# Introduction

This book is divided into six main parts. The first part (lectures 1–5) presents the definitions and the first three comparison results. The second part (lectures 6–10) presents the étale version of the theory, focussing on coefficients, $1/m \in k$. As Suslin's Rigidity Theorem 7.20 demonstrates, a key role is played by locally constant étale sheaves such as $\mu_m^{\otimes i}$, which are quasi-isomorphic to the motivic $\mathbb{Z}/m(i)$ by theorem 10.3. The tensor triangulated category $\mathbf{DM}^{-}_{\acute{e}t}(k, \mathbb{Z}/m)$ of étale motives is constructed in lecture 9 and shown to be equivalent to the derived category of discrete $\mathbb{Z}/m$-modules over the Galois group $G = \mathrm{Gal}(k_{sep}/k)$ in theorem 9.35.

The first main goal of the lecture notes, carried out in lectures 11–16, is to introduce the tensor triangulated category $\mathbf{DM}^{\mathrm{eff},-}_{Nis}(k,R)$ of effective motives and its subcategory of effective geometric motives $\mathbf{DM}^{\mathrm{eff}}_{gm}$. The motive $M(X)$ of a scheme $X$ is an object of $\mathbf{DM}^{\mathrm{eff},-}_{Nis}(k,R)$, and belongs to $\mathbf{DM}^{\mathrm{eff}}_{gm}$ if $X$ is smooth. This requires an understanding of the cohomological properties of sheaves associated with homotopy invariant presheaves with transfers for the Zariski, Nisnevich and cdh topologies. This is addressed in the third part (lectures 11–13). Lecture 11 introduces the technical notion of a standard triple, and uses it to prove that homotopy invariant presheaves with transfers satisfy a Zariski purity property. Lecture 12 introduces the Nisnevich and cdh topologies, and lecture 13 considers Nisnevich sheaves with transfers and their associated cdh sheaves.

A crucial role in this development is played by theorem 13.8: if $F$ is a homotopy invariant presheaf with transfers, and $k$ is a perfect field, then the associated Nisnevich sheaf $F_{Nis}$ is homotopy invariant, and so is its cohomology. For reasons of exposition, the proof of this result is postponed and occupies lectures 21 to 24.

In the fourth part (lectures 14–16) we introduce the categories $\mathbf{DM}^{\mathrm{eff},-}_{Nis}(k,R)$ and $\mathbf{DM}^{\mathrm{eff}}_{gm}$. The main properties of these categories — homotopy, Mayer-Vietoris, projective bundle decomposition, blow-up triangles, Gysin sequence, the Cancellation Theorem, and the connection with Chow motives — are summarized in 14.5. We also show (in 15.9) that the product on motivic cohomology (defined in 3.12) is graded-commutative and in agreement (for coefficients $\mathbb{Q}$) with the étale theory presented in lectures 9 and 10 (see 14.30).

Lecture 16 introduces equidimensional algebraic cycles. These are used to construct the Suslin-Friedlander motivic complexes $\mathbb{Z}^{SF}(i)$, which are quasi-isomorphic to the motivic complexes $\mathbb{Z}(i)$; this requires the field to be perfect (see 16.7). They are also used to define motives with compact support $M^c(X)$. The basic theory with compact support complements the theory presented in lecture 14; this requires the field to admit resolution of singularities. This lecture concludes with the use of Friedlander-Voevodsky duality (see 16.24) to establish the Cancellation Theorem 16.25; this lets us embed effective motives into the triangulated category of all motives.

The second main goal of this book is to establish the final comparison (theorem 19.1) with Bloch's higher Chow groups: for any smooth separated scheme $X$ over a perfect field $k$, we have

$$H^{p,q}(X,\mathbb{Z}) \cong CH^q(X, 2q-p).$$

This is carried out in the fifth part (lectures 17–19). In lecture 17, we introduce Bloch's higher Chow groups and show (in 17.21) that they are presheaves with transfers over any field. Suslin's comparison (18.3) of higher Chow groups with equidimensional cycle groups over any affine scheme is given in lecture 18, and the link between equidimensional cycle groups and motivic cohomology is given in lecture 19.

We briefly revisit the triangulated category $\mathbf{DM}_{gm}$ of geometric motives in lecture 20. We work over a perfect field which admits resolution of singularities. First we embed Grothendieck's classic category of Chow motives as a full subcategory. We then construct the dual of any geometric motive and use it to define internal Hom objects $\underline{Hom}(X,Y)$. The lecture culminates in theorem 20.17, which states that this structure makes $\mathbf{DM}_{gm}$ a rigid tensor category.

The final part (lectures 21–24) is dedicated to the proof of theorem 13.8. Using technical results from lecture 21, we prove (in 22.3) that $F_{Nis}$ is homotopy invariant. The proof that its cohomology is homotopy invariant (24.1) is given in lecture 24. We conclude with a proof that the sheaf $F_{Nis}$ admits a "Gersten" resolution.

During the writing of the book, we received many suggestions and comments from the mathematical community. One popular suggestion was that we include some of the more well known and useful properties of motives that were missing from the original lectures, in order to make the exposition of the theory more complete. For this reason, a substantial amount of material has been added to lectures 12–14, 16 and 20. Another suggestion was that we warn the reader that the exercises vary in difficulty and content, from the concrete to the abstract; some are learning exercises and some augment the ideas presented in the text.

In Figure 1 we give a rough bird's eye view of the structure of the book and how the various lectures depend upon each other. Lectures 1 and 2 are missing from the figure because they are prerequisites for all other lectures. We split lecture 13 into two parts to clarify that the results in the second half of the lecture crucially depend on theorem 13.8, which is proven in lecture 24. The dependency chart (and this Introduction) should serve as a guide to the reader.

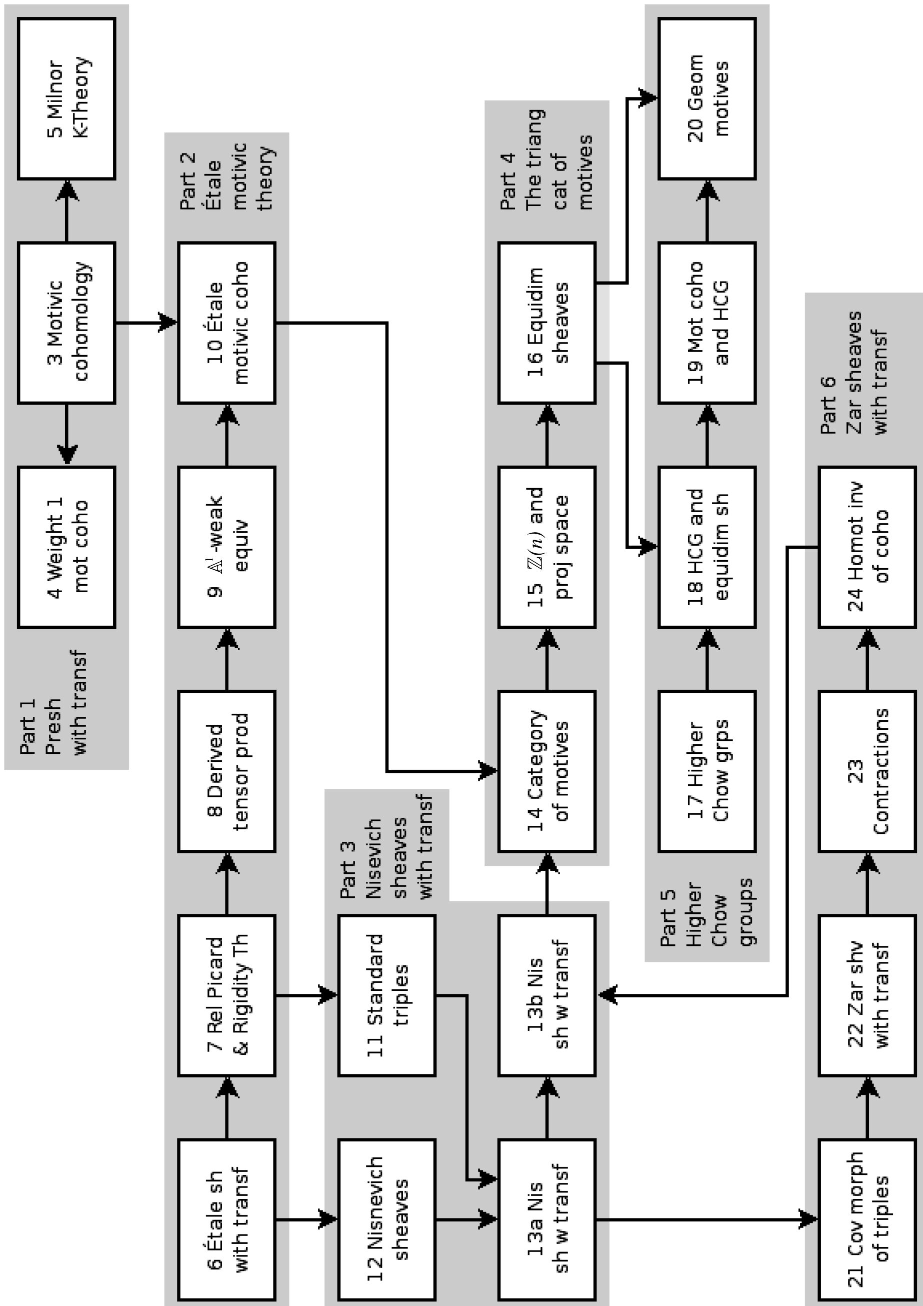


FIGURE 1. Dependency graph of the lectures

## Acknowledgements

The authors are deeply indebted to the Institute for Advanced Study, the Clay Mathematics Institute and Rutgers University for the support provided by these institutions during the writing of this book. In addition, we are grateful to Thomas Geisser, who carefully proofread an earlier version and provided us with his lecture notes on this subject, which were used as an outline for the additions in lectures 12, 16 and 20.

In addition, the authors were supported on numerous grants during the writing phase (2000–2005) of this book. This includes support from the NSF, NSA, INDAM and the institutions named above.

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December 2005