Preface

This book was written by Carlo Mazza and Charles Weibel on the basis of the lectures on motivic cohomology which I gave at the Institute for Advanced Study in Princeton in 1999/2000.

From the point of view taken in these lectures, motivic cohomology with coefficients in an abelian group A is a family of contravariant functors

$$H^{p,q}(-,A) : Sm/k \to Ab$$

from smooth schemes over a given field k to abelian groups, indexed by integers p and q. The idea of motivic cohomology goes back to P. Deligne, A. Beilinson and S. Lichtenbaum.

Most of the known and expected properties of motivic cohomology (predicted in [**ABS87**] and [**Lic84**]) can be divided into two families. The first family concerns properties of motivic cohomology itself – there are theorems about homotopy invariance, Mayer-Vietoris and Gysin long exact sequences, projective bundles, etc. This family also contains conjectures such as the Beilinson-Soulé vanishing conjecture ($H^{p,q} = 0$ for $p < 0$) and the Beilinson-Lichtenbaum conjecture, which can be interpreted as a partial étale descent property for motivic cohomology. The second family of properties relates motivic cohomology to other known invariants of algebraic varieties and rings. The power of motivic cohomology as a tool for proving results in algebra and algebraic geometry lies in the interaction of the results in these two families; applying general theorems of motivic cohomology to the specific cases of classical invariants, one gets new results about these invariants.

The idea of these lectures was to define motivic cohomology and to give careful proofs for the elementary results in the second family. In this sense they are complementary to the study of [**VSF00**], where the emphasis is on the properties of motivic cohomology itself. The structure of the proofs forces us to deal with the main properties of motivic cohomology as well (such as homotopy invariance). As a result, these lectures cover a considerable portion of the material of [**VSF00**], but from a different point of view.

One can distinguish the following "elementary" comparison results for motivic cohomology. Unless otherwise specified, all schemes below are assumed to be smooth or (in the case of local or semilocal schemes) limits of smooth schemes.

(1) $H^{p,q}(X,A) = 0$ for $q < 0$, and for a connected X one has

$$H^{p,0}(X,A) = \begin{cases} A & \text{for } p = 0 \\ 0 & \text{for } p \neq 0; \end{cases}$$

(2) one has

$$H^{p,1}(X,\mathbb{Z}) = \begin{cases} \mathcal{O}^*(X) & \text{for } p = 1 \\ \mathrm{Pic}(X) & \text{for } p = 2 \\ 0 & \text{for } p \neq 1,2; \end{cases}$$

(3) for a field k, one has $H^{p,p}(\mathrm{Spec}(k),A) = K_p^M(k) \otimes A$ where $K_p^M(k)$ is the p-th Milnor K-group of k (see [**Mil70**]);

(4) for a strictly Hensel local scheme S over k and an integer n prime to $\mathrm{char}(k)$, one has

$$H^{p,q}(S,\mathbb{Z}/n) = \begin{cases} \mu_n^{\otimes q}(S) & \text{for } p = 0 \\ 0 & \text{for } p \neq 0 \end{cases}$$

where $\mu_n(S)$ is the groups of n-th roots of unity in S;

(5) one has $H^{p,q}(X,A) = CH^q(X, 2q-p; A)$. Here $CH^i(X,j;A)$ denotes the higher Chow groups of X introduced by S. Bloch in [**Blo86**], [**Blo94**]. In particular,

$$H^{2q,q}(X,A) = CH^q(X) \otimes A,$$

where $CH^q(X)$ is the classical Chow group of cycles of codimension q modulo rational equivalence.

The isomorphism between motivic cohomology and higher Chow groups leads to connections between motivic cohomology and algebraic K-theory, but we do not discuss these connections in the present lectures. See [**Blo94**], [**BL94**], [**FS02**], [**Lev98**] and [**SV00**].

Deeper comparison results include the theorem of M. Levine comparing $CH^i(X,j;\mathbb{Q})$ with the graded pieces of the gamma filtration in $K_*(X) \otimes \mathbb{Q}$ [**Lev94**], and the construction of the spectral sequence relating motivic cohomology and algebraic K-theory for arbitrary coefficients in [**BL94**] and [**FS02**].

The lectures in this book may be divided into two parts, corresponding to the fall and spring terms. The fall term lectures contain the definition of motivic cohomology and the proofs for all of the comparison results listed above except the last one. The spring term lectures include more advanced results in the theory of sheaves with transfers and the proof of the final comparison result (5).

The definition of motivic cohomology used here goes back to the work of Andrei Suslin in about 1985. As I understand it, when he came up with this definition he was able to prove the first three of the comparison results stated above. In particular, the proof of comparison (3) between motivic cohomology and Milnor's K-groups given in these lectures is exactly Suslin's original proof. The proofs of the last two comparison results, (4) and (5), are also based on results of Suslin. Suslin's formulation of the Rigidity Theorem ([**Sus83**]; see theorem 7.20) is a key result needed for the proof of (4), and Suslin's moving lemma (theorem 18A.1 below) is a key result needed for the proof of (5).

It took ten years and two main new ideas to finish the proofs of the comparisons (4) and (5). The first one, which originated in the context of the qfh-topology and was later transferred to sheaves with transfers (definition 2.1), is that the sheaf of finite cycles $\mathbb{Z}_{tr}(X)$ is the *free* object generated by X. This idea led to a group of results, the most important of which is lemma 6.23. The second idea, which is the main result of [**CohTh**], is represented here by theorem 13.8. Taken together they allow one to efficiently do homotopy theory in the category of sheaves with transfers.

A considerable part of the first half of the lectures is occupied by the proof of (4). Instead of stating it in the form used above, we prove a more detailed theorem. For a given weight q, the motivic cohomology groups $H^{p,q}(X,A)$ are defined as the hypercohomology (in the Zariski topology) of X with coefficients in a complex of sheaves $A(q)|_{X_{Zar}}$. This complex is the restriction to the small Zariski site of X (i.e., the category of open subsets of X) of a complex $A(q)$ defined on the site of all smooth schemes over k with the Zariski and even the étale topology. Restricting $A(q)$ to the small étale site of X, we may consider the étale version of motivic cohomology,

$$H_L^{p,q}(X,A) := \mathbb{H}^p_{ét}(X, A(q)|_{X_{ét}}).$$

The subscript L is in honor of Steve Lichtenbaum, who first envisioned this construction in [**Lic94**].

Theorem 10.2 asserts that the étale motivic cohomology of any X with coefficients in $\mathbb{Z}/n(q)$ where n is prime to char(k) are isomorphic to $H^p_{ét}(X,\mu_n^{\otimes q})$. This implies comparison result (4), since the Zariski and the étale motivic cohomology of a strictly Hensel local scheme X agree. There should also be an analog of (4) for the case of $\mathbb{Z}/\ell^r$ coefficients where $\ell = \text{char}(k)$, involving the logarithmic de Rham-Witt sheaves $\nu_r^q[-q]$, but I do not know much about it. We refer the reader to [**GL00**] for more information.

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