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Preface

The mathematical language of classical physics is based upon real numbers. Configuration spaces and phase spaces of classical systems are differentiable manifolds, and physical laws are expressed by differential equations in the real domain.

The mathematical language of quantum physics is based upon complex numbers, and it would be natural to expect that the complex analytic and the algebraic geometry should replace the differential geometry of the classical period. In a sense, this is what has been happening during the last two or three decades, with the advent of scattering matrices, twistors, strings propagating in the ten–dimensional space–time, quantum cohomology, and M–theory. The mathematical physics of the dawning New Age sets as its ultimate goal construction of the universal quantum theory of all interactions including gravity. In the meantime it distanced itself from the traditional preoccupations of experimental particle physics and cosmology and did not just become heavily mathematicized, but in fact almost merged with mathematics. What made this development so exciting for mathematicians was that physicists brought not only a wealth of fresh insights, ideas, and problems, but also heuristic tools of great power and a certain freedom of expression which supplanted a rather strait–laced mood in the mathematical community of the fifties and sixties.

This book summarizes some of the mathematical developments that took place in the last decade or so and that focus on the notion of Quantum Cohomology, introduced by Cumrun Vafa (see [Va]) and Edward Witten. However, this is a mathematical monograph, and the reader who is interested in physical motivation and history will have to refer to other sources: see [MirS1], [MirS2], and the references therein.

Quantum Cohomology is a construction which endows with an additional highly non–linear structure the usual cohomology space $H = H^*(V)$ with complex coefficients of any projective algebraic (or symplectic) manifold V. The resulting structure, suitably axiomatized by B. Dubrovin, is called the *Frobenius manifold.* Interest in this axiomatization depends on the fact that there exist several general constructions of Frobenius manifolds, seemingly quite different, and unexpected isomorphisms between Frobenius manifolds of various classes (dualities, including Mirror duality). The first part of the book, Chapters I–IV, is dedicated to this notion and its multiple interconnections with geometry, differential equations, operads, and perturbation formalism. A more detailed summary can be found in the Introduction.

Although Quantum Cohomology in the proper sense of the word is invoked in several places in the first part of the book (Introduction, examples in Chapter II, axiomatic exposition in Chapter III), its systematic treatment is postponed until

Chapters V and VI. But whereas Chapters I–IV are reasonably self–contained and provide complete proofs of the main results, the final part of the book is meant as an introduction to the original papers and cannot replace them. In fact, the construction of Quantum Cohomology requires considerable algebraic geometric technique: the machinery of the Deligne–Mumford and Artin stacks, including intersection theory and the deformation theory for them. Already for schemes, this machinery takes hundreds of pages in standard sources: see [Ful] for intersection theory and [Il] for the deformation formalism. A monograph exhaustively treating the algebraic–geometric background for Quantum Cohomology is highly desirable. Hopefully, this book might stimulate its appearance.

A word of warning is in order: although the Mirror Conjecture initially provided the main stimulus for studying Quantum Cohomology, it is not treated in this book. On the one hand, this subject is still in a state of flux and rapid change. On the other hand, the body of firmly established facts, among which Givental's proof of the Mirror Identity of [COGP] for quintics occupies the prominent position (see [Giv2], [BiCPP], [Pa3], and the further development in [LiLY]), still constitutes only a fraction of the extremely varied and fascinating insights into what might be called the Mirror Phenomenon, which is an ambitious collective project bridging the physical and the mathematical communities.

Acknowledgements. Work on this book started in 1992–93, when Iz Singer and I led a seminar on the Mirror Conjecture at MIT. Contacts with Cumrun Vafa and Ed Witten were crucial at this stage.

The book took its present form after several lecture courses given at the Max–Planck–Institut für Mathematik in Bonn in 1994–98, and many shorter lecture courses delivered at various summer schools and conferences.

The vision of Quantum Cohomology expounded here was greatly influenced by Maxim Kontsevich, with whom I collaborated at the Max–Planck–Institut in 1994 and later. A part of the results in this book, including the axiomatic treatment of Gromov–Witten invariants, the theory of operadic tensor products in Chapter III, and the treatment of gravitational descendants in Chapter VI, is based on our joint work. Boris Dubrovin's papers, in particular his lecture notes [D2], provide the basic source of information about Frobenius manifolds, and most of the key definitions and theorems of Chapters I–II are due to him. The notion of weak Frobenius manifolds was introduced in my joint paper with Claus Hertling. Ralph Kaufmann's study of tensor products in the categories of local and global (as opposed to the operadic and formal) Frobenius manifolds is also incorporated in Chapter III. Chapter IV can serve as a brief introduction to operads and perturbation series. Our presentation owes much to the work of Misha Kapranov and Ezra Getzler. The final part of the book prepares and presents the construction of Gromov–Witten invariants which in genus zero are the coefficients of the formal series (potential) embodying Quantum Cohomology, and in higher genus provide a far–reaching extension of this theory in which much work remains to be done. This construction is due to Kai Behrend and Barbara Fantechi: see [Beh] and [BehF]. It was motivated by the earlier construction of the Gromov–Witten invariants in the symplectic and complex–analytic context due to J. Li and G. Tian: see [LiT1] and [LiT2]. The Behrend–Fantechi theory uses in essential ways stacks and their intersection theory, which are reviewed in Chapter V of this book. It is based on the work of Pierre Deligne, David Mumford, Mike Artin, Vistoli, and many others.

During the course of the work, I profited from many enlightening conversations and/or correspondence with my colleagues, friends, and collaborators mentioned above, and with Victor Batyrev, Sergei Barannikov, Alexander Givental, Vadim Schechtman, Sergey Merkulov, Markus Rosellen, and Don Zagier. Their contributions are gratefully acknowledged.

CHAPTER 0

Introduction: What Is Quantum Cohomology?

0.1. An overview. We start with a rather detailed overview of the two central themes of this book: Quantum Cohomology and Frobenius Manifolds.

Let $H = H^*(V, k)$ be the cohomology space of a projective algebraic manifold V with coefficients in a field k of characteristic zero.

The quantum cohomology $H^*_{quant}(V)$ consists of H *plus* an additional piece of data which can be described in at least three seemingly unrelated ways:

i) As a formal series ("potential") Φ in coordinates on H whose third derivatives can be used to define on $K \otimes H$ the structure of a $\mathbf{Z}_2$–graded commutative associative algebra, K being the ring of all formal series in the coordinates.

ii) As a family of polylinear cohomological operations $[m]: \; H^{\otimes n} \to H, \; n \geq 2$, indexed by all homology classes $m \in H_*(\overline{M}_{0,n+1}, k)$. Here $\overline{M}_{0,n+1}$ denotes the moduli space of stable $(n+1)$–marked algebraic curves of genus zero (cf. 0.2, Chapter III, §3, and [Ke]).

iii) As a "completely integrable system" on the tangent sheaf of the formal spectrum Spf(K) (i.e. a formal completion of H at the origin considered as a linear supermanifold). In this context, the system itself consists of a one–parametric family of flat connections on the tangent bundle of Spf(K).

The structures i)–iii) can and must first be described abstractly. We will do it in more detail in 0.2–0.4, and then discuss in what sense they are equivalent in 0.5. In the main body of the book and in 0.4.1 below, they are called *formal Frobenius manifolds*: cf. Chapter III, §4. Chapters I and II introduce and study Frobenius manifolds in more geometric categories (differentiable, analytic, algebraic).

A constructive description of these structures on cohomology spaces, i.e. quantum cohomology of V in the proper sense, involves counting (parametrized) rational curves on V (Gromov–Witten invariants) and is thus related to some classical problems of enumerative algebraic geometry. In 0.6 and 0.7, we will give two examples of the potential Φ constructed in this way, for $V = \mathbf{P}^2$ and for $V =$ a quintic hypersurface in $\mathbf{P}^4$. For more systematic treatment, see Chapter III, §5, and Chapter VI. The geometry underlying these constructions leads naturally to the descriptions of types i) and ii). Quantum cohomology is a functor from the category of smooth projective algebraic manifolds *and their isomorphisms* to the category of formal Frobenius manifolds. The study of its properties with respect to the more general morphisms has not been carried out systematically as yet, and remains an important problem. However, the quantum Künneth formula is reasonably well understood via the general construction of the tensor product of Frobenius manifolds.

In the language of physicists, quantum cohomology is a mathematical theory of the topological quantum sigma–model with target space V (in the tree approximation). In the context of the Mirror Conjecture (cf. below) it is also referred to as the *A–model.*

Algebraic geometry also furnishes several constructions of (formal and non–formal) Frobenius manifolds of different nature. They are supported by the moduli spaces of various kinds: versal deformations of an isolated singularity (Kyoji Saito's theory, physicist's Landau–Ginzburg models), Hurwitz spaces, and moduli spaces of Calabi–Yau manifolds and their extended (formal) versions constructed by S. Barannikov and M. Kontsevich. The relevant potentials and metrics are constructed via periods of algebraic integrals and variations of Hodge structure. In the context of the Mirror Conjecture, we call such constructions *B–models.* We will discuss an example in 0.8, cf. also 0.9. For more general constructions see I.4.5 (A_n–singularities), III.8 (an axiomatic version of K. Saito's construction), III.9 and III.10 (Barannikov–Kontsevich's theory.)

If a potential Φ obtained by counting curves on a manifold can be identified with another potential Ψ related to the periods on another manifold, this gives a strong hold on the analytical properties of Φ and on the behavior of its coefficients. Existence of such an identification first suggested for quintics in [COGP] and proved in [Giv2] is (a part of) the famous Mirror Conjecture for Calabi–Yau manifolds. It is already clear that it constitutes a part of a much vaster mirror pattern, whose formulation suggested by M. Kontsevich might involve identification of a triangulated category related to coherent sheaves (A–model) with another triangulated category related to the Lagrange and Kähler geometry (B–model.)

We will now fix notation for the remaining part of the Introduction. Denote by (H, g) a $\mathbf{Z}_2$–graded finite–dimensional k–linear space H endowed with an even non–degenerate graded symmetric bilinear form g. Let $\{\Delta_a\}$ be a basis of H, $g_{ab} = g(\Delta_a, \Delta_b)$, $(g^{ab}) = (g_{ab})^{-1}$, and $\Delta = \sum \Delta_a g^{ab} \otimes \Delta_b \in H \otimes H$. Denote by $\{x^a\}$ the dual basis of the dual space of H. We will consider x^a as formal independent graded commuting variables of the same parity as Δ_a. Put $K = k[[x^a]]$; this is the same as the completed symmetric algebra of the dual space. Put $\partial_a = \partial/\partial x^a : K \to K$. We will write Φ_a instead of $\partial_a \Phi$, etc.

0.2. Definition. *A formal solution Φ to the associativity equations on (H, g), or simply a potential, is a formal series $\Phi \in K$ satisfying the following differential equations:*

$$\forall a, b, c, d : \sum_{ef} \Phi_{abe} g^{ef} \Phi_{fcd} = (-1)^{\tilde{x}_a(\tilde{x}_b + \tilde{x}_c)} \sum_{ef} \Phi_{bce} g^{ef} \Phi_{fad} \tag{0.1}$$

where generally $\tilde{x}$ denotes the $\mathbf{Z}_2$–parity of x.

Define a K–linear multiplication $\circ$ on $H_K := K \otimes_k H$ by the rule

$$\Delta_a \circ \Delta_b = \sum_{cd} \Phi_{abc} g^{cd} \Delta_d. \tag{0.2}$$

Clearly, it is supercommutative.

0.2.1. Proposition. *a) $(H_K, \circ)$ is associative iff Φ is a potential. Multiplication $\circ$ does not change if one adds to Φ a polynomial of degree ≤ 2 in x^a.*

b) An element Δ_0 of the basis is a unit with respect to $\circ$ iff it is even and $\Phi_{0bc} = g_{bc}$ for all b, c. Equivalently:

$$\Phi = \frac{1}{6}\, g_{00}(x^0)^3 + \frac{1}{2}\sum_{c\neq 0} x^0 x^b x^c g_{bc} + \textit{terms independent of } x^0. \tag{0.3}$$

If $H = H^*(V,k), g =$ Poincaré pairing ($g_{ab} = \int_V \Delta_a \wedge \Delta_b$), and Φ is obtained via the Gromov–Witten counting of rational curves on V, then $(H_K, \circ)$ is called *the quantum cohomology ring of V.*

0.3. Moduli spaces $\overline{M}_{0n}$. Before giving the next definition, we recall some basic facts about stable curves of genus 0 with $n \geq 3$ labeled pairwise distinct non–singular points $(x_1, \dots, x_n)$ (cf. [Kn1], [Ke]). Such a curve is a tree of $\mathbf{P}^1$'s: any two irreducible components are either disjoint or intersect transversely at one point. Each component must contain at least three special (singular or labeled) points.

The space $\overline{M}_{0n}$ is a smooth projective algebraic manifold of dimension $n-3$ supporting a universal family $X_n \to \overline{M}_{0n}$ of stable curves whose labeled points are given by n structure sections $x_i :\ \overline{M}_{0n} \to X_n$. An open subset ("big cell") parametrizes $\mathbf{P}^1$ with n pairwise distinct points on it. The boundary, or infinity, of $\overline{M}_{0n}$ is stratified according to the degeneration type of fibers of X_n: the combinatorics of the incidence tree of the curve and the distribution of labeled points among the components. The number of the components diminished by one is the codimension of the stratum. Of course, the closure of such a stratum includes its own boundary corresponding to further degeneration.

In particular, the irreducible boundary divisors D_σ of $\overline{M}_{0n}$ correspond to the stable (unordered) 2–partitions $\sigma :\ \{1, \dots, n\} = S_1 \coprod S_2, |S_i| \geq 2$, describing the distribution of the labeled points among the two $\mathbf{P}^1$'s at the generic point of D_σ. A choice of the ordering of the partition defines an identification of D_σ with $\overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1}, n_i = |S_i|$: on each $\mathbf{P}^1$ add to the labeled points the intersection point of the two components. Thus we have a family of closed embeddings

$$\varphi_\sigma :\ \overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1} \to \overline{M}_{0n} \tag{0.4}$$

inducing the restriction morphisms of the cohomology groups with coefficients in k,

$$\varphi_\sigma^* :\ H^*(\overline{M}_{0n}) \to H^*(\overline{M}_{0,n_1+1}) \otimes H^*(\overline{M}_{0,n_2+1}). \tag{0.5}$$

Besides, S_n acts on $\overline{M}_{0n}$, $H^*(\overline{M}_{0n})$ and on the partitions σ by renumbering the labeled points, and (0.5) is compatible with this action.

0.3.1. Definition. *A structure of the Cohomological Field Theory (CohFT) (or an algebra over the operad $H_*\overline{M}_0$, cf. Chapter IV and* [GeK2]*) on (H, g) consists of a family of S_n–equivariant $\mathbf{Z}_2$–even polylinear maps*

$$I_n :\ H^{\otimes n} \to H^*(\overline{M}_{0n}, k),\ n \geq 3, \tag{0.6}$$

satisfying the following conditions. For every stable 2–partition σ of $\{1, \dots, n\}$ and all homogeneous $\gamma_1, \dots, \gamma_n \in H$ we have

$$\varphi_\sigma^*(I_n(\gamma_1 \otimes \dots \otimes \gamma_n)) = \epsilon(\sigma)(I_{n_1+1} \otimes I_{n_2+1}) \left(\bigotimes_{i\in S_1} \gamma_i \otimes \Delta \otimes (\bigotimes_{i\in S_2} \gamma_i) \right), \tag{0.7}$$

where $\epsilon(\sigma)$ *is the sign of the permutation induced by* σ *on the odd–dimensional classes* γ_i*.*

Another way of looking at such a structure is to make a partial dualization with the help of the Poincaré pairing on $\overline{M}_{0,n+1}$ and g on H. Then one can rewrite $(0.6)_{n+1}$ as

$$H_*(\overline{M}_{0,n+1}) \otimes H^{\otimes n} \to H, \ n \geq 2, \tag{0.8}$$

that is, to interpret every class $m \in H_*(\overline{M}_{0,n+1})$ as an n–ary multiplication $[m]$ on H linearly depending on $[m]$. Then (0.7) gives a complex system of quadratic identities between these multiplications which are best described in the operadic formalism.

However, the situation simplifies considerably if we restrict ourselves to looking only at those multiplications that correspond to the fundamental classes $[\overline{M}_{0,n+1}] \in H_*(\overline{M}_{0,n+1})$ and denote them simply by

$$[\overline{M}_{0,n+1}] \otimes (\gamma_1 \otimes \cdots \otimes \gamma_n) \mapsto (\gamma_1, \ldots, \gamma_n), \ n \geq 2. \tag{0.9}$$

These multiplications are supercommutative. Moreover:

0.3.2. Proposition. *The identities (0.7) imply the following generalized associativity equations for these multiplications: for any* $\alpha, \beta, \gamma, \delta_1, \ldots, \delta_n \in H, \ n \geq 0$, *we have*

$$\sum_\sigma \epsilon'(\sigma)((\alpha, \beta, \delta_i \,|\, i \in S_1), \gamma, \delta_j \,|\, j \in S_2)$$

$$= \sum_\sigma \epsilon''(\sigma)(\alpha, (\beta, \gamma, \delta_i \,|\, i \in S_1), \delta_j \,|\, j \in S_2) \tag{0.10}$$

where σ *runs over 2–partitions* $\sigma : \ \{1, \ldots, n\} = S_1 \coprod S_2$ *(non–necessarily stable), and* ϵ *are the standard signs.*

In particular, for $n = 0, 1$ *we get respectively*

$$((\alpha, \beta), \gamma) = (\alpha, (\beta, \gamma)),$$

$$((\alpha, \beta), \gamma, \delta) + (-1)^{\tilde{\gamma}\tilde{\delta}}((\alpha, \beta, \delta), \gamma) = (\alpha, (\beta, \gamma, \delta)) + (\alpha, (\beta, \gamma), \delta). \tag{0.11}$$

Remarkably, this family of n–ary multiplications is actually equivalent to the whole structure described in 0.3.1: cf. sketch of the proof of Theorem 0.5 below and its complete version in III.4.

In conclusion, let us formally compare the system of operations (0.8) on $H = H^*(V, k)$ (in the situation of quantum cohomology) with the more traditional Steenrod operations.

i) Steenrod powers are defined on the cohomology with coefficients in $\mathbf{F}_p$ whereas we allow characteristic zero coefficients.

ii) Steenrod powers generate an *algebra* whereas $[m], \ m \in H_*(\overline{M}_{0,n+1})$, are elements of an *operad*.

iii) Steenrod powers are defined solely in terms of topology of V, whereas to construct $[m]$ we need additionally the structure of algebraic (or symplectic) manifold, in order to be able to define holomorphic curves on V.

0.4. Frobenius manifolds. The term "completely integrable system" is used rather indiscriminately in a wide variety of contexts. The notion relevant here was introduced by B. Dubrovin (cf. [D1], [D2]) under the name of Frobenius manifold. We start with the formal version.

0.4.1. Definition. *a) The structure of a formal Frobenius manifold on (H, g) is a one–parametric system of flat connections on the module of derivations of K/k given by its covariant derivatives*

$$\nabla_{\lambda,\partial_a}(\partial_b) := \lambda \sum_{cd} A_{abc} g^{cd} \partial_d = \lambda \sum_d A_{ab}{}^d \partial_d, \tag{0.12}$$

where $A_{abc} \in K$ is a symmetric tensor, λ an even parameter.

b) This structure is called a potential one if the tensor $\partial_d A_{abc}$ is totally symmetric.

More generally, a Frobenius manifold (M, g, A) (in any of the standard geometric categories: smooth, analytic, algebraic (super)manifolds) is a manifold M endowed with a flat metric g and a tensor field A of rank 3 such that if we write the components of A in local g–flat coordinates, the conditions of 0.4.1 a) and eventually b) are satisfied.

0.5. Theorem. *For a given (H, g), there exists a natural bijection between the sets of the additional structures described above:*

i) Formal solutions of the associativity equations on (H, g), modulo terms of degree ≤ 2.

ii) Structures of the CohFT on (H, g).

iii) Structures of the formal potential Frobenius manifold on (H, g).

Easy part of the proof (sketch). We will first describe maps $ii) \to i) \to iii)$.

$ii) \to i)$. Assume that we have on (H, g) the structure of CohFT given by some maps I_n as in (0.6). Construct first the symmetric polynomials

$$Y_n : \ H^{\otimes n} \to k, \ Y_n(\gamma_1 \otimes \dots \otimes \gamma_n) := \int_{\overline{M}_{0n}} I_n(\gamma_1 \otimes \dots \otimes \gamma_n) \tag{0.13}$$

and form the series

$$\Phi(x) := \sum_{n \geq 3} \frac{1}{n!} Y_n \left((\sum_a x^a \Delta_a)^{\otimes n} \right). \tag{0.14}$$

Keel ([Ke]) has described the linear relations between the cohomology classes of the boundary divisors D_σ defined in 0.3. Namely, choose a quadruple of pairwise distinct indices $i, j, k, l \in \{1, \dots, n\}, \ n \geq 4$. For a stable 2–partition $\sigma = \{S_1, S_2\}$ write $ij\sigma kl$ if $i, j \in S_1, \ k, l \in S_2$ for some ordering of the parts. Then the $\{ijkl\}$–th Keel's relation is

$$\sum_{\sigma : ij\sigma kl} D_\sigma \cong \sum_{\sigma : ik\sigma jl} D_\sigma \quad \text{in } H^*(\overline{M}_{0n}). \tag{0.15}$$

Geometrically, it follows from the fact that the two sides of (0.15) are the two fibers of the projection

$$\overline{M}_{0n} \to \overline{M}_{0,\{ijkl\}} \cong \overline{M}_{0,4} = \mathbf{P}^1$$

forgetting all the labeled points except for x_i, x_j, x_k, x_l. The space $\overline{M}_{0,4}$ has exactly three boundary points corresponding to the three stable partitions of $\{i, j, k, l\}$. In (0.15) we use two of them.

Notice that the existence of the forgetful morphism is a non–trivial geometric fact, because on the level of fibers of X_n (i.e. geometric points of the moduli) it involves contracting those components that become unstable: cf. [Kn1] and V.4.4.

If we restrict $I_n(\gamma_1 \otimes \cdots \otimes \gamma_n)$ to D_σ using (0.7) then integrate over D_σ and take into account (0.15), we will get a series of bilinear identities: $\forall i, j, k, l$

$$\sum_{\sigma:ij\sigma kl} \epsilon(\sigma)(Y_{|S_1|+1} \otimes Y_{|S_2|+1}) \left(\bigotimes_{p\in S_1} \gamma_p \otimes \Delta \otimes (\bigotimes_{q\in S_2} \gamma_q) \right)$$

$$(0.16) \qquad = \sum_{\sigma:ik\sigma jl} \epsilon(\sigma)(Y_{|S_1|+1} \otimes Y_{|S_2|+1}) \left(\bigotimes_{p\in S_1} \gamma_p \otimes \Delta \otimes (\bigotimes_{q\in S_2} \gamma_q) \right).$$

On the other hand, writing the associativity equations (0.1) for the series (0.14), one can directly show that they reduce to a subfamily of the relations (0.16), which implies the whole family by the standard polarization argument. Thus Φ encodes the same amount of information as $\{Y_n\}$ and (0.16).

i) $\to$ *iii)*. Given a potential Φ, we simply put $A_{abc} = \partial_a \partial_b \partial_c \Phi$. This is in fact a bijection, because given (H, g, A), the symmetry of A_{abc} and $\partial_d A_{abc}$ implies the existence of Φ with $A_{abc} = \partial_a \partial_b \partial_c \Phi$, and the curvature vanishing equation $\nabla_\lambda^2 = 0$ implies the associativity equations for Φ.

Difficult part of the proof. It remains to show that nothing is lost or gained in the passage from I_n to Y_n, i.e., that the arrow *ii)* $\to$ *i)* is both injective and surjective. Injectivity is again easy, because using (0.7) consecutively one sees that the knowledge of Y_n allows us to reconstruct integrals of I_n along all the boundary strata, whose classes span $H^*(\overline{M}_{0n})$. But surjectivity requires considerable work. Basically, it reduces to showing that the ad hoc formulas for the integrals over the boundary strata do define a cohomology class, i.e., satisfy all the linear relations between the classes. A remarkable reformulation of this property asserts that the homology of moduli spaces forms a Koszul operad. For details, see the main text.

0.5.1. Remark. What this last argument additionally shows is that the structure of a CohFT on (H, g) can be replaced by the structure of a $Comm_\infty$–algebra given by a family of n–ary operations, one for each $n \geq 2$, satisfying the generalized associativity relations (0.10). This structure looks simpler because it does not involve the moduli spaces $\overline{M}_{0n}$ which look completely irrelevant also for the remaining two descriptions. However, there are at least three reasons not to eliminate the moduli spaces, and even to consider *ii)* as the most important structure.

a) In the applications to quantum cohomology, the geometry of the Gromov–Witten invariants naturally involves total maps I_n, not just their top–dimensional terms Y_n describing the physicists' correlation functions.

b) The higher genus theory of Gromov–Witten invariants furnishes cohomological operations parametrized by all homology classes of the moduli spaces of stable curves $\overline{M}_{gn}$, and unlike the genus zero case, they cannot be reconstructed from the operations corresponding to the fundamental classes, because there exist cohomology classes vanishing on the boundary.

c) The whole theory can be extended to include the so–called gravitational descendants. The respective correlation functions can be calculated, if we know the complete Gromov–Witten invariants, but not if we know only its top degree parts. For details, see Chapter VI.

d) Returning to the genus zero case, in the abstract framework of $Comm_\infty$–algebras, there exists an operation of their tensor product. It can be defined as follows:

$$(H', g', I'_n) \otimes (H'', g'', I''_n) = (H' \otimes H'', g' \otimes g'', I_n),$$

where I_n are given by

$$I_n(\gamma'_1 \otimes \gamma''_1 \otimes \ldots \otimes \gamma'_n \otimes \gamma''_n) := \epsilon(\gamma', \gamma'') I'_n(\gamma'_1 \otimes \ldots \otimes \gamma'_n) \wedge I''_n(\gamma''_1 \otimes \ldots \otimes \gamma''_n).$$

This is an important and natural operation necessary e.g. for the formulation of the quantum Künneth formula. However, it seems impossible to construct this product without invoking $\overline{M}_{0n}$. In fact, its existence is a reflection of the fact that $H_*(\overline{M}_{0n})$ forms an operad of coalgebras, and not just linear spaces.

In particular, consider C_∞–algebras of rank 1 (i.e. $\dim(H)$=1). In terms of potentials, they correspond to arbitrary power series in one variable $\Phi(x) = \sum_{n\geq 3} \frac{C_n}{n!} x^n$ because the associativity equations in one variable are satisfied identically. Hence we can define a tensor multiplication of such series. It turns out to be given by quite non–trivial polynomials in the coefficients involving a generalization of the Petersson–Weil volumes of $\overline{M}_{0n}$ (see Theorem III.6.5.)

We will now describe some examples.

0.6. Quantum cohomology of $\mathbf{P}^2$. First, we have

$$H^{2i}(\mathbf{P}^2, k) = k\Delta_i,\ \Delta_i = c_1(\mathcal{O}(1))^i,\ i = 0, 1, 2.$$

Denote by $N(d)$ (for $d \geq 1$) the number of rational curves of degree d in $\mathbf{P}^2$ passing through $3d-1$ points in general position. The first few values of $N(d)$ starting with $d = 1$ are

$$1,\ 1,\ 12,\ 620,\ 87304,\ 26312976,\ 14616808192.$$

The potential $\Phi^{\mathbf{P}^2}$, by definition, is

$$\Phi^{\mathbf{P}^2}(x\Delta_0 + y\Delta_1 + z\Delta_2) = \frac{1}{2}(xy^2 + x^2 z) + \sum_{d=1}^{\infty} N(d) \frac{z^{3d-1}}{(3d-1)!} e^{dy}$$

$$:= \frac{1}{2}(xy^2 + x^2 z) + \varphi(y, z). \tag{0.17}$$

A direct computation shows:

0.6.1. Proposition. *The associativity equations (0.1) for the potential (0.17) are equivalent to one differential equation for φ:*

$$\varphi_{zzz} = \varphi_{yyz}^2 - \varphi_{yyy}\varphi_{yzz} \tag{0.18}$$

which is in turn equivalent to the family of recursive relations uniquely defining $N(d)$ *starting with* $N(1) = 1$*:*

$$N(d) = \sum_{k+l=d} N(k)N(l)k^2 l \left[l \binom{3d-4}{3k-2} - k \binom{3d-4}{3k-1} \right], \ d \geq 2. \tag{0.19}$$

0.6.2. Geometry. The identities (0.19) showing that $(H^*(\mathbf{P}^2, \mathbf{Q}), g, \Phi^{\mathbf{P}^2})$ is actually an instance of the structure described above were first proved by M. Kontsevich. He skillfully applied an old trick of enumerative geometry: in order to understand the number of solutions of a numerical problem, try to devise a degenerate case of the problem where it becomes easier. In this setting, Kontsevich starts with a new problem having *one–dimensional space of solutions* and looks at two different degeneration points in the line of solutions.

More precisely, fix $d \geq 2$ and consider a generic configuration in $\mathbf{P}^2$ consisting of two labeled points y_1, y_2, two labeled lines l_1, l_2, and a set of $3d-4$ unlabeled points Y. Look at the space of quintuples $(\mathbf{P}^1, x_1, x_2, x_3, x_4, f)$ where $x_i \in \mathbf{P}^1$ are pairwise distinct points, $f: \mathbf{P}^1 \to \mathbf{P}^2$ is a map of degree d such that $f(x_i) = y_i$ for $i = 1, 2$, $f(x_i) \in l_i$ for $i = 3, 4$, and $Y \subset f(\mathbf{P}^1)$. We identify such diagrams if they are isomorphic (identically on $\mathbf{P}^2$). Then we can assume that $(x_1, x_2, x_3, x_4) = (1, 0, \infty, \lambda)$. If λ is fixed and generic, the number of maps does not depend on it. Kontsevich counts it by first letting $\lambda \to \infty$, and then letting $\lambda \to 1$. In the stable limit, $\mathbf{P}^1$ degenerates into two projective lines, and we must sum over all possible distributions of $\{x_i\} \cup f^{-1}(Y)$ on these components. Comparison of the two limits furnishes (0.19).

To make all of this rigorous, one must introduce not only the moduli spaces of stable curves, but also the moduli spaces of stable maps $\overline{M}_{0n}(\mathbf{P}^2)$ parametrizing Kontsevich–stable maps to $\mathbf{P}^2$. Then it will become clear that the calculation we sketched above furnishes a particular case of the identities (0.16).

Chapter V is a systematic introduction to the study of stable maps.

0.7. Quantum cohomology of a three–dimensional quintic. Let $V \subset \mathbf{P}^4$ be a smooth quintic hypersurface. Its even cohomology has rank four and is spanned by the powers of a hyperplane section, the odd cohomology has rank 204 and consists of three–dimensional classes. For a generic even element $\gamma = \sum x^a \Delta_a \in H^*(V)$, denote by y the coefficient at $\Delta_1 := c_1(\mathcal{O}(1))$ and put

$$\Phi^V(\gamma) = \frac{1}{6}(\gamma^3) + \sum_{d \geq 1} n(d) Li_3(e^{dy}) \tag{0.20}$$

where (γ^3) means the triple self–intersection index, $Li_3(z) = \sum_{m \geq 1} z^m/m^3$, and $n(d)$ is the appropriately defined number of rational curves of degree d on V.

Before we turn to the definition of $n(d)$, let us notice that in this case the associativity equations are satisfied with whatever choice of these coefficients! This can be checked by a direct calculation. An arguably more enlightening argument runs as follows: in quantum cohomology of any V, the associativity equations must reflect the degeneration properties of rational curves on V as was the case with $\mathbf{P}^2$. Now, on a quintic, the rational curves are typically rigid so that there is nothing to degenerate. (See, however, the discussion in 0.7.3.)

Algebraically, the quantum cohomology ring of the projective plane with $\circ$–multiplication (cf. 0.2 above) is semisimple whereas that of the quintic is nilpotent. B. Dubrovin has developed a rich theory of the Frobenius manifolds with pointwise semisimple multiplication in a tangent sheaf: see I.3, II.4 below, and the recent preprint [DZh3]. This should eventually provide analytic tools for the numerical theory of rational curves on Fano varieties. On the contrary, potentials of the Calabi–Yau threefolds are conjecturally constrained by the Mirror identities rather than associativity equations. Nevertheless, there are at least two contexts in which the Calabi–Yau quantum cohomology can be understood as a limiting case of the semisimple situation. First, Givental's approach via equivariant cohomology produces a family of Frobenius manifolds, whose generic fiber is semisimple, and a special fiber is the relevant quantum cohomology. Second, Gepner's approach via Landau–Ginzburg models conjecturally realizes quantum cohomology of certain Calabi–Yau hypersurfaces as a closed Frobenius submanifold of a generically semisimple Frobenius manifold: see III.8.7.2.

0.7.1. A definition of the numbers $n(d)$. A naive argument showing that the number of rational curves of degree d on V must be finite runs as follows. The space of maps $f:\ \mathbf{P}^1 \to \mathbf{P}^4,\ (t_0,t_1) \mapsto (f_0(t_0,t_1),\dots,f_4(t_0,t_1))$ of degree d is a Zariski open subset in the space $\mathbf{P}^{5d+4}$ of the coefficients of forms f_i. The condition $F(f_0(t_0,t_1),\dots,f_4(t_0,t_1)) = 0$ where $F=0$ is the equation of V furnishes $5d+1$ equations on these coefficients. If these equations were independent, the space of solutions would be 3–dimensional. It is acted upon effectively by $\mathrm{Aut}(\mathbf{P}^1)$ (linear reparametrizations) which leaves us with finitely many equivalence classes of unparametrized curves.

Unfortunately, it is unknown whether there exists a sufficiently generic V for which these equations actually are independent after deleting degenerating maps. The symplectic approach to this problem going back to M. Gromov uses a drastic deformation of the complex structure of V destroying its integrability. In this way the problem is put into general position. More precisely, only isolated non–singular pseudoholomorphic spheres in V with normal sheaf $\mathcal{O}(-1)+\mathcal{O}(-1)$ survive; they can be counted directly, and their number is stable.

Another strategy which we will sketch below does not leave the algebraic geometric framework and even allows one to calculate $n(d)$ using the same degeneration philosophy as in example 0.6, although in a rather different setting. This construction is also due to M. Kontsevich ([Ko7]).

Consider a pair (C,f) where C is a connected curve of genus 0 (a tree of $\mathbf{P}^1$'s), and $f:\ C\to\mathbf{P}^4$ is a map of degree d such that the inverse image of any point in $f(C)$ is either 0–dimensional, or a stable curve of genus zero whose labeled points are intersection points with non–contracted components. Such pairs (C,f) are called (Kontsevich–)stable maps (of genus zero, to $\mathbf{P}^4$). There exists a diagram

$$\overline{M}(\mathbf{P}^4,d) \leftarrow \overline{C}_d \to \mathbf{P}^4$$

where $\overline{M}(\mathbf{P}^4,d)$ is the moduli space (or rather stack) of stable maps of degree d, and $\overline{C}_d$ is the universal curve on it. Denote the right arrow (the universal map) by φ_d, and the left arrow by π. Put $\mathcal{E}_d=\varphi_d^*(\mathcal{O}(5)),\ E_d=\pi_*(\mathcal{E}_d)$.

0.7.2. *a) $\overline{M}(\mathbf{P}^4,d)$ is a smooth orbifold of dimension $5d+1$.*

b) E_d is a locally free sheaf on it of rank $5d+1$.

0.7.3. Definition. $n(d) := c_{5d+1}(E_d)$.

Motivation for this definition is simple: if a quintic V is defined by $s=0,\ s \in \Gamma(\mathbf{P}^4, \mathcal{O}(5))$, then s produces a section $\overline{s} \in \Gamma(\overline{M}(\mathbf{P}^4, d), E_d)$, and

$$c_{5d+1}(E_d) = \text{the number of zeroes of } \overline{s}$$

calculated with appropriated multiplicities. But $\overline{s}([\varphi]) = 0$ for $[\varphi] \in \overline{M}(\mathbf{P}^4, d)$ iff $\varphi_d(\overline{C}_{d,[\varphi]}) \subset V$. Thus we simply avoided the problem of assigning ad hoc multiplicities to actual rational curves on V (which may have a "wrong" normal sheaf, singularities, or come in families) by reducing it to a calculation of Chern numbers on orbifolds.

Moreover, we simultaneously created a setting in which degeneration can easily occur. In fact, instead of considering curves in a fixed quintic V, we are now looking at curves in $\mathbf{P}^4$ lying in V, i.e., treat V as an "incidence condition", similar to $3d-1$ points in $\mathbf{P}^2$ in 0.6 above. We may now freely change the equation $s=0$ for V and can take, e.g., $s = \prod_{i=0}^4 s_i$ where $s_i \in \Gamma(\mathbf{P}^4, \mathcal{O}(1))$ are coordinates in $\mathbf{P}^4$.

To make sense of the problem of "counting rational curves on the algebraic symplex $V_\infty := \bigcup_{i=0}^4 \{s_i = 0\}$" Kontsevich proceeds as follows. Consider the G_m–action on the whole setting $(\mathbf{P}^4, \mathcal{O}(5), \overline{M}(\mathbf{P}^4, d))$ given by $s_i \mapsto e^{\lambda_i t} s_i,\ i = 0, \dots, 4$, where λ_i are the parameters of this action considered as independent variables.

0.7.4. Claim. *a) V_∞ is the only reduced quintic fixed with respect to this action.*

b) Fixed points of this action in $\overline{M}(\mathbf{P}^4, d)$ consist of stable pairs (C, f) where C is a tree of $\mathbf{P}^1$'s mapped by f to the 1–skeleton of V_∞ (consisting of 10 projective lines).

Each such (C, f) has a combinatorial invariant (τ, λ) which is, roughly speaking, the dual tree τ of C, each vertex of which is labeled either by zero (if the respective component of C is contracted by f), or by the name of the line in the skeleton to which it is mapped and the degree of this map.

Bott's formula for Chern numbers of a bundle E in a situation where G_m acts upon the whole setting involves a sum of local contributions over the connected components of the set of G_m–fixed points, each contribution depending on the weights of G_m on the normal sheaf of the component and on the restriction of E upon it.

Kontsevich shows that in our case we get a sum

$$n(d) = \sum w(\tau, \lambda) \tag{0.21}$$

where the Bott multiplicities $w(\tau, \lambda)$ of the parametrized curves in the 1–skeleton of V_∞ are explicit but complex rational functions on the parameters λ of the G_m–action. Since $n(d)$ must be a rational or even integral *number*, miraculous cancellations must take place in the r.h.s. of (0.21) which are not at all evident algebraically.

Computer calculations furnish the following values for the first four $n(d)$'s:

$$2875,\ 609250,\ 317206375,\ 242467530000. \tag{0.22}$$

More direct methods of counting rational curves lead to the same numbers.

Although in a sense the potential (0.20) is now explicitly known, it is still difficult to identify it with its conjectural mirror image which we will shortly describe.

0.8. Moduli spaces of Calabi–Yau threefolds as weak Frobenius manifolds. As the discussion in 0.4 and 0.5 shows, the geometry of a Frobenius manifold on M is basically defined by a flat structure and a symmetric cubic tensor which is the third Taylor differential of a potential in flat coordinates. A flat metric is then used in order to raise indices and write the associativity equations.

If we are interested in a class of potentials for which the associativity equations are trivial, like (0.20), we may as well forget about the metric, and call the resulting structure *weak Frobenius.* For a precise definition of weak Frobenius manifolds, see I.5. This geometry naturally arises from the theory of variation of Hodge structure of Calabi–Yau threefolds.

Let $\pi:\ W \to Z$ be a complete local family of Calabi–Yau threefolds. Recall that each fiber W_z is a projective algebraic manifold with trivial canonical bundle and $h^{i,0}=0$ for $i=1,2$. Denote by $\mathcal{L}=\pi_*\Omega^3_{W/Z}$ the invertible sheaf of holomorphic volume forms on the fibers of π. We will construct an $\mathcal{L}^{-2}$–valued cubic differential form $G:\ S^3(\mathcal{T}_Z)\to\mathcal{L}^{-2}$ in the following way. First, according to Bogomolov–Todorov–Tian, the Kodaira–Spencer map (following from $0\to\mathcal{T}_{W/Z}\to\mathcal{T}_W\to\pi^*(\mathcal{T}_Z)\to 0$)

$$KS:\ \mathcal{T}_Z\to R^1\pi_*\mathcal{T}_{W/Z}$$

is actually an isomorphism so that the tangent space at $z\in Z$ can be identified with $H^1(W_z,\mathcal{T}_{W_z})\cong H^1(W_z,\Omega^2_z)\otimes\mathcal{L}(z)^{-1}$. Second, the convolution $i:\ \mathcal{T}_{W/Z}\times\Omega^p_{W/Z}\to\Omega^{p-1}_{W/Z}$ induces the pairings

$$R^1\pi_*(i):\ R^1\pi_*\mathcal{T}_{W/Z}\times R^q\pi_*\Omega^p_{W/Z}\to R^{q+1}\pi_*\Omega^{p-1}_{W/Z}$$

or else

$$R^1\pi_*\mathcal{T}_{W/Z}\to\mathcal{E}nd^{(-1,1)}\left(\bigoplus_{p,q}R^q\pi_*\Omega^p_{W/Z}\right)$$

which is essentially the graded symbol of the Gauss–Manin connection defined thanks to the Griffiths' transversality condition. Iterating it three times and using Serre's duality we get finally:

$$G:\ S^3(\mathcal{T}_Z)\cong S^3(R^1\pi_*\mathcal{T}_{W/Z})\to\mathcal{H}om(\pi_*\Omega^3_{W/Z},\pi_*\mathcal{O}_W)\cong\mathcal{L}^{-2}.$$

In order to identify $\mathcal{L}^{-2}$ with $\mathcal{O}_Z$ (which we need to define a weak Frobenius structure) we must choose a trivialization of the volume form sheaf. In the context of the Mirror Conjecture, this is achieved by postulating that Z can be partially compactified by $\dim(Z)$ divisors with normal intersection in such a way that the family W can be extended to a family of "degenerate Calabi–Yau's" and the zero–dimensional stratum of the boundary W_∞ becomes a maximally degenerate manifold, like the simplex V_∞ in the family of quintics. A precise description of this condition is fairly technical, and we omit it here; but see Deligne's paper [De2], [Mo1], and [Pea].

Then the monodromy invariant part of $H_3(W_z,\mathbf{Z})/(tors)$ around zero will be generated by one cycle γ defined up to sign (more or less by the definition of maximal degeneration), and we locally trivialize $\mathcal{L}$ by choosing a volume form ω_z on W_z in such a way that $\int_{\gamma_z}\omega_z=(2\pi i)^3$.

The flat coordinates in which G is the third Taylor differential of a potential Ψ can be constructed in the same context as the action variables of the algebraically completely integrable system whose phase space is the family of Griffiths Jacobians of W_z: cf. [DoM].

A family W is called the mirror family for V if one can identify the weak Frobenius manifold structure on $H^2(V)$ obtained via curve counting on V (A–model) with that corresponding to the variation of Hodge structure for W (B–model).

For the particular case of quintics considered in 0.7 the mirror family depends on one parameter z, and W_z is obtained by resolving singularities of the spaces $\widetilde{W}_z/(\mathbf{Z}/5\mathbf{Z})^3$ where $\widetilde{W}_z \subset \mathbf{P}^4$ is given by the equation $\sum_{j=1}^5 x_j^5 = z\prod_{j=1}^5 x_j$, and $(\mathbf{Z}/5\mathbf{Z})^3$ acts by $x_j \mapsto \xi_j x_j,\ \xi_j^5 = 1,\ \prod_{j=1}^5 \xi_j = 1$.

All the periods $\psi(z) := \int_{\gamma_z} \nu_z$ of an explicit algebraic volume form along $\gamma_z \in H_3(W_z, \mathbf{Z})$ (any horizontal cycle) satisfy the Picard–Fuchs differential equation $\partial := zd/dz)$:

$$[\partial^4 - 5z(5\partial+1)(5\partial+2)(5\partial+3)(5\partial+4)]\psi(z) = 0.$$

It has four linearly independent solutions near $z = 0$:

$$\psi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n,$$

$$\psi_1(z) = \log(z)\psi_0(z) + 5\sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5}\left(\sum_{k=n+1}^{5n} k^{-1}\right) z^n,$$

and two more for which we give only the top terms:

$$\psi_2(z) = \frac{1}{2}(\log z)^2\psi_0(z) + \dots, \psi_3(z) = \frac{1}{6}(\log z)^3\psi_0(z) + \dots.$$

An appropriate flat coordinate on the z–line by definition is $\frac{\psi_1}{\psi_0}(z)$. Under the mirror correspondence, it becomes y in (0.20), thus locally identifying $H^2(V, \mathbf{C})$ (where V is a generic quintic) to the moduli space of the dual family W. Putting

$$F(y) := \Phi^V(y) = \frac{5}{6}y^3 + \sum_{d=1}^{\infty} n(d) Li_3(e^{dy}) \tag{0.23}$$

we have the following mirror identity:

$$F'''\left(\frac{\psi_1}{\psi_0}\right) = \frac{5}{2}\frac{\psi_1\psi_2 - \psi_0\psi_3}{\psi_0^2}. \tag{0.24}$$

Since ψ_i are explicitly known, one can check that the first coefficients agree with (0.22).

However, conceptually (0.24) looks baffling. In order to reduce our problem to the proof of an explicit identity, we have oversimplified the geometry. In particular, the mirror pattern must involve some operator of parity change or an odd scalar product on the full Frobenius supermanifold, because an even part of $H^*(V)$ becomes identified with an odd part of $H^*(W)$. E. Witten and M. Kontsevich suggested that generally one should extend the moduli space of the model B rather than restrict (to H^2) the moduli of the problem A. This is crucially important for understanding the mirror picture for the higher–dimensional Calabi–Yau manifolds where rational curves cease to be isolated and a considerably larger (depending on $\dim(V)$) portion of $H^*(V)$ becomes affected by the instanton corrections. According to one suggestion due to M. Kontsevich, one should construct deformations of a Calabi–Yau manifold in a mysterious universe of non–commutative and/or non–associative objects like A_∞–categories (cf. [Ko4]). A less ambitious construction due to S. Barannikov and M. Kontsevich produces extended *formal* moduli spaces with Frobenius structure using solutions of formal Maurer–Cartan equations: see III.9 and III.10.

A. Givental proved the mirror identity (0.24) in [Giv2] by refining Kontsevich's approach, passing to the equivariant cohomology, and completing the geometric picture by extremely ingenious calculations (cf. also [BiCPP], [Pa3] and the further extension in [LiLY]).

V. Batyrev developed a theory of mirror correspondence between complete Calabi–Yau intersections in toric varieties which conjecturally should serve as the background for a multitude of mirror identities: see [Ba1], [BaBo1], [Babo2]. It is also expected that mirror identities reflect only a part of a much richer geometric picture, which still remains rather mysterious. For some additional insights, see [Va], [Ko4], [Ko5], [Bor2].

0.9. Weil–Petersson volumes as rank 1 Cohomological Field Theory. The rank of the CohFT on (H, g) is, by definition, $\dim(H)$. Let it be 1. Assume for simplicity that $g(\Delta_0, \Delta_0) = 1$ for a basis vector $\Delta_0 \in H$ and fix it. Then the whole structure boils down to a sequence of (non–necessarily homogeneous) cohomology classes

$$c_n := I_n(\Delta_0^{\otimes n}) \in H^*(\overline{M}_{0n})^{S_n},\ n \geq 3, \tag{0.25}$$

satisfying the identities

$$\phi_\sigma^*(c_n) = c_{n_1+1} \otimes c_{n_2+1},\ n = n_1 + n_2,\ n_i \geq 2 \tag{0.26}$$

(cf. (0.6) and (0.7)).

By Theorem 0.5, we see that each such theory is uniquely determined by the coefficients of its potential

$$\Phi(x) := \sum_{n\geq 3} \frac{C_n}{n!}, \quad C_n = \int_{\overline{M}_{0n}} c_n \tag{0.27}$$

(cf. (0.14)), which can be totally arbitrary because any series in one variable satisfies the associativity equations. Therefore, rank one theories seem to be rather trivial objects. However, this is not so for at least two reasons: first, there are quite interesting specific theories of algebro–geometric origin; second, the behavior of $\Phi(x)$ with respect to the tensor product of theories is non–trivial.

Here we give an example (the first term of a hierarchy) of algebro–geometric theories.

There is a standard Weil–Petersson hermitian metric on the non–compact moduli spaces M_{0n} parametrizing irreducible curves. On the boundary this metric becomes singular. Nevertheless, its Kähler form extends to a closed L^2–current on $\overline{M}_{0n}$, thus defining a real cohomology class $\omega_n^{WP} \in H^2(\overline{M}_{0n})^{S_n}$. There is also a purely algebro–geometric definition of this class (see [AC1]). Consider the universal curve $p_n: \ X_n \to \overline{M}_{0n}$. Let $x_i \subset X_n$ be the divisors corresponding to the structure sections, and $\omega = \omega_{X_n/\overline{M}_{0n}}$ the relative dualizing sheaf. Then

$$\omega_n^{WP} = 2\pi^2 p_{n*}\left(c_1(\omega(\sum_{i=1}^{n} x_i))^2\right). \tag{0.28}$$

The main property of ω_n^{WP} is

$$\varphi_\sigma^*(\omega_n^{WP}) = \omega_{n_1+1}^{WP} \otimes 1 + 1 \otimes \omega_{n_2+1}^{WP}. \tag{0.29}$$

Comparing this with (0.26) one sees that

$$c_n := \exp(\omega_n^{WP}/2\pi^2) \in H^*(\overline{M}_{0n}, \mathbf{Q}) \tag{0.30}$$

is a rank one CohFT. Its potential is a generating function for the Weil–Petersson volumes considered in [Zo1]:

$$\Phi^{WP}(x) := \sum_{n=3}^{\infty} \frac{v_n}{n!(n-3)!} x^n, \tag{0.31}$$

$$\frac{v_n}{(n-3)!} := \frac{1}{\pi^{2(n-3)}} \int_{\overline{M}_{0n}} \frac{(\omega_n^{WP})^{n-3}}{(n-3)!}. \tag{0.32}$$

P. Zograf proved that $v_4 = 1, \ v_5 = 5, \ v_6 = 61, \ v_7 = 1379$, and generally

$$v_n = \frac{1}{2} \sum_{i=1}^{n-3} \frac{i(n-i-2)}{n-1} \binom{n-4}{i-1} \binom{n}{i+1} v_{i+2} v_{n-i}, \ n \geq 4. \tag{0.33}$$

This is equivalent to a non–linear differential equation for $\Phi^{WP}(x)$. What is more remarkable, the inverse function for the second derivative of the potential satisfies a linear (modified Bessel) equation:

$$y = \sum_{n=3}^{\infty} \frac{v_n}{(n-2)!\,(n-3)!}\, x^{n-2} \quad \Longleftrightarrow \quad x = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!\,(m-1)!}\, y^m\ . \tag{0.34}$$

This can be considerably generalized to the complete description of the tensor product of invertible rank one CohFT's: see III.6. Thus, in addition to the associativity equations for the quantum cohomology of plane (and other Fano manifolds) and the hypergeometric equations for Calabi–Yau (made non–linear by a coordinate change) we have one more differential equation of a seemingly different origin. In fact, this is a reincarnation of the (partly conjectural) Virasoro constraints of a fuller theory, involving correlators of all genera with gravitational descendants: see Chapter VI.

0.10. Plan of the book. From this sketchy overview, it must be clear that the quantum cohomology is an exceptionally rich and tightly woven structure.

Chapters I and II develop the local and global geometric and analytic theory of Frobenius manifolds. Chapter III introduces the more algebraic aspects: formal Frobenius manifolds, moduli spaces and their homology operads. Besides, Chapter III contains the theory of tensor products and several constructions of large classes of Frobenius manifolds of algebraic geometric origin.

Chapters V and VI focus on the algebraic geometric constructions of the Gromov–Witten invariants. In the first part they figure only as examples or in axiomatic form. The theory of quantum cohomology is thereby considerably enriched, but its relationship to the basic substructure of Frobenius manifolds needs further clarification. To be more precise, it is clear that a considerable part of the higher genus theory with gravitational descendants can be extended to more general Frobenius manifolds than actual quantum cohomology (cf. the notion of Frobenius manifolds of qc–type, III.5.4.) However, the exact scope of such an extension remains unclear.

There is one more structure that keeps appearing in all the ramifications of this subject: trees and more general graphs, eventually with labels. They enumerate the strata and cells of $\overline{M}_{g,n}$, help to visualize the composition laws of operads and operadic algebras, and govern the counting of curves on quintics via Kontsevich's construction. Many generating functions and potentials Φ, when they can be explicitly calculated, often appear in the guise of sums over labeled graphs of rather special type, perturbation series, which are well known in statistical physics and quantum field theory.

One can look at graphs as a mere book–keeping device and treat them in an *ad hoc* manner whenever they appear. However, I thought it worthwhile to pay them more respect as a combinatorial skeleton of the theory. Chapter IV summarizes some of their applications.