CHAPTER 1

Introduction

This book builds upon and is an extension of [**22**]. Here, we complete a proof of the following:

Geometrization Conjecture: Any closed, orientable, prime[1] 3-manifold M contains a disjoint union of embedded incompressible[2] 2-tori and Klein bottles such that each connected component of the complement admits a complete, locally homogeneous Riemannian metric of finite volume.

Geometric 3-manifolds. Let us briefly review the nature of *geometric* 3-manifolds, that is to say complete, locally homogeneous Riemannian 3-manifolds of finite volume. Any such manifold is *modelled on* a complete, simply connected homogeneous manifold; that is to say, it is isometric to the quotient of a complete, simply connected homogeneous Riemannian manifold by a discrete group of symmetries acting freely. Here, *homogeneous* means that the isometry group of the manifold acts transitively on the manifold. Geometric 3-manifolds come in eight classes or types depending on the complete, simply connected homogeneous manifold they are modelled on. Here is the list, where, for simplicity we have restricted attention to the orientable case.

(1) **Hyperbolic:** These are manifolds of constant negative sectional curvature. The complete, simply connected example of this geometry is hyperbolic 3-space. It can be presented as $\mathbb{C} \times (0, \infty)$ with coordinates (z, y) with $z \in \mathbb{C}$ and $y \in \mathbb{R}^+$ and with the metric being $(|dz|^2 + dy^2)/y^2$. Complete hyperbolic manifolds are the quotients of hyperbolic 3-space by discrete, torsion-free, co-finite volume subgroups of its isometry group $PSL(2, \mathbb{C})$. These manifolds can be non-compact; a neighborhood of any end is diffeomorphic to $T^2 \times [0, \infty)$, and the torus cross-sections are all conformally equivalent and have areas that are decaying exponentially fast as we go to infinity.

[1]Not diffeomorphic to S^3 and with the property that every separating 2-sphere in the manifold bounds a 3-ball.

[2]Meaning the fundamental group of the surface injects into the fundamental group of the 3-manifold.

(2) **Flat:** These are manifolds with 0 sectional curvature. They are quotients of $\mathbb{R}^3$ by discrete torsion-free, co-finite volume subgroups of its isometry group. All such manifolds are compact and are finitely covered by a flat 3-torus.
(3) **Round:** These are manifolds with constant positive sectional curvature. They are quotients of S^3 with its natural round metric by finite groups of isometries acting freely. Examples are lens spaces and the Poincaré dodecahedral space.
(4) **Modelled on hyperbolic 2-space times $\mathbb{R}$:** At every point two of the sectional curvatures are 0 and the third is negative. These manifolds are finitely covered by the product of a hyperbolic surface of finite area with S^1. There are non-compact examples but every neighborhood of an end of one of these manifolds is diffeomorphic to $T^2 \times [0, \infty)$ and the torus cross sections have areas that decay exponentially as we go to infinity; one direction is of constant length and the other decays exponentially fast.
(5) **Modelled on $S^2 \times \mathbb{R}$:** There are exactly two examples here: $S^2 \times S^1$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$.
(6) **Modelled on Nil, the 3-dimensional nilpotent group:**

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Any example here is compact and is finitely covered by a non-trivial circle bundle over T^2.
(7) **Modelled on the universal covering of $PSL_2(\mathbb{R})$:** Any example is finitely covered by a circle bundle over a hyperbolic surface of finite area. Non-compact examples have ends that are diffeomorphic to $T^2 \times [0, \infty)$.
(8) **Modelled on Solv, the 3-dimensional solvable group**

$$\mathbb{R}^2 \rtimes \mathbb{R}^*.$$

All examples here are compact and are finitely covered by non-trivial T^2-bundles over S^1 with gluing diffeomorphism being an element of $SL(2, \mathbb{Z})$ of whose trace has absolute value > 2.

Manifolds of the last seven types are easily classified and their classifications have been long known, see [**31**]. Finite volume hyperbolic 3-manifold are not classified. It was only recently ([**9**]) that the hyperbolic 3-manifold of smallest volume was definitely established. It is known that the set of real numbers which are volumes of complete hyperbolic 3-manifolds is totally ordered by the usual order on $\mathbb{R}$ and also that the function that associates to a hyperbolic 3-manifold its volume is finite-to-one but not one-to-one. The Geometrization Conjecture reduces the problem of completely classifying 3-manifolds to the problem of classifying complete, finite volume hyperbolic

3-manifolds, or equivalently to classifying torsion-free, co-finite volume lattices in $PSL(2,\mathbb{C})$. These problems remain open.

There is another way to organize the list of eight types of geometric 3-manifolds that fits better with what Ricci flow with surgery produces:

(1) **Semi-positive type:** compact and modelled on either S^3 or $S^2 \times \mathbb{R}$.
(2) **Flat:** compact with a flat metric.
(3) **Essentially 1-dimensional:** geometric and modelled on Solv.
(4) **Essentially 2-dimensional:** the interior of a compact Seifert fibered 3-manifold with incompressible boundary; the interior of the base 2-dimensional orbifold of this Seifert fibration admits a complete hyperbolic or Euclidean metric of finite area. The manifold is geometric and modelled on either the universal covering of $PSL(2,\mathbb{R})$, the product of the hyperbolic plane with $\mathbb{R}$, or Nil.
(5) **Essentially 3-dimensional:** diffeomorphic to a complete, finite volume hyperbolic 3-manifold.

We shall use information about the structure of the *cusps* or neighborhoods of the ends of a finite volume hyperbolic 3-manifold. For any such orientable manifold H and any end $\mathcal{E}$ of H there is a neighborhood of $\mathcal{E}$ that is isometric to the quotient of subset of the upper half-space

$$\{(z,y)\in\mathbb{C}\times[y_0,\infty)\big|y_0>0\}$$

by a lattice subgroup of $\mathbb{C}$ acting on the first factor by translations and acting trivially on the second factor. The quotients of the slices $\{y=y_1\}$ are *horospherical tori* in the end. They foliate the neighborhood of the end. Each one of them cuts off a neighborhood of the end that is diffeomorphic to $T^2\times[0,\infty)$. A *truncation* of a complete hyperbolic 3-manifold of finite volume is the compact submanifold obtained by cutting off a neighborhood of each end of the manifold along some horospherical torus in that end.

Interpretation of the Geometrization Conjecture. The Geometrization Conjecture can be viewed as saying that any closed, orientable, prime 3-manifold M maps to a graph Γ in such a way that:

- The map is transverse to the midpoints of the edges and the pre-image of the mid-point of each edge is an incompressible torus in M.
- Let $\widehat{\mathcal{T}}$ be the union of the tori that are the pre-images of the mid-points of the edges of the graph, and let N be the result of cutting M open along $\widehat{\mathcal{T}}$ (so that N is a compact manifold whose boundary consists of two copies of $\widehat{\mathcal{T}}$). The manifold N naturally maps to the result $\widehat{\Gamma}$ of cutting Γ open along the midpoints of its edges. This map induces a bijection from the connected components of N to those of $\widehat{\Gamma}$, the latter being naturally indexed by the vertices of Γ.

- Each connected component of N is either a twisted I-bundle over the Klein bottle or its interior admits a complete, locally homogeneous metric of finite volume (automatically of one of the eight types listed above).

Statement for a general closed 3-manifold. The statement for a general closed, orientable 3-manifold is that there is a two-step process. The first step is to cut the manifold open along a maximal family of essential 2-spheres (essential in the sense that none of the 2-spheres bounds a 3-ball in the manifold and no two of the 2-spheres are parallel in the manifold), and then attach a 3-ball to each boundary component to produce a new closed 3-manifold, each component of which is automatically prime. The second step is to remove a disjoint family of incompressible tori and Klein bottles so that each connected component of the result has a complete, locally homogeneous metric of finite volume. Notice that there is a fundamental difference in these two steps in that in the first one one has to add material (the 3-balls) by hand whereas in the second step nothing is added. By definition, a closed, orientable, connected 3-manifold M satisfies the Geometrization Conjecture if and only if each of its prime factors does.

Uniqueness of the decomposition. Every closed 3-manifold has a decomposition into prime factors and these factors are unique up to order (and diffeomorphism). Given an orientable, prime 3-manifold M, consider all families of disjointly embedded tori and Klein bottles in M for which the conclusion of the Geometrization Conjecture holds. We choose one such family $\widehat{\mathcal{T}}$ with a minimal number of connected surfaces. Then for any other such family $\widehat{\mathcal{T}}'$ with the same number of connected surfaces as $\widehat{\mathcal{T}}$ there is isotopy of M carrying $\widehat{\mathcal{T}}'$ to $\widehat{\mathcal{T}}$. Thus, families $\widehat{\mathcal{T}}$ which satisfy the Geometrization Conjecture and have a minimal number of connected surfaces are unique up to isotopy. The geometric structures on the complementary components are not unique. For example, for those components that fiber over surfaces or Seifert fiber over two-dimensional orbifolds, there are the moduli of the geometric structure on those surfaces or orbifolds. In addition, there are non-compact examples of types (4) and (7) that are diffeomorphic.

1.1. Outline of the proof

The basic ingredient for the proof of the Geometrization Conjecture is the existence and properties of a Ricci flow with surgery. In [**22**], following Perelman's arguments, we showed that for any closed, oriented Riemannian 3-manifold $(M_0, g(0))$ there is a Ricci flow with surgery defined for all time with $(M_0, g(0))$ as the initial condition. This flow consists of a one-parameter family of compact, Riemannian 3-manifolds $(M_t, g(t))$, defined for $0 \leq t < \infty$. The underlying smooth manifolds are locally constant and the Riemannian metrics are varying smoothly except for a discrete set $\{t_i\}$ of surgery times. At these times the topological type of the M_t and Riemannian metrics $g(t)$ undergo discontinuous (but highly controlled) changes. One consequence of the nature of these changes is that if M_{t_0} satisfies the Geometrization Conjecture for some $t_0 < \infty$, then M_t satisfies the Geometrization Conjecture for all $0 \le t < \infty$, and in particular, M_0 satisfies the Geometrization Conjecture.

The strategy for proving the Geometrization Conjecture should now be clear. Start with any closed, oriented 3-manifold M_0. Impose a Riemannian metric $g(0)$ and construct the Ricci flow with surgery defined for all $0 \le t < \infty$ with $(M_0, g(0))$ as initial condition. Then show, for all t sufficiently large, that M_t satisfies the Geometrization Conjecture. This manuscript concentrates on the topology and geometry of the manifolds $(M_t, g(t))$ for all t sufficiently large.

The nicest statement one can imagine is that (after an appropriate rescaling) the Riemannian manifolds $(M_t, g(t))$ converge smoothly as $t \to \infty$ (meaning there are no surgery times for t sufficiently large and up to diffeomorphism as $t \to \infty$ the metrics $g(t)$ converge smoothly to a limiting metric $g(\infty)$) to a locally homogeneous metric, which is automatically complete and of finite volume since the M_t are compact. As we shall see, this essentially happens under certain topological assumptions, namely infinite fundamental group which (i) is not a non-trivial free product and (ii) does not contain a non-cyclic abelian subgroup. In this case the limiting metric is hyperbolic. But in general this scenario is too optimistic, not all 3-manifolds are geometric – somehow Ricci flow with surgery must allow for the cutting of the manifold into its prime factors and also allow for the torus decomposition.

A more accurate picture of what happens in general goes as follows. First of all the discontinuities (or surgeries) perform the connected sum decomposition including possibly redundant (i.e., trivial) such decompositions which simply split off new components diffeomorphic to the 3-sphere without changing the topology of the already existing components. For sufficiently large t, every connected component of M_t is either prime or diffeomorphic to S^3. Also, the surgeries remove all components with round metrics and with metrics modelled on $S^2 \times \mathbb{R}$. This is the full extent of the topological changes wrought by the surgeries. All of these statements follow from what was established in [**22**]. Thus, for all sufficiently large t we have the following: Each connected component of M_t either is prime or is diffeomorphic to S^3. Furthermore, if a connected component of M_t has finite fundamental group or has a fundamental group with an infinite cyclic subgroup of finite index, then it is diffeomorphic to S^3. As we shall show in Part I here, it turns out that given $(M_0, g(0))$, there is a finite list of complete hyperbolic manifolds $\mathcal{H} = H_1 \coprod \cdots \coprod H_k$ of finite volume such that for any truncation $\overline{\mathcal{H}}$ of $\mathcal{H}$ along horospherical tori the following holds. For all t sufficiently large, there is an embedding $\varphi_t\colon \overline{\mathcal{H}} \to M_t$ such that the rescaled pulled back metrics $\frac{1}{t}\varphi_t^* g(t)$ converge to the restriction of the hyperbolic metric on $\overline{\mathcal{H}}$. Furthermore, the image of the boundary tori $\widehat{\mathcal{T}}$ of $\overline{\mathcal{H}}$ under φ_t are incompressible tori in M_t. Lastly, the complement $\big(M_t \setminus \varphi_t(\mathrm{int}(\overline{\mathcal{H}}), g(t))\big)$ is locally volume collapsed on the negative curvature scale (details on this notion below). Actually, $\mathcal{H}$ depends only on the diffeomorphism type of M_0. The proof of the existence of $\mathcal{H}$ and the embeddings as required are rescaled versions, valid near infinity, of the main finite-time results that were used in [**22**] in the construction of a Ricci flow with surgery and an understanding of its singularity development. These deal with non-collapsing and bounded curvature at bounded distance for the rescaled metrics $\frac{1}{t}g(t)$ as $t \to \infty$.

To complete the proof of the Geometrization Conjecture we must show that the locally volume collapsed pieces satisfy the appropriate relative version of the Geometrization Conjecture.

The Relative Version of the Geometrization Conjecture: Let M be a compact, orientable 3-manifold whose boundary components are incompressible tori. Suppose that M is prime in the sense that every separating 2-sphere in M bounds a 3-ball and no component of M is diffeomorphic to S^3. Then there is a finite disjoint union $\widehat{\mathcal{T}}$ of incompressible tori and Klein bottles in $\mathrm{int}\, M$ such that every connected component of $\mathrm{int}\, M \setminus \widehat{\mathcal{T}}$ is either diffeomorphic to $T^2 \times \mathbb{R}$ or admits a complete, locally homogeneous metric of finite volume.

It is a direct argument to see that the relative version of the conjecture implies the original version of the conjecture when the manifold in question is closed.

Locally Volume Collapsed manifolds.

Definition 1.1.1. Suppose that M is a complete n-dimensional Riemannian manifold and $w > 0$ and $\psi\colon M \to [0,\infty)$ are given. Then we say that M is w *locally volume collapsed on scale* ψ if for every $x \in M$ we have

$$\mathrm{Vol}\, B(x, \psi(x)) \leq w\psi(x)^n.$$

Definition 1.1.2. Suppose that M is a complete, connected Riemannian manifold and that M does not have everywhere non-negative sectional curvature. Then we define

$$\rho\colon M \to (0,\infty)$$

such that for each $x \in M$ the infimum of the sectional curvatures on $B(x,\rho(x))$ is $-\rho^{-2}(x)$. Then $\rho(x)$ is the *negative curvature scale* at x. We say that M is w *locally volume collapsed on the negative curvature scale* if it is w locally volume collapsed on scale ρ.

The results on Ricci flow as $t \to \infty$ indicated above produce truncations of hyperbolic manifolds in $(M_t, g(t))$ whose complements are locally volume collapsed on the negative curvature scale. In fact, given $w > 0$ for all t sufficiently large the complement of the hyperbolic pieces in $(M_t, g(t))$ is w locally volume collapsed on the negative curvature scale. The idea for studying the complement is to first understand the balls $B(x, \rho(x)) \subset M_t$. Rescaling $g(t)$ by $\rho^{-2}(x)$ gives us a unit ball on which the sectional curvatures are bounded below by -1. This uniform lower curvature bound implies that any sequence of such balls with $t \to \infty$ has a subsequence which converges in a weak sense (the Gromov-Hausdorff sense) to a metric space that is weaker than a Riemannian manifold but still has some curvature structure, a so-called Alexandrov space.

Let us list the local models for the limit and the corresponding 3-dimensional models. By general results the Gromov-Hausdorff limit of a sequence of rescaled ballsρ $\rho^{-1}(x_n)B(x_n, \rho(x_n))$ is an Alexandrov ball of dimension ≤ 3 and curvature ≥ -1. The fact that the volume of the $\rho^{-1}(x_n)B(x_n, \rho(x_n))$ are tending to zero as $n \to \infty$, means that the limit has dimension ≤ 2. Also, it turns out that we can assume that $\rho(x_n) \leq$ diameter$(M_n)/2$. This implies that the limit is not a point and hence has dimension ≥ 1. Thus, the Gromov-Hausdorff limit is either 1- or 2-dimensional.

Let us describe what happens when the limit Alexandrov space is 1-dimensional. In this case the limit is either an interval (open, half-closed or closed) or a circle. The local structure of the 3-manifolds converging to such Alexandrov space near points converging to an interior point is a product of $S^2 \times (0,1)$ or $T^2 \times (0,1)$ where the surface fibers are of diameter converging to zero and the interval has length bounded away from zero. In fact we can view neighborhoods in the M_n as fibering over the limiting open interval with fibers of small diameter which are either S^2-fibers or T^2-fibers. Near an end point the structure is either a 3-ball or a punctured $\mathbb{R}P^3$ (when the fibers over nearby interior points are S^2) or a solid torus or a twisted I-bundle over the Klein bottle (when the fibers over the nearby interior points are 2-tori).

Now we consider the second possibility when the limiting Alexandrov space is 2-dimensional. As we shall see, we write a 2-dimensional Alexandrov space as a union four types of points for an appropriately chosen $\delta_0 > 0$:

- interior points that are the center of neighborhoods close to open balls in $\mathbb{R}^2$,
- points at which the space is an almost circular cone of cone angle $\leq 2\pi - \delta_0$,
- boundary points that are the center of neighborhoods close to open balls centered at boundary points of half-space, and

- boundary points at which is space is almost isometric to flat cone in $\mathbb{R}^2$ of cone angle $\leq \pi - \delta_0$.

The local models for neighborhoods of $x \in M_n$ in these four cases are:

- $S^1 \times \mathbb{R}^2$ with a Riemannian metric that is almost a product of a Riemannian metric on S^1 with a flat Riemannian metric on $\mathbb{R}^2$;
- a solid torus;
- $D^2 \times \mathbb{R}$;
- a 3-ball.

It turns out that these neighborhoods are glued together in a completely standard way. It then is an elementary problem in 3-dimensional topology to show that a 3-manifold covered by such neighborhoods intersecting in standard ways satisfies the relative version of the Geometrization Conjecture.

Thus, for all t sufficiently large, the $\big(M_t \setminus \Phi_t(\mathrm{int}(\overline{\mathcal{H}})), t^{-1}g(t)\big)$ satisfies the relative version of the Geometrization Conjecture. This then completes the proof of the Geometrization Conjecture for M_t for t sufficiently large, and consequently also for M_0.

1.2. Outline of manuscript

This manuscript has three parts. In Part I we cover the material in Sections 6 and 7 of [**28**], in particular the material from Lemma 6.3 through Section 7.3. This preliminary study of the limits as $t \to \infty$ of the t time-slices $(M_t, g(t))$ of a 3-dimensional Ricci flow with surgery produces a dichotomy. For any $w > 0$ and for all t sufficiently large (given w), the t time-slice is divided along incompressible tori into two pieces. The first piece is a disjoint union of components each of which is an almost complete hyperbolic manifold implying in particular that its interior is diffeomorphic to a complete hyperbolic manifold of finite volume. The second piece, $M_t(w, -)$, is locally w volume collapsed on the negative curvature scale. Then we turn to the manifolds $M_t(w, -)$ for w sufficiently small and t sufficiently large. The result we need to handle this case is stated by Perelman as Theorem 7.4 in [**28**], but no proof is provided in [**28**]. Part II of this work is devoted to giving a proof of Theorem 7.4 from [**28**] which is stated as Theorem 6.2.1 below. We review the background material from the theory of Gromov-Hausdorff convergence of metric spaces and the theory of Alexandrov needed to establish this result. In Part III we state and sketch the proof of the equivariant version of the Geometrization Conjecture for compact group actions on compact 3-manifolds.

1.3. Other approaches

The Geometrization Conjecture was proposed by W. Thurston in early 1980s. It includes the Poincaré Conjecture as a special case. Thurston himself established this conjecture for a large class of 3-manifolds, namely those containing an incompressible surface; i.e., an embedded surface of genus ≥ 1 whose fundamental group injects into the fundamental group of the 3-manifold, see [**26**].

While Perelman's approach is the most direct, there are other approaches to the Geometrization Conjecture using Ricci flow with surgery and variations of Theorem 6.2.1. As was indicated above, if a 3-manifold M admits an incompressible torus, then it falls into the class of 3-manifolds for which the Geometrization Conjecture had been established by Thurston himself. A detailed proof of the Geometrization Conjecture for those 3-manifolds was given in [**25**] and [**26**]. In view of this, it suffices to prove Theorem 6.2.1 for closed manifolds (again appealing to the Ricci flow results from [**27**] and the material in [**28**] preceding Theorem 7.4). This is route followed in [**16**] and [**4**]. A version of Theorem 6.2.1 for closed 3-manifolds has been proved in a series of papers of Shioya-Yamaguchi ([**33**], [**34**]). They did not make use of Assumption 3 of Theorem 6.2.1 on bounds on derivatives of curvature[3], so their result is more general and can be applied to 3-manifolds that do not necessarily arise from Ricci flow. However, because they are not relying on estimates on higher derivatives of the curvature as stated in Assumption 3, to prove their result, Shioya-Yamaguchi need to use a stability theorem on Alexandrov spaces. This stability theorem is due to Perelman and its proof was given in an unpublished manuscript in 1993. Recently, V. Kapovitch posted a preprint, [**15**], which proposes a more readable proof for this stability theorem of Perelman. Putting all these together, one has a Perelman-Shioya-Yamaguchi-Kapovitch proof of Theorem 6.2.1 for closed manifolds without the assumption of higher curvature bounds. As we have indicated, this proof requires a more knowledge about Alexandrov spaces, in particular knowledge about 3-dimensional Alexandrov spaces than the proof we present. It also relies on Thurston's result for manifolds with incompressible tori to give a complete proof of Geometrization.

Our presentation of the collapsing space theory is motivated by, and to a large extent follows, the Shioya-Yamaguchi paper [**34**], however it differs from theirs in two fundamental aspects. First of all, as indicated above, we follow Perelman and add the assumption concerning the control of the higher derivatives of the curvature, thus allowing us to simplify the argument and in particular avoid the use of the stability theorem for Alexandrov spaces. Also, again following Perelman, we directly treat the case of non-empty boundary so that we do not have to appeal to Thurston's proof of the Geometrization Conjecture for manifolds containing an incompressible surface.

[3]Their proof was mostly for manifolds with curvature bounded from below, but the extension to the case of curvature locally bounded from below is not difficult as they point out in an appendix.

There is another approach to the proof of the Geometrization Conjecture due to Bessières et al [**2**] which avoids using Theorem 6.2.1 below. This argument also relies on Thurston's theorem that 3-manifolds with incompressible surfaces satisfy the Geometrization Conjecture, so that one only needs to consider the case when the entire closed 3-manifold is collapsed. Rather than appealing to the theory of Alexandrov spaces, this approach relies on other deep works in geometry and topology, e.g., results on the Gromov norms of 3-manifolds.

1.4. Acknowledgements

This work grew out of our attempt over the last 10 years to understand Perelman's original preprints on Ricci flow with surgery and its consequences – the Poincaré Conjecture and Geometrization of 3-manifolds. During that time we benefited from many conversations with others engaged in a similar quest. Of special importance to us were numerous conversations with Bruce Kleiner and John Lott. Discussions with Michel Boileau were also helpful, especially on the topological aspects which occupy the Part II of this book More recently several long conversations with Richard Bamler help us clarify some of the issues and arguments in the long-time analytic arguments that appear in Chapter 4 and 5. He also carefully read these sections pointing out many corrections. It is a pleasure to thank all these people for their help in the preparation of this book.