Preface

The Clay Mathematics Institute 2014 Summer School, "Periods and Motives: Feynman amplitudes in the 21st century" took place from June 30 to July 25, 2014, at the Instituto de Ciencias Matemáticas – ICMAT (Institute of Mathematical Sciences) in Madrid, Spain.

The central theme of the summer school was, as the title indicates, periods and motives and their interplay with Feynman amplitudes in perturbative quantum field theory. The school brought together more than 80 selected students from around the world to attend to several courses presenting background as well as advanced tools and techniques necessary to play an active role in future mathematics research on periods and motives.

The first three weeks were organized into four grand lecture series supplemented by exercise and problem sessions. The four series were

- **Spencer Bloch** (University of Chicago, USA): *Lectures on the Mathematics of Feynman Amplitudes*,
- **José Ignacio Burgos Gil** (ICMAT-CSIC, Spain): *Motivic multiple zeta values*,
- **Lars Kindler** and **Kay Rülling** (Freie Universität Berlin, Germany): *ℓ-adic Galois representations of function fields over finite fields I*,
- **Matilde Marcolli** (Caltech, USA): *Feynman integrals, periods and motives.*

The fourth week of the school consisted of five mini-courses which were more advanced and focused on specific topics. They also included presentations of some open problems which are under current investigation, ranging from pure mathematics to theoretical physics:

- **Joseph Ayoub** (University of Zürich, Switzerland): *Motives, motivic Galois groups and periods,*
- **Francis Brown** (Oxford University, UK and CNRS, France): *Multiple modular values,*
- **James Drummond** (CNRS, France): *Bootstraps for scattering amplitudes in $N = 4$ super Yang–Mills theory,*
- **Claude Duhr** (Durham University, UK): *Feynman integrals, scattering amplitudes and the Hopf algebra of multiple polylogarithms,*
- **Hélène Esnault** (Freie Universität Berlin, Germany): *ℓ-adic Galois representations of function fields over finite fields II.*

This is the first of two volumes containing articles that grew out of the lectures presented at the summer school. The volume at hand includes the following three lecture notes:

- **Feynman Integrals in Mathematics and Physics,**
  by Spencer Bloch,
- **Feynman integrals and periods in configuration spaces,**
  by Özgür Ceyhan and Matilde Marcolli,
- **Introductory course on ℓ-adic sheaves and their ramification theory on curves,**
  by Lars Kindler and Kay Rülling

Spencer Bloch's contribution is based on his summer school lectures. The focus lies on some mathematical questions associated to Feynman integrals, including Hodge structures, relations with string theory, and monodromy (Cutkosky rules).

The notes by Lars Kindler and Kay Rülling resulted from their joint introductory lecture series on ℓ-adic Galois representations of function fields and ramification theory over curves. They also provided the necessary foundations for the advanced course of Prof. Esnault's in the last week of the school.

The contribution by Özgür Ceyhan and Matilde Marcolli is based on the lectures given by Matilde Marcolli. In her course she explained the occurrence of motives, in particular mixed Tate motives, and periods in the computation of Feynman integrals in two different settings: in momentum space and in configuration space.

The second volume contains the single manuscript

- **Multiple zeta values: from numbers to motives,**
  by José Ignacio Burgos Gil and Javier Fresán. First chapter in collaboration with Ulf Kühn.

It is based on the series of lectures given by José Ignacio Burgos Gil, which aimed at explaining the results of Francis Brown, Pierre Deligne, Alexander Goncharov and Tomohide Terasoma on the algebra of multiple zeta values, as well as presenting some of the required prerequisites needed to get familiar with the fundamental works of the aforementioned authors.

The term "period", understood as the integral of an algebraic differential form over a topological cycle on an algebraic variety, goes back to the study of elliptic integrals. The term originated from the period of a periodic elliptic function, which can be computed as an elliptic integral. Since then, periods have played an important role in algebraic geometry. Kontsevich and Zagier have formalized the notion of "ring of periods", that is, the set of complex numbers whose real as well as imaginary part are given in terms of absolutely convergent multiple integrals of a rational differential form with rational coefficients, over a domain in $\mathbb{R}^n$, which is defined by polynomial equations and inequalities with rational coefficients. They conjectured, roughly speaking, that the only relations among periods are the "obvious" ones coming from geometry, that is linearity, change of variables formula and Stoke's theorem. This conjecture encompasses many transcendence conjectures.

The theory of motives was introduced by Grothendieck as a "universal cohomology theory" to explain the common properties of different cohomology theories. One may describe motives as an intermediate step between algebraic varieties and their linear invariants (cohomology). The motives associated to smooth projective varieties are called "pure", while the motives associated to quasi-projective and singular varieties are called "mixed" (they consist of different pure pieces). A complete and satisfactory theory of mixed motives is not yet available, although some pieces of it have already been developed. For instance the category of mixed Tate motives over number fields (the simplest mixed motives) is well understood. Motives form an active area of current research in pure mathematics. As an example we mention a recent letter of Deligne to Drinfeld, in which he proved the finiteness of the number of irreducible lisse $\mathbb{Q}_\ell$-sheaves with bounded ramification, up to isomorphism and up to twist, on a smooth variety defined over a finite field. This result is inspired by motivic considerations, and its proof relies on Lafforgue's Langlands correspondence over curves. This was the topic of the series of lectures by Esnault, Kindler and Rülling.

Being intermediate between algebraic varieties and cohomology theories, the theory of motives provides a conceptual framework for the study of periods. For instance the periods that arise from mixed Tate motives over $\mathbb{Z}$ are combinations of multiple zeta values.

Multiple zeta values (MZVs) are defined by iterated sums. Their analysis can be traced back to the work of Euler in the 18th century, but it was not until the last decades that the theory of MZVs has been intensively developed. Indeed, a systematic exploration of these objects only started with the seminal work of Zagier in the early 1990s. MZVs are special values of two related families of special functions, i.e., multiple zeta functions and multiple polylogarithms. These two families of functions stand at the intersection of several mathematical areas, including algebraic geometry, Lie group theory (Kashiwara–Vergne conjecture, recently proved by Alekseev and Meinrenken following an approach initiated by Torossian), algebra and combinatorics.

Kontsevich discovered a representation of MZVs in terms of iterated integrals, which shows that they appear as periods of mixed Tate motives. Moreover, the fact that they originate from iterated integrals as well as from iterated sums gives rise to one of the most fundamental properties of them, i.e., they satisfy three families of nontrivial polynomial relations, regrouped under the terminology "double shuffle and regularization". One of the prominent open problems (roughly speaking, the conjectural algebraic independence of MZVs over the rational numbers, modulo double-shuffle and regularization relations) is still far from being solved, and is related to Kontsevich's and Zagier's conjecture on the structure of the ring of periods.

The study of MZV's through the theory of motives is the topic of the course by Burgos Gil and will be the main theme of the second volume.

Recently, the notion of Feynman diagrams entered the picture of motives and periods. In perturbative quantum field theory (QFT) Feynman diagrams appear as coefficients of power expansions in coupling parameters. These diagrams encode highly intricate integrals over a large number of variables. The correspondence between diagrams and integrals is established via the Feynman rules of a QFT, which associate to any Feynman diagram a Feynman amplitude. Their efficient calculation is of foremost importance for theoretical predictions, however the integrals thus obtained are complicated and hard to compute. Moreover, the occurrence of ultraviolet and infrared singularities demands the introduction of regularization and renormalization procedures in order to obtain meaningful results. The former procedure replaces a divergent integral with a function of an additional—non-physical—parameter, which has a pole or singularity at the particular value of the parameter that corresponds to the original integral. It is otherwise well defined and finite at nearby values of the parameter. Renormalization provides a mean for extracting finite values from the regularized expressions in a way that is consistent with physics as well as the combinatorics of nested diagrams. Over the years, several regularization methods and renormalization schemes have been developed.

Thanks to the pathbreaking work of Bloch, Esnault, Kreimer, Connes, Marcolli and others, a beautiful relation between Feynman integrals and periods has emerged. In particular Bloch, Esnault, and Kreimer showed that Feynman integrals (modulo divergences) can be written as integrals of algebraic differential forms on an algebraic variety, integrated over a cycle with boundary contained on a divisor in the variety. It turns out that, in many simple situations, this period corresponds to a mixed Tate motive, and thus to multiple zeta values. The interplay between Feynman integrals and motives, and some consequences of this relation are the topic of the courses by Bloch and Marcolli.

Two remarks are in order. First, recent computations involving more complicated Feynman diagrams clearly indicate that we are starting to uncover a much richer interplay between the theory of motives and QFT, one that will involve other kinds of motives and special functions. Second, we should emphasize that the relation between QFT and the motivic point of view is not only of "theoretical" interest, but has already given rise to practical consequences. As an example we mention the recent work of Goncharov et al. on the amplitude associated to a particular type of Feynman diagram, whose original expansion spanned an expression covering several pages. Thanks to the motivic interpretation and the development of MPL functional equations, this expansion has been simplified to just two lines, which significantly reduced the time needed to compute it. This result underlines the importance that these developments are having in theoretical physics and is a prelude to future progress.

It goes without saying that we are most thankful for the help and expertise we received from the referees in preparing this volume. A large summer school like this one can only succeed in a place centered around the idea of bringing together established and aspiring mathematicians to fruitfully discuss and do research. ICMAT is such a place and we greatly acknowledge the availability of its facilities that were key in making this a fruitful event. We are as well thankful for the warm hospitality and professionality that we have experienced during the summer school.

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