

Contents

Introduction	1
Chapter 1. Statements of the Main Results	17
1.0. Measures attached to spacings of eigenvalues	17
1.1. Expected values of spacing measures	23
1.2. Existence, universality and discrepancy theorems for limits of expected values of spacing measures: the three main theorems	24
1.3. Interlude: A functorial property of Haar measure on compact groups	25
1.4. Application: Slight economies in proving Theorems 1.2.3 and 1.2.6	25
1.5. Application: An extension of Theorem 1.2.6	26
1.6. Corollaries of Theorem 1.5.3	28
1.7. Another generalization of Theorem 1.2.6	30
1.8. Appendix: Continuity properties of “the i 'th eigenvalue” as a function on $U(N)$	32
Chapter 2. Reformulation of the Main Results	35
2.0. “Naive” versions of the spacing measures	35
2.1. Existence, universality and discrepancy theorems for limits of expected values of naive spacing measures: the main theorems bis	37
2.2. Deduction of Theorems 1.2.1, 1.2.3 and 1.2.6 from their bis versions	38
2.3. The combinatorics of spacings of finitely many points on a line: first discussion	42
2.4. The combinatorics of spacings of finitely many points on a line: second discussion	45
2.5. The combinatorics of spacings of finitely many points on a line: third discussion: variations on $\text{Sep}(a)$ and $\text{Clump}(a)$	49
2.6. The combinatorics of spacings of finitely many points of a line: fourth discussion: another variation on $\text{Clump}(a)$	54
2.7. Relation to naive spacing measures on $G(N)$: Int, Cor and TCor	54
2.8. Expected value measures via INT and COR and TCOR	57
2.9. The axiomatics of proving Theorem 2.1.3	58
2.10. Large N COR limits and formulas for limit measures	63
2.11. Appendix: Direct image properties of the spacing measures	65
Chapter 3. Reduction Steps in Proving the Main Theorems	73
3.0. The axiomatics of proving Theorems 2.1.3 and 2.1.5	73
3.1. A mild generalization of Theorem 2.1.5: the φ -version	74
3.2. M -grid discrepancy, L cutoff and dependence on the choice of coordinates	77
3.3. A weak form of Theorem 3.1.6	89

3.4.	Conclusion of the axiomatic proof of Theorem 3.1.6	90
3.5.	Making explicit the constants	98
Chapter 4.	Test Functions	101
4.0.	The classes $\mathcal{T}(n)$ and $\mathcal{T}_0(n)$ of test functions	101
4.1.	The random variable $Z[n, F, G(N)]$ on $G(N)$ attached to a function F in $\mathcal{T}(n)$	103
4.2.	Estimates for the expectation $E(Z[n, F, G(N)])$ and variance $\text{Var}(Z[n, F, G(N)])$ of $Z[n, F, G(N)]$ on $G(N)$	104
Chapter 5.	Haar Measure	107
5.0.	The Weyl integration formula for the various $G(N)$	107
5.1.	The $K_N(x, y)$ version of the Weyl integration formula	109
5.2.	The $L_N(x, y)$ rewriting of the Weyl integration formula	116
5.3.	Estimates for $L_N(x, y)$	117
5.4.	The $L_N(x, y)$ determinants in terms of the sine ratios $S_N(x)$	118
5.5.	Case by case summary of explicit Weyl measure formulas via S_N	120
5.6.	Unified summary of explicit Weyl measure formulas via S_N	121
5.7.	Formulas for the expectation $E(Z[n, F, G(N)])$	122
5.8.	Upper bound for $E(Z[n, F, G(N)])$	123
5.9.	Interlude: The $\sin(\pi x)/\pi x$ kernel and its approximations	124
5.10.	Large N limit of $E(Z[n, F, G(N)])$ via the $\sin(\pi x)/\pi x$ kernel	127
5.11.	Upper bound for the variance	133
Chapter 6.	Tail Estimates	141
6.0.	Review: Operators of finite rank and their (reversed) characteristic polynomials	141
6.1.	Integral operators of finite rank: a basic compatibility between spectral and Fredholm determinants	141
6.2.	An integration formula	143
6.3.	Integrals of determinants over $G(N)$ as Fredholm determinants	145
6.4.	A new special case: $O_-(2N + 1)$	151
6.5.	Interlude: A determinant-trace inequality	154
6.6.	First application of the determinant-trace inequality	156
6.7.	Application: Estimates for the numbers $\text{eigen}(n, s, G(N))$	159
6.8.	Some curious identities among various $\text{eigen}(n, s, G(N))$	162
6.9.	Normalized “ n ’th eigenvalue” measures attached to $G(N)$	163
6.10.	Interlude: Sharper upper bounds for $\text{eigen}(0, s, SO(2N))$, for $\text{eigen}(0, s, O_-(2N + 1))$, and for $\text{eigen}(0, s, U(N))$	166
6.11.	A more symmetric construction of the “ n ’th eigenvalue” measures $\nu(n, U(N))$	169
6.12.	Relation between the “ n ’th eigenvalue” measures $\nu(n, U(N))$ and the expected value spacing measures $\mu(U(N), \text{sep. } k)$ on a fixed $U(N)$	170
6.13.	Tail estimate for $\mu(U(N), \text{sep. } 0)$ and $\mu(\text{univ}, \text{sep. } 0)$	174
6.14.	Multi-eigenvalue location measures, static spacing measures and expected values of several variable spacing measures on $U(N)$	175
6.15.	A failure of symmetry	183
6.16.	Offset spacing measures and their relation to multi-eigenvalue location measures on $U(N)$	185

6.17.	Interlude: “Tails” of measures on \mathbb{R}^r	189
6.18.	Tails of offset spacing measures and tails of multi-eigenvalue location measures on $U(N)$	192
6.19.	Moments of offset spacing measures and of multi-eigenvalue location measures on $U(N)$	194
6.20.	Multi-eigenvalue location measures for the other $G(N)$	195
Chapter 7.	Large N Limits and Fredholm Determinants	197
7.0.	Generating series for the limit measures $\mu(\text{univ, sep.'s } a)$ in several variables: absolute continuity of these measures	197
7.1.	Interlude: Proof of Theorem 1.7.6	205
7.2.	Generating series in the case $r = 1$: relation to a Fredholm determinant	208
7.3.	The Fredholm determinants $E(T, s)$ and $E_{\pm}(T, s)$	211
7.4.	Interpretation of $E(T, s)$ and $E_{\pm}(T, s)$ as large N scaling limits of $E(N, T, s)$ and $E_{\pm}(N, T, s)$	212
7.5.	Large N limits of the measures $\nu(n, G(N))$: the measures $\nu(n)$ and $\nu(\pm, n)$	215
7.6.	Relations among the measures μ_n and the measures $\nu(n)$	225
7.7.	Recapitulation, and concordance with the formulas in [Mehta]	228
7.8.	Supplement: Fredholm determinants and spectral determinants, with applications to $E(T, s)$ and $E_{\pm}(T, s)$	229
7.9.	Interlude: Generalities on Fredholm determinants and spectral determinants	232
7.10.	Application to $E(T, s)$ and $E_{\pm}(T, s)$	235
7.11.	Appendix: Large N limits of multi-eigenvalue location measures and of static and offset spacing measures on $U(N)$	235
Chapter 8.	Several Variables	245
8.0.	Fredholm determinants in several variables and their measure-theoretic meaning (cf. [T-W])	245
8.1.	Measure-theoretic application to the $G(N)$	248
8.2.	Several variable Fredholm determinants for the $\sin(\pi x)/\pi x$ kernel and its \pm variants	249
8.3.	Large N scaling limits	251
8.4.	Large N limits of multi-eigenvalue location measures attached to $G(N)$	257
8.5.	Relation of the limit measure $\text{Off } \mu(\text{univ, offsets } c)$ with the limit measures $\nu(c)$	263
Chapter 9.	Equidistribution	267
9.0.	Preliminaries	267
9.1.	Interlude: zeta functions in families: how lisse pure \mathcal{F} 's arise in nature	270
9.2.	A version of Deligne's equidistribution theorem	275
9.3.	A uniform version of Theorem 9.2.6	279
9.4.	Interlude: Pathologies around (9.3.7.1)	280
9.5.	Interpretation of (9.3.7.2)	283
9.6.	Return to a uniform version of Theorem 9.2.6	283
9.7.	Another version of Deligne's equidistribution theorem	287

Chapter 10. Monodromy of Families of Curves	293
10.0. Explicit families of curves with big G_{geom}	293
10.1. Examples in odd characteristic	293
10.2. Examples in characteristic two	301
10.3. Other examples in odd characteristic	302
10.4. Effective constants in our examples	303
10.5. Universal families of curves of genus $g \geq 2$	304
10.6. The moduli space $\mathcal{M}_{g,3K}$ for $g \geq 2$	307
10.7. Naive and intrinsic measures on $USp(2g)^{\#}$ attached to universal families of curves	315
10.8. Measures on $USp(2g)^{\#}$ attached to universal families of hyperelliptic curves	320
Chapter 11. Monodromy of Some Other Families	323
11.0. Universal families of principally polarized abelian varieties	323
11.1. Other “rational over the base field” ways of rigidifying curves and abelian varieties	324
11.2. Automorphisms of polarized abelian varieties	327
11.3. Naive and intrinsic measures on $USp(2g)^{\#}$ attached to universal families of principally polarized abelian varieties	328
11.4. Monodromy of universal families of hypersurfaces	331
11.5. Projective automorphisms of hypersurfaces	335
11.6. First proof of 11.5.2	335
11.7. Second proof of 11.5.2	337
11.8. A properness result	342
11.9. Naive and intrinsic measures on $USp(\text{prim}(n, d))^{\#}$ (if n is odd) or on $O(\text{prim}(n, d))^{\#}$ (if n is even) attached to universal families of smooth hypersurfaces of degree d in \mathbb{P}^{n+1}	346
11.10. Monodromy of families of Kloosterman sums	347
Chapter 12. GUE Discrepancies in Various Families	351
12.0. A basic consequence of equidistribution: axiomatics	351
12.1. Application to GUE discrepancies	352
12.2. GUE discrepancies in universal families of curves	353
12.3. GUE discrepancies in universal families of abelian varieties	355
12.4. GUE discrepancies in universal families of hypersurfaces	356
12.5. GUE discrepancies in families of Kloosterman sums	358
Chapter 13. Distribution of Low-lying Frobenius Eigenvalues in Various Families	361
13.0. An elementary consequence of equidistribution	361
13.1. Review of the measures $\nu(c, G(N))$	363
13.2. Equidistribution of low-lying eigenvalues in families of curves according to the measure $\nu(c, USp(2g))$	364
13.3. Equidistribution of low-lying eigenvalues in families of abelian varieties according to the measure $\nu(c, USp(2g))$	365
13.4. Equidistribution of low-lying eigenvalues in families of odd-dimensional hypersurfaces according to the measure $\nu(c, USp(\text{prim}(n, d)))$	366

13.5.	Equidistribution of low-lying eigenvalues of Kloosterman sums in evenly many variables according to the measure $\nu(c, USp(2n))$	367
13.6.	Equidistribution of low-lying eigenvalues of characteristic two Kloosterman sums in oddly many variables according to the measure $\nu(c, SO(2n+1))$	367
13.7.	Equidistribution of low-lying eigenvalues in families of even-dimensional hypersurfaces according to the measures $\nu(c, SO(\text{prim}(n, d)))$ and $\nu(c, O_-(\text{prim}(n, d)))$	368
13.8.	Passage to the large N limit	369
Appendix: Densities		373
AD.0.	Overview	373
AD.1.	Basic definitions: $W_n(f, A, G(N))$ and $W_n(f, G(N))$	373
AD.2.	Large N limits: the easy case	374
AD.3.	Relations between eigenvalue location measures and densities: generalities	378
AD.4.	Second construction of the large N limits of the eigenvalue location measures $\nu(c, G(N))$ for $G(N)$ one of $U(N)$, $SO(2N+1)$, $USp(2N)$, $SO(2N)$, $O_-(2N+2)$, $O_-(2N+1)$	381
AD.5.	Large N limits for the groups $U_k(N)$: Widom's result	385
AD.6.	Interlude: The quantities $V_r(\varphi, U_k(N))$ and $V_r(\varphi, U(N))$	386
AD.7.	Interlude: Integration formulas on $U(N)$ and on $U_k(N)$	390
AD.8.	Return to the proof of Widom's theorem	392
AD.9.	End of the proof of Theorem AD.5.2	399
AD.10.	Large N limits of the eigenvalue location measures on the $U_k(N)$	401
AD.11.	Computation of the measures $\nu(c)$ via low-lying eigenvalues of Kloosterman sums in oddly many variables in odd characteristic	403
AD.12.	A variant of the one-level scaling density	405
Appendix: Graphs		411
AG.0.	How the graphs were drawn, and what they show	411
	Figure 1	413
	Figure 2	414
	Figure 3	415
	Figure 4	416
References		417