

Contents

Introduction	ix
1. Overview of Perelman's argument	x
2. Background material from Riemannian geometry	xvi
3. Background material from Ricci flow	xix
4. Perelman's advances	xxv
5. The standard solution and the surgery process	xxxix
6. Extending Ricci flows with surgery	xxxiv
7. Finite-time extinction	xxxvii
8. Acknowledgements	xl
9. List of related papers	xlii
Part 1. Background from Riemannian Geometry and Ricci flow	1
Chapter 1. Preliminaries from Riemannian geometry	3
1. Riemannian metrics and the Levi-Civita connection	3
2. Curvature of a Riemannian manifold	5
3. Geodesics and the exponential map	10
4. Computations in Gaussian normal coordinates	16
5. Basic curvature comparison results	18
6. Local volume and the injectivity radius	19
Chapter 2. Manifolds of non-negative curvature	21
1. Busemann functions	21
2. Comparison results in non-negative curvature	23
3. The soul theorem	24
4. Ends of a manifold	27
5. The splitting theorem	28
6. ϵ -necks	30
7. Forward difference quotients	33
Chapter 3. Basics of Ricci flow	35
1. The definition of Ricci flow	35
2. Some exact solutions to the Ricci flow	36
3. Local existence and uniqueness	39
4. Evolution of curvatures	41

5. Curvature evolution in an evolving orthonormal frame	42
6. Variation of distance under Ricci flow	45
7. Shi's derivative estimates	50
8. Generalized Ricci flows	59
Chapter 4. The maximum principle	63
1. Maximum principle for scalar curvature	63
2. The maximum principle for tensors	65
3. Applications of the maximum principle	67
4. The strong maximum principle for curvature	69
5. Pinching toward positive curvature	75
Chapter 5. Convergence results for Ricci flow	83
1. Geometric convergence of Riemannian manifolds	83
2. Geometric convergence of Ricci flows	90
3. Gromov-Hausdorff convergence	92
4. Blow-up limits	99
5. Splitting limits at infinity	100
Part 2. Perelman's length function and its applications	103
Chapter 6. A comparison geometry approach to the Ricci flow	105
1. \mathcal{L} -length and \mathcal{L} -geodesics	105
2. The \mathcal{L} -exponential map and its first-order properties	112
3. Minimizing \mathcal{L} -geodesics and the injectivity domain	116
4. Second-order differential inequalities for $\tilde{L}^{\bar{\tau}}$ and $L_x^{\bar{\tau}}$	119
5. Reduced length	129
6. Local Lipschitz estimates for l_x	133
7. Reduced volume	140
Chapter 7. Complete Ricci flows of bounded curvature	149
1. The functions L_x and l_x	149
2. A bound for $\min l_x^{\bar{\tau}}$	152
3. Reduced volume	164
Chapter 8. Non-collapsed results	169
1. A non-collapsing result for generalized Ricci flows	169
2. Application to compact Ricci flows	176
Chapter 9. κ -non-collapsed ancient solutions	179
1. Preliminaries	179
2. The asymptotic gradient shrinking soliton for κ -solutions	183
3. Splitting results at infinity	203
4. Classification of gradient shrinking solitons	206
5. Universal κ	220
6. Asymptotic volume	221

7. Compactness of the space of 3-dimensional κ -solutions	225
8. Qualitative description of κ -solutions	230
Chapter 10. Bounded curvature at bounded distance	245
1. Pinching toward positive: the definitions	245
2. The statement of the theorem	245
3. The incomplete geometric limit	247
4. Cone limits near the end \mathcal{E} for rescalings of U_∞	255
5. Comparison of the two types of limits	263
6. The final contradiction	265
Chapter 11. Geometric limits of generalized Ricci flows	267
1. A smooth blow-up limit defined for a small time	267
2. Long-time blow-up limits	271
3. Incomplete smooth limits at singular times	279
4. Existence of strong δ -necks sufficiently deep in a 2ϵ -horn	287
Chapter 12. The standard solution	293
1. The initial metric	293
2. Standard Ricci flows: The statement	295
3. Existence of a standard flow	296
4. Completeness, positive curvature, and asymptotic behavior	297
5. Standard solutions are rotationally symmetric	300
6. Uniqueness	306
7. Solution of the harmonic map flow	308
8. Completion of the proof of uniqueness	322
9. Some corollaries	325
Part 3. Ricci flow with surgery	329
Chapter 13. Surgery on a δ -neck	331
1. Notation and the statement of the result	331
2. Preliminary computations	334
3. The proof of Theorem 13.2	339
4. Other properties of the result of surgery	341
Chapter 14. Ricci Flow with surgery: the definition	343
1. Surgery space-time	343
2. The generalized Ricci flow equation	348
Chapter 15. Controlled Ricci flows with surgery	353
1. Gluing together evolving necks	353
2. Topological consequences of Assumptions (1) – (7)	356
3. Further conditions on surgery	359
4. The process of surgery	361
5. Statements about the existence of Ricci flow with surgery	362

6. Outline of the proof of Theorem 15.9	365
Chapter 16. Proof of non-collapsing	367
1. The statement of the non-collapsing result	367
2. The proof of non-collapsing when $R(x) = r^{-2}$ with $r \leq r_{i+1}$	368
3. Minimizing \mathcal{L} -geodesics exist when $R(x) \leq r_{i+1}^{-2}$: the statement	368
4. Evolution of neighborhoods of surgery caps	369
5. A length estimate	375
6. Completion of the proof of Proposition 16.1	391
Chapter 17. Completion of the proof of Theorem 15.9	395
1. Proof of the strong canonical neighborhood assumption	395
2. Surgery times don't accumulate	408
Part 4. Completion of the proof of the Poincaré Conjecture	413
Chapter 18. Finite-time extinction	415
1. The result	415
2. Disappearance of components with non-trivial π_2	420
3. Components with non-trivial π_3	429
4. First steps in the proof of Proposition 18.18	432
Chapter 19. Completion of the Proof of Proposition 18.24	437
1. Curve-shrinking	437
2. Basic estimates for curve-shrinking	441
3. Ramp solutions in $M \times S^1$	445
4. Approximating the original family Γ	449
5. The case of a single $c \in S^2$	453
6. The completion of the proof of Proposition 18.24	461
7. Proof of Lemma 19.31: annuli of small area	464
8. Proof of the first inequality in Lemma 19.24	481
Appendix. 3-manifolds covered by canonical neighborhoods	497
1. Shortening curves	497
2. The geometry of an ϵ -neck	497
3. Overlapping ϵ -necks	502
4. Regions covered by ϵ -necks and (C, ϵ) -caps	504
5. Subsets of the union of cores of (C, ϵ) -caps and ϵ -necks.	508
Bibliography	515
Index	519