Preface to the second edition

This revision has two main purposes: first to correct various errors that crept into the first edition and second to update our discussions of current work in the field. Since the first edition of this book appeared in 2004, symplectic geometry has developed apace. It has found new applications in low dimensional topology, via Heegaard Floer theory [**318**] and the newly understood relations of embedded contact homology to gauge theory [**74, 219**]. Several important books have been published that develop powerful new ideas and techniques: Seidel [**371**] on the Fukaya category, Fukaya–Oh–Ohta–Ono [**128**] on Lagrangian Floer homology, and Cieliebak and Eliashberg [**63**] on the relations between complex and symplectic manifolds. Another exciting development is the introduction of sheaf-theoretic methods for proving fundamental rigidity results in symplectic geometry by Tamarkin [**388**] and Guillermou–Kashiwara–Shapira [**166**]. There has also been great progress on particular problems; for example Taubes [**394**] solved the Weinstein conjecture for 3-dimensional contact manifolds using Seiberg–Witten–Floer theory, Hingston [**176**] and Ginzburg [**143**] solved the Conley conjecture by new advances in Hamiltonian dynamics and Floer theory, and the nearby Lagrangian conjecture has been partially solved (by Fukaya–Seidel–Smith [**131**] and Abouzaid [**1**] among others) using Fukaya categories. A comprehensive exposition of Hamiltonian Floer theory is now available with the book by Audin–Damian [**24**], which presents all the basic analysis needed to set up Hamiltonian Floer theory for manifolds with $c_1 = 0$ as well as in the monotone case. Finally, the long series of papers and books by Hofer–Wysocki–Zehnder [**184, 185, 186, 187, 188, 189**] develops a new functional analytic approach to the theory of J-holomorphic curves. Their work will eventually give solid foundations to Lagrangian Floer theory and the various forms of Symplectic Field Theory.

We do not say much about the details of these developments. However, we have updated the introductions to the chapters where relevant, and also have extended the discussions of various applications of J-holomorphic curves in Chapters 9, 11 and 12, aiming to give a sense of the main new developments and the main new players rather than to be comprehensive.

Many of the corrections are rather minor. However, we have rewritten Section 4.4 on the isoperimetric inequality, the proof of Theorem 7.2.3, the proof of Proposition 7.4.8, and the proof of the sum formula for the Fredholm index in Theorem C.4.2. In Chapter 10 we added Section 10.9 with a new geometric formulation of the gluing theorem for z-independent almost complex structures, in Appendix C we expanded Section C.5 to include a proof of integrability of almost complex structures in dimension two, and in Appendix D we expanded Section D.4 to include the material previously in Sections D.4 and D.5 and added a new Section D.6 on the cohomology of the moduli space of stable curves of genus zero.

We warmly thank everyone who pointed out mistakes in the earlier edition, but particularly Aleksei Zinger who sent us an especially thorough and useful list of comments.

Dusa McDuff and Dietmar Salamon, April 2012

Preface

The theory of J-holomorphic curves has been of great importance to symplectic topologists ever since its inception in Gromov's paper of 1985. Its applications include many key results in symplectic topology, and it was one of the main inspirations for the creation of Floer homology. It has caught the attention of mathematical physicists since it provides a natural context in which to define Gromov–Witten invariants and quantum cohomology, which form the so-called A-side of the mirror symmetry conjecture. Insights from physics have in turn inspired many fascinating developments, for example highlighting as yet little understood connections between the theory of integrable systems and Gromov–Witten invariants.

Several years ago the two authors of this book wrote an expository account of the field that explained the main technical steps in the theory of J-holomorphic curves. The present book started life as a second edition of that book, but the project quickly grew. The field has been developing so rapidly that there has been little time to consolidate its foundations. Since these involve many analytic subtleties, this has proved quite a hindrance. Therefore the main aim of this book is to establish the fundamental theorems in the subject in full and rigorous detail. We also hope that the book will serve as an introduction to current work in symplectic topology. These two aims are, of course, somewhat in conflict, and in different parts of the book different aspects are predominant.

We have chosen to concentrate on setting up the foundations of the theory rather than attempting to cover the many recent developments in detail. Thus, we limit ourselves to genus zero curves (though we do treat discs as well as spheres). A more serious limitation is that we restrict ourselves to the semipositive case, where it is possible to define the Gromov–Witten invariants in terms of pseudocycles. Our main reason for doing this is that an optimal framework for the general case (which would involve constructing a virtual moduli cycle) has not yet been worked out. Rather than cobbling together a definition that would do for many applications but not suffice in broader contexts such as symplectic field theory, we decided to show what can be done with a simpler, more geometric approach. On the other hand, we give a very detailed proof of the basic gluing theorem. This is the analytic foundation for all subsequent work on the virtual moduli cycle and is the essential ingredient in the proof of the associativity of quantum multiplication. There are also five extensive appendices, on topics ranging from standard results such as the implicit function theorem, elliptic regularity and the Riemann–Roch theorem to lesser known subjects such as the structure of the moduli space of genus zero stable curves and positivity of intersections for J-holomorphic curves in dimension four. We have adopted the same approach to the applications, giving complete proofs of the foundational results and illustrating more recent developments by describing some key examples and giving a copious list of references.

The book is written so that the subject develops in logical order. Chapters 2 through 5 establish the foundational Fredholm theory and compactness results for J-holomorphic spheres and discs; Chapter 6 introduces the concepts need to define the Gromov–Witten pseudocycle for semipositive manifolds; Chapter 7 is the pivotal chapter in which the invariants are defined; and the later chapters discuss various applications. Since there is more detail in Chapters 2 through 6 than can possibly be absorbed at a first reading, we have written the introductory Chapter 1 to describe the outlines of the theory and to introduce the main definitions. It serves as a detailed guide to this book, pointing out where the key arguments occur and where to find the background details needed to understand various examples. Each chapter also has an introduction describing its main contents, which should help to orient the more knowledgeable readers. Wherever possible we have written the sections and chapters to be independent of each other. Hence the reader should feel free to skip parts that seem excessively technical.

We hope that Chapter 1 (supplemented by suitable parts of Chapters 2–6) will provide beginning students with enough of the essential background for understanding the main definitions in Chapter 7. Here is a brief outline of the contents of the remaining chapters. After the basic invariants are defined in Section 7.1 (with important supplemental ideas in Section 7.2 and Section 7.3), Section 7.4 discusses the fundamental example of rational curves in projective space. The chapter ends with a discussion of the Kontsevich–Manin axioms for the genus zero Gromov–Witten invariants, and deduces from them Kontsevich's beautiful iterative formula for the number of degree d rational curves in the projective plane.

Chapter 8 sets up the theory of locally Hamiltonian fibrations over Riemann surfaces and shows how to count sections of such fibrations. This allows us to define Gromov–Witten invariants of arbitrary genus (but where the complex structure of the domain is fixed). It also provides the background for some important applications, for example Gromov's result that every Hamiltonian system on a symplectically aspherical manifold has a 1-periodic orbit (see Theorem 9.1.1), and results about the group of Hamiltonian symplectomorphisms: a taste of Hofer geometry in Section 9.6 and a discussion of the Seidel representation in Sections 11.4 and 12.5.

Chapter 9 describes some of the main applications of J-holomorphic curve techniques in symplectic geometry. Besides the examples mentioned above and a discussion of the basic properties of Lagrangian submanifolds, it gives full proofs of McDuff's results on the structure of rational and ruled symplectic 4-manifolds as well as Gromov's results on the symplectomorphism group of the projective plane and the product of 2-spheres.

The other main application, quantum cohomology, requires a further deep analytic technique, that of gluing. The first rigorous gluing arguments are due to Floer (in the somewhat easier context of Floer homology) and Ruan–Tian (in the context relevant to quantum homology). In Chapter 10 we present a different, perhaps easier, method of gluing and derive from it a proof of the splitting axiom for the Gromov–Witten invariants in semipositive manifolds.

With this in hand, Chapter 11 defines quantum cohomology and explains some of the structures arising from it, such as the Gromov–Witten potential and Frobenius manifolds. As is clear from the examples in Section 11.3, this is the place where symplectic topology makes the deepest contact with other areas such as algebraic geometry, conformal field theory, mirror symmetry, and integrable systems. This chapter should be accessible after Chapter 7. Finally, Chapter 12 is a survey that formulates the main outlines of Floer theory, omitting the analytic underpinnings. It explains the relations between Floer theory and quantum cohomology, using a geometric approach, and also indicates the directions of further developments, both analytic (the vortex equations) and geometric (Donaldson's quantum category).

There are five appendices. The first three set up the foundations of the classical theory of linear elliptic operators that is generalized in Chapters 3 and 4: Fredholm theory and the implicit function theorem for Banach manifolds in Appendix A, Sobolev spaces and elliptic regularity in Appendix B, and the Riemann–Roch theorem for Riemann surfaces with boundary in Appendix C. Appendix D provides background for Chapter 5. It explains the structure of the Grothendieck–Knudson moduli space of genus zero stable curves using cross ratios rather than the usual algebro-geometric approach. Appendix E was written jointly with Laurent Lazzarini. It contains a complete proof of positivity of intersections and the adjunction inequality for J-holomorphic curves in four-dimensional manifolds. Lazzarini provided the first draft of this appendix with complete proofs and we then worked together on the exposition. The results of Appendix E provide the basis for the structure theorems for rational and ruled symplectic 4-manifolds.

Those who wish to use this book as the basis for a graduate course must make some firm decisions about what kind of course they want to teach. As we know from experience, it is impossible in one semester to prove all the main analytic techniques as well as to describe interesting examples. One possibility, explained in more detail in Chapter 1, would be to concentrate on Chapter 1 (amplified by small parts of Chapter 2), Chapter 3 through Section 3.3 (together perhaps with some extra analysis from Appendices B and C), the basic compactness result for spheres with minimal energy in Section 4.2, very selected parts of Chapter 6 (the definition of pseudocycle), and then move to Section 7.1. Then either one could go directly to some of the geometric applications in Chapter 9 (for example, prove the nonsqueezing theorem or some of the results about symplectic 4-manifolds in Section 9.4) or one could discuss the Kontsevich–Manin axioms for Gromov–Witten invariants in Section 7.5 and then move to Chapter 11 to set up quantum cohomology. The idea here would be to develop a familiarity with the main analytic setup, prove some of the basic techniques, and then set them in context by discussing one set of applications

The above outline is perhaps still too ambitious, but there are ways to shorten the preliminaries. For example, it is possible to discuss many of the applications in Chapter 9 directly after the foundational material of Chapters 2–4 (and relevant parts of Chapter 8), without any reference to Chapters 5, 6 and 7. For if one considers only the simplest cases of these applications, rather than proving them in their most general form, the relevant moduli spaces are compact and so the results become accessible without any formal definition of the Gromov–Witten invariants. Alternatively, those aiming at quantum cohomology could state the results on Fredholm theory without proof and instead concentrate on explaining some of the compactness (bubbling) results in Chapters 4 and 5. These combine well with a study of the moduli space of stable maps and hence lead naturally to the Kontsevich–Manin axioms.

As indicated above, a first course, unless it moves incredibly fast or contains almost no applications, cannot both cover Fredholm theory and come to grips with the analytic details of the compactness proof, even less go through all the details of gluing. Even though this proof in the main needs the same analytic background as Chapter 3, the proof of the surjectivity of the gluing map hinges on the deepest result from Chapter 4 (the behaviour of long cylinders with small energy) and relies on several technical estimates. We have written the gluing chapter to try to make accessible the outlines of the construction, together with the main analytic ideas. (These are summarized in Section 10.5.) Hence, for a more analytically sophisticated audience, one might base a course on Chapters 3, 4 and 10, with motivation taken from some of the examples in Chapter 9 or 11.

Despite the length of this book, its subject is so rich that it is impossible to treat all its aspects. We have given many references throughout. Here are some books on related areas that the reader might wish to consult both on their own account and for the further references that they contain: Cox–Katz [**76**] on mirror symmetry and algebraic geometry, Donaldson [**87**] on Floer homology and gauge theory, Manin [**286**] on Frobenius manifolds and quantum cohomology, Polterovich [**330**] on the geometry of the symplectomorphism group, and the paper by Eliashberg–Givental–Hofer [**101**] on symplectic field theory.

This book has been long in the making and would not have been possible without help from many colleagues who shared their insights and knowledge with us. In particular, Coates, Givental, Lalonde, Lazzarini, Polterovich, Popescu, Robbin, Ruan, and Seidel all gave crucial help with various parts of this book. We also wish to thank the many students and others who pointed out various typos and inaccuracies, and especially Eduardo Gonzalez, Sam Lisi, Jake Solomon, and Fabian Ziltener for their meticulous attention to detail.

Dusa McDuff and Dietmar Salamon, December 2003