote privately to d'Alembert in part to alert his colleague that this part of his paper may be in error. Regarding the logarithms of negative numbers, Euler wrote:

For me, I believe I have demonstrated [in another work, not yet published] that it [the logarithm of -1] is imaginary, and that it is $= \pi(1 \pm 2n)\sqrt{-1}$, where π denotes the circumference of a circle in which the diameter $= 1$, and n is any integer.[2]

Because I have shown that, just as each value of sine corresponds to an infinite number of arcs of the circle, so the logarithm of each number has an infinite number of values, among which there is but one that is real when the number is positive, and when the number is negative all the values are imaginary. Thus, $\log(1) = \pi(0 \pm 2n)\sqrt{-1}$ where n is any integer. Setting $n = 0$, we have the ordinary logarithm $\log 1 = 0$. And in the same manner, we have $\log a = \log a + \pi(0 \pm 2n)\sqrt{-1}$, where $\log a$ on the right side of the equation denotes the ordinary logarithm of a, but $\log(-a) = \log a + \pi(1 \pm 2n)\sqrt{-1}$, where all the values are imaginary. All of this follows from the formula $\log(\cos\theta + \sin\theta\sqrt{-1})^k = (k\theta \pm 2mk\pi \pm 2n\pi)\sqrt{-1}$, where m and n are any integers, the truth of which is easy to demonstrate.[3]

---

[2]Euler would eventually use the symbol i to represent $\sqrt{-1}$, but not until 1777.

[3]Euler may have been understating the difficulty in establishing this equation. It didn't appear until the final page of the paper that he was working on at the time on the logarithms of negative and imaginary numbers. We will take a look at part of this paper in the final section of this project.

Task 7 What did Euler mean when he wrote "corresponds to an infinite number of arcs of the circle"? Use this idea to give an example of a function with infinitely many values for one input. Does this function match the definition given above Task 2?

Task 8 Assuming that Euler is correct, give three different values of log 1.

The next two tasks ask you to investigate Euler's formula

$$\log(\cos\theta + \sin\theta\sqrt{-1})^k = (k\theta \pm 2mk\pi \pm 2n\pi)\sqrt{-1}.$$

Task 9 Using this formula, pick values of θ, k, m, and n that allow you to calculate $\log(-1)$. What do you find?

Task 10 Now set $\theta = \pi$, $k = 1$, $m = 0$, and $n = 0$ in this formula. Then exponentiate both sides and rewrite the equation so that you have 0 on the right side, and all other terms on the left. What do you get?

25.4 The Fight Escalates

D'Alembert was unconvinced by Euler's explanation, and he sent back no fewer than six arguments concerning why Euler was wrong in his next letter, dated March 14, 1747. Some of these arguments were quite weak (amounting to little more than "I can't really imagine logarithms being imaginary"), but one was quite interesting. D'Alembert wrote

All the difficulty is reduced, it seems to me, to knowing what $\log(-1)$ is. But cannot we prove that it is $= 0$ by this reasoning? $-1 = 1/-1$, therefore $\log(-1) = \log(1) - \log(-1)$. Thus $2\log(-1) = \log(1) = 0$. Therefore $\log(-1) = 0$.

Task 11 Explain why d'Alembert can claim that $\log(-1) = \log 1 - \log(-1)$.

Task 12 Can you find any flaw in d'Alembert's reasoning? Are you convinced? Why or why not?

Euler, as it turns out, was not convinced. He was thinking more about the earlier work of Johann Bernoulli. In his investigation of the logarithms of imaginary numbers [Bernoulli, 1702], Bernoulli had demonstrated a rather non-obvious result (stated here in modern notation):

Theorem (Johann Bernoulli). Given a circle of radius a, the area of a sector of the circle with boundaries of the x-axis and the line from the origin to the point (x, y) is given by

$$\frac{a^2}{4\sqrt{-1}} \log\left(\frac{x + y\sqrt{-1}}{x - y\sqrt{-1}}\right).$$

Task 13 Taking Bernoulli's Theorem as a given, use his formula to find an expression for the area of a sector with a central angle of 90°. Then find the same area using geometry. What would the value of $\log(-1)$ have to be so that these two answers were equal?

With Bernoulli's theorem in hand, Euler responded to d'Alembert's argument on April 15, 1747 with the following reasoning.

By the reasoning that you used to prove that $\log(-1) = 0$, you could equally prove that $\log(\sqrt{-1}) = 0$ because, since $\sqrt{-1}\sqrt{-1} = 1$, you would have $\log(\sqrt{-1}) + \log(\sqrt{-1}) = \log(-1)$, that is to say, $2\log(\sqrt{-1}) = \log(-1) = \frac{1}{2}\log(+1)$, and so $\log(\sqrt{-1}) = \frac{1}{4}\log(1) = 0$, and if you do not approve this reasoning, you will agree with me that the first reasoning [that $\log(-1) = 0$] is not more convincing. But you will at least be in agreement that the logarithms of imaginary numbers are not real, or else this expression: $\log(\sqrt{-1})/\sqrt{-1}$ would not express the quadrature[4] of the circle.

Task 14 Explain how the reasoning Euler used above is analogous to d'Alembert's claim.

Task 15 Explain why Euler believed that Bernoulli's result implied that the expression $\log(\sqrt{-1})/\sqrt{-1}$ gives a real number, and why therefore the logarithm of imaginary numbers must not be real.

Sadly, the two mathematicians never agreed on the logarithm of -1. Indeed, this was one of the issues that broke up their friendship years later (the two would eventually attack each other in print over a series of arguments). For more on this fascinating story, see Robert Bradley's article "Euler, D'Alembert and the Logarithm Function" [Bradley, 2007].[5]

[4] For Euler, "expressing the quadrature" of a circle is analogous to finding its area.

[5] This article also inspired and provided the central ideas for much of this project, for which the project author is grateful to Bradley. While the most important ideas discussed in his article have been encapsulated into the project, a lot of Bradley's rich discussion of eighteenth-century views of logarithms has been left out — another reason why readers (both students and instructors!) are strongly encouraged to read it.

25.5 The Formalization of the Logarithm Function

In a 1747 paper,[6] Euler definitively showed that in order to extend the domain of the logarithm function to negative and imaginary numbers, it was necessary to assume that the logarithm function took multiple (in fact, infinitely many) values for each argument. His arguments were so convincing that the mathematical community quickly accepted his definition of the logarithm as true; indeed, it's the definition we still use today. Euler summarized his work as follows:

> This equation that we just found, expressing the relationship between the arc ϕ and its sine and cosine values, will also hold for other arcs that have the same sine x and cosine y. Consequently, we will have
>
> $$\phi \pm 2n\pi = \frac{1}{\sqrt{-1}} \log(y + x\sqrt{-1}),$$
>
> from which it follows that
>
> $$\log(y + x\sqrt{-1}) = (\phi \pm 2n\pi)\sqrt{-1}.$$

Task 16 Using this claim, find three values of log(1).

Task 17 Return to d'Alembert's argument that $\log(-1) = 0$, and fix it using Euler's correct interpretation of the logarithm function. How would you answer someone now who asked you what the logarithm of -1 is? What about the logarithm of 1?

Task 18 You have discovered that the logarithm function is a *multi-valued function* — a useful idea that seems to contradict the standard definition of a function. Give some other examples of functions that can be thought of as having several values. Ideally, try to find examples for functions defined over both the real numbers and the complex numbers.

[6] This paper, entitled "Sur les logarithmes des nombres négatifs et imaginaires" ("On the logarithms of negative and imaginary numbers"), was not published until 1862, in a two-volume publication by the St. Petersburg Academy of Sciences containing all the works of Euler that had not yet been published prior to his death in 1783. In the Eneström index that enumerates all of Euler's works, based on a comprehensive survey conducted by Swedish mathematician Gustaf Eneström between 1910–1913, Euler's 1847 paper is [E807]. The complete Eneström index includes 866 distinct works by Euler.

Bibliography

Bernoulli, Jo. (1702). Solution d'un probléme concernant le calcul intégral (Solution of a problem concerning integral calculus). *Histoire de l'Académie Royale des Sciences, avec les Mémoires de Mathématique et de Physique*, pages 289–297.

Bradley, R. E. (2007). Euler, D'Alembert and the Logarithm Function. In Bradley, R. E. and Sandifer, E., editors, *Leonhard Euler: Life, Work and Legacy*, volume 5 of *Studies in the History and Philosophy of Mathematics*, pages 255–277. Elsevier, Amsterdam.

Euler, L. (1862). Sur les logarithmes des nombres négatifs et imaginaires (On the logarithms of negative and imaginary numbers). In *Opera Postuma mathematica et physica*, pages 269–281. Imperial Academy of Sciences, St. Petersburg. Also in *Leonhardi Euleri Opera Omnia*, series 1, volume 19, Birkhäuser, Basel, pages 417–438. This posthumously published paper was written by Euler in 1747.

Euler, L. (1980). *Correspondence de Leonhard Euler avec A. C. Clairaut, J. d'Alembert et J. L. Lagrange* (*Correspondance of Leonhard Euler with A. C. Clairaut, J. d'Alembert and J. L. Lagrange*). Volume 5 of *Leonhardi Euleri Opera Omnia*, series IVA: *Commercium Epistolicum*, Birkhäuser, Basel.

Euler, L. (1988). *Correspondence de Leonhard Euler avec Johann* (*I*) *and Niklaus* (*I*) *Bernoulli* (*Correspondance of Leonhard Euler with Johann* (*I*) *and Niklaus* (*I*) *Bernoulli*). Volume 2 of *Leonhardi Euleri Opera Omnia*, series IVA: *Commercium Epistolicum*, Birkhäuser, Basel.

Notes to Instructors

PSP Content: Topics and Goals. This PSP is intended to be used in a course on complex variables, and introduces students to the definition of the logarithm function on negative and complex numbers via an epistolary argument between Leonhard Euler and Jean d'Alembert. Its primary goal is to motivate the definition of the logarithm over $\mathbb{C}$, in particular its property of having multiple values. Students familiar with the definition of a function from a real analysis class (or even the vertical line test in calculus) have been taught that a function can have only one output for every input, a lesson belied by the behavior of natural logarithm (along with several other functions) over the complex numbers.

The biggest pedagogical hurdle that this PSP tries to help students clear is thus the idea of a multi-valued function, which might seem a contradiction in terms based on their previous experience. In fact, it was this exact property of the logarithms that made extending the domain of this function to negative and natural numbers so difficult in the eighteenth century. Johann Bernoulli and Jean d'Alembert both seemed unable to make the leap to multi-valued functions, and even Euler (who eventually extended the domain of the logarithm function in the way we think of it today) struggled for several years.

Student Prerequisites. Ideally, students should have seen a formal definition of a function. They should have a level of mathematical maturity sufficient to look for their own example and to be able to reach conclusions using properties of logarithms.

PSP Design and Task Commentary. Through this project, students have an opportunity to recapitulate some of the early work on the logarithm of negative and imaginary values, and to try to solve an early paradox, proposed by d'Alembert, that seems to indicate that $\log(-1) = 0$.

Task 3 can be tricky for students; a multi-valued function doesn't exist by definition. The hope here is that they may think of $f(x) = \sqrt{x}$, which we can think of as a two-valued function, or $g(x) = \arcsin x$, which could (through appropriate choices of domain and range) take more. The question of arcsine is explored in Task 7.

Task 7 is designed to help students start thinking about multi-valued functions by reminding them of sine and arcsine. As they know, there are infinitely many angles θ for which the $\sin(\theta) = \pi$, and thus $\arcsin(\pi)$ can be thought of as a multi-valued function. We usually avoid this issue and calculus and precalculus classes by restricting the domain and range of arcsine, but one could imagine not doing so.

Task 13 asks students to use a formula for the area of a sector of a circle that leads to an expression involving the logarithm of a complex number. They should find an imaginary value for this logarithm, leading to a real result for the circular sector.

Suggestions for PSP Implementation and Sample Schedule (based on a 50-minute class period). One possible implementation, using just 1.5 days of class time, follows. If time allows, the homework can be eliminated and replaced with in-class time, in which case it may take 2–2.5 days to complete the full project.

Day 0. Students work together in groups for about 15 minutes to read the introduction and complete Task 1. **Day 0 Homework:** Assign Section 1, the reading in Section 2 up to the Task 7, and Task 7. This is the longest section of reading in the PSP, and it may be better done as homework than during class.

Day 1. Students meet in groups to discuss their answers to Task 7, and work together to complete Section 2 and Section 3 through Task 13. The remaining tasks can be completed as homework. **Day 1 Homework:** Assign Tasks 14–18.

Day 2. I would encourage beginning Day 2 with a short full-class discussion of some of the juicier tasks. Task 17 would be a really good one to work through, and the instructor should feel free to use Task 18 to discuss multi-valued functions in complex analysis. If it's appropriate for the current state of your course, this could lead to a discussion of branch cuts.