CHAPTER 1

Quantum fields, noncommutative spaces, and motives

1. Introduction

The main goal of Chapter one is to unveil the mathematical conceptual meaning of some of the sophisticated computations performed by physicists in the domain of particle physics. It is divided into two parts dealing, respectively, with

(1) Renormalization
(2) The Standard Model

We try to keep close contact with the way the computations are actually performed by physicists and to bridge the gap between, on the one hand, the lessons physicists learned from their constant dialogue with experimental results and, on the other hand, with the elaborate mathematical concepts involved in allowing one to understand the meaning of these computations (if any). The bare data we start with are, respectively,

- Computations of cross sections and scattering amplitudes from the perturbative expansion of the Feynman integral using renormalized values of Feynman graphs in the dimensional regularization and minimal subtraction scheme.
- The detailed expression of the full Standard Model Lagrangian with neutrino mixing, the see-saw mechanism and coupling to gravity.

We start the chapter with a presentation of quantum field theory (QFT) that ought to be understandable to mathematicians. In particular we recall in §2.1 the Lagrangian and Hamiltonian formalisms of classical mechanics and explain in §2.2 how the Lagrangian formulation of quantum field theory leads to Feynman's path integral. The path integral prescription is ambiguous (even ignoring all the divergence problems) and the removal of the ambiguity by Feynman's $i\epsilon$ trick can only be properly understood after an excursion into the Hamiltonian formulation and canonical quantization. We do this in §2.3 where we base the discussion on the three main physical properties of a quantum field theory, which are

- Causality
- Positivity of energy
- Unitarity

We discuss the simplest example of QFT in §2.4: it is the free bosonic field on the space-time

$$X = \mathbb{R} \times S^1$$

with the Lorentz metric. This gives a good occasion to describe this basic example of algebraic quantum field theory, and to explain what are the vacuum states and the temperature states which fulfill the KMS_β condition relative to the Heisenberg time evolution. Even though algebraic quantum field theory is an interesting formalism involving deep mathematical structures such as von Neumann algebras it falls short of what is essential to our purpose: concrete physics computations. It is, however, essential in clarifying the conceptual meaning of the boundary conditions on Green's functions by clearly separating the kinematical relations from the construction of the vacuum states. We turn to the Green's functions in §2.5 and give their formal perturbative expansion in terms of free fields as the Gell-Mann–Low formula. We then show in §2.6 how Wick rotation allows one to encompass Feynman's $i\epsilon$ prescription for how to go around the pole of the propagator in the analytic continuation to imaginary time. We give all the details on that point since it removes the first ambiguity in the perturbative computation of the functional integral and shows from the start the merit of the Euclidean formulation.

The Feynman graphs are dealt with in §3. We start with a detailed account of a concrete example in §3.1 and show how the various pairings coming from the integration by parts under a Gaussian are labeled by graphs and yield integrals. The simplest graphs, such as self-energy graphs, give rise to integrals which diverge when an ultraviolet cutoff is removed so that one is confronted with the problem of renormalization. The physics origin of the problem was already understood by Green in 1830 and we explain the computation of the self-energy in hydrodynamics in §3.2 as a first example of mass renormalization. We then use the analogy between hydrodynamics and electromagnetism to explain how the crucial distinction between the bare parameters and the observed ones makes it possible to eliminate the divergence of the simplest self-energy graph by adding counterterms to the Lagrangian. We give a precise mathematical definition of Feynman graphs and of the rules which associate a formal integral to a graph in §3.3. We then describe the standard procedures that allow one to simplify the combinatorics of the Feynman graphs. First, by taking the logarithm of the partition function with a source term one reduces to connected graphs (§3.4). Then after applying the Legendre transform one obtains the effective action. Both the action's role as the basic unknown of QFT and its expansion in terms of one-particle irreducible (1PI) graphs is explained in §3.5. With this tool at hand we come to a precise definition of the physical parameters, such as the mass, and observables, such as the scattering matrix, in terms of the effective action in §3.6. Finally we describe the physical ideas of mass, field strength and coupling constant renormalization in §3.7.

In §4 we recall the basic dimensional regularization and minimal subtraction procedures (DimReg+MS). We begin with the very simple example of the self-energy graph for the scalar ϕ^3 theory. We show explicitly, in this example, how to implement the dimensional regularization of the divergent integral using Schwinger parameters and the formal rules for Gaussian integration in a complexified dimension $D - z$. We then discuss the existence of an analytic continuation of the Feynman integrals to a meromorphic function on the complex plane. We prove in Theorem 1.9 that the dimensionally regularized unrenormalized values $U^z(\Gamma(p_1, \dots, p_N))$ have the property that their Taylor coefficients at $p = 0$ admit a meromorphic continuation to the whole complex plane $z \in \mathbb{C}$. In §5.1 we show a simple example of a *subdivergence* for a 1PI (one-particle irreducible) graph of the massless ϕ^3 theory in dimension 6. This example shows the need, in addition to the regularization scheme (here DimReg+MS), for a renormalization procedure that accounts for the combinatorics of nested subdivergences.

In §5, we introduce the Bogoliubov–Parasiuk–Hepp–Zimmermann renormalization (BPHZ) procedure. This takes care of eliminating the divergences step by step in the perturbative expansion, by repeatedly adjusting the bare parameters in the Lagrangian by suitable counterterms that cancel the divergences. The BPHZ procedure also takes care of the problem of non-local counterterms associated to subdivergences. We show this in detail in an explicit example in §5.1. We also explain the role of the external structure of Feynman graphs. The counting of the degree of divergence is described in §5.2. The BPHZ preparation of Feynman graphs and the extraction of the renormalized value and the inductive definition of the counterterms are discussed in §5.3.

In §6.1 we give some mathematical background on commutative Hopf algebras and affine group schemes, while in § 6.2 we introduce the Connes–Kreimer Hopf algebra of Feynman graphs, first only in its discrete combinatorial version. Then in §6.3 we refine the construction, by taking also into account the external structure. Theorem 1.39 gives the recursive formula of Connes–Kreimer for the Birkhoff factorization in a graded connected Hopf algebra, which gives a clear conceptual interpretation to the BPHZ procedure, when applied to the Hopf algebra of Feynman graphs.

In §6.5 we recall another result of the Connes–Kreimer theory, relating the affine group scheme of the Hopf algebra of Feynman graphs, called the group of diffeographisms of the physical theory, to formal diffeomorphisms of the coupling constants of the theory. We explain in §6.6 the dependence of the $U^z(\Gamma(p_1, \dots, p_N))$ on a mass parameter μ and how to recover in the Connes–Kreimer theory the notion of renormalization group lifted to the level of the group of diffeographisms, with the β-function given by an element in the corresponding Lie algebra.

This singles out the data of perturbative renormalization as describing a certain class of loops in the affine group scheme of diffeographisms, satisfying some conditions on the dependence on the mass parameter μ, with the renormalization procedure consisting of their Birkhoff factorization.

We give in §7 a reinterpretation of these data in terms of a Riemann–Hilbert correspondence. We begin in §7.1 with the expression of the counterterms as a "time-ordered exponential" (iterated integral). We then introduce in §7.2 the notion of flat equisingular connection, which reformulates geometrically the conditions satisfied by the class of loops corresponding to the data of perturbative renormalization. The corresponding equivariant principal bundles are described in §7.3.

In §7.4 we recall some general facts about Tannakian categories, the Tannakian formalism, and representations of affine group schemes. In §7.5 we show how this formalism is variously used in the context of differential Galois theory to classify categories of differential systems with prescribed singularities through the Riemann–Hilbert correspondence, a broad generalization of the original Riemann–Hilbert problem on the reconstruction of differential equations from their monodromy representations.

In §7.6 we apply this general strategy to the case of renormalization. We introduce a category of flat equisingular vector bundles, and we obtain in Theorem 1.100 an identification with the category $\mathrm{Rep}_{\mathbb{U}^*}$ of finite-dimensional linear representations of the affine group scheme $\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$ with Hopf algebra $\mathcal{H}_{\mathbb{U}} := U(\mathcal{F}(1,2,3,\cdots)_\bullet)^\vee$ where $\mathcal{F}(1,2,3,\cdots)_\bullet$ is the free graded Lie algebra with one generator e_{-n} in each degree $n > 0$.

In §8 we give a brief introduction to the theory of motives, which plays a role in many different ways throughout the book. In particular we mention the fact that the affine group scheme $\mathbb{U}^*$ also has an incarnation as a motivic Galois group for a category of mixed Tate motives. Of the general aspects of the theory of motives, we recall briefly the general interpretation as a universal cohomology theory, the construction of the category of pure motives, the role of algebraic cycles and equivalence relations, the relation to zeta functions (which plays an important role in Chapter 2), the Weil conjectures and the Grothendieck standard conjectures, the role of the Tannakian formalism and motivic Galois groups, the special case of Artin motives (which plays a role in Chapter 4), mixed motives and, in particular, the mixed Tate motives that seem to be deeply related to quantum field theory and their relation to mixed Hodge structures.

This completes the first part of Chapter 1. In the second part of the chapter we deal with the Standard Model of elementary particle physics and an approach to a simple mathematical understanding of its structure via noncommutative geometry. The second part of Chapter 1 follows closely our joint work with Chamseddine [**52**], which is based on the model introduced by Connes in [**73**], as well as on the previous work of Chamseddine and Connes on the spectral action [**45**], [**46**], [**47**].

Since we do not assume that the reader has much familiarity with particle physics, we begin §9 by giving a brief overview of the *Standard Model.* We introduce the various parameters, the particles and interactions, symmetries in §§9.1, 9.2, 9.3 and we reproduce in full in §9.4 the very complicated expression of the Lagrangian. We discuss in §9.5 other aspects, such as the ghost terms and gauge fixing, that become relevant at the quantum rather than the semi-classical level. In § 9.6 we distinguish between the *minimal* Standard Model, which has only left-handed massless neutrinos, and the extension that is required in order to account for the experimental evidence of neutrino mixing. We describe in §9.6.2 the corresponding modifications of the Lagrangian. In §§9.7 and 9.8 we describe the Lagrangian that gives the Standard Model minimally coupled to gravity, where the gravity part can be considered as an effective field theory by including higher-derivative terms.

The problem of realizing the symmetries of particle interactions as diffeomorphisms (pure gravity) on a suitable space suggests the idea that noncommutative geometry, where inner symmetries are naturally present, should provide the correct framework. We discuss this in §9.9.

In §10 we then recall the main notions of (metric) noncommutative geometry developed by Connes, based on the structure of *spectral triple* that generalizes Riemannian geometry to the noncommutative setting. We introduce spectral geometry in §10.1, we recall the definition and basic properties of spectral triples in §10.2, including the *real part* defined in §10.3. We recall some well known facts on Hochschild and cyclic cohomology in §10.4. The local index formula of Connes–Moscovici is recalled briefly in §10.5. The Yang–Mills and Chern–Simons actions are described in terms of Hochschild cohomology in §10.6 and §10.7 following work of Chamseddine–Connes.

The important notion of inner fluctuations of the metric associated to self-Morita equivalences of a noncommutative space is discussed in §10.8.

In §11 we introduce the *spectral action principle* of Chamseddine–Connes. This plays a crucial role in recovering the Standard Model Lagrangian from noncommutative geometry and is one of the main tools in metric noncommutative geometry. A careful discussion of the terms arising in the asymptotic expansion of the spectral action functional is given in §§11.1, 11.2, 11.3 using Seeley–DeWitt coefficients and Gilkey's theorem.

In §11.4 we recall a result of [**46**] that illustrates how to recover the Einstein–Yang–Mills action from the spectral action on the very simple noncommutative space given by the product of an ordinary 4-manifold by a noncommutative space described by the algebra $M_N(\mathbb{C})$ of $N \times N$ matrices.

We then analyze the terms that appear in the asymptotic expansion of the spectral action in §11.5 and their behavior under inner fluctuations of the metric. We also recall briefly in §11.6 the modification of the spectral action by a *dilaton field* introduced by Chamseddine and Connes in [**47**].

In §12 we begin the discussion of the noncommutative geometry of the Standard Model, following [**52**]. We introduce in §13 a finite noncommutative geometry F, derived from the basic input of the model, which is the finite-dimensional associative algebra $\mathbb{C} \oplus \mathbb{H} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with $\mathbb{H}$ the real division algebra of quaternions. We explain how to construct canonically an *odd spin* representation $\mathcal{H}_F$ of this algebra and, upon imposing a very natural condition on possible Dirac operators for this geometry, we identify in §13.1 a maximal subalgebra compatible with the existence of Dirac operators with the required properties. The subalgebra is of the form

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \subset \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{H} \oplus M_3(\mathbb{C}).$$

In §13.2 we identify the bimodule $\mathcal{H}_F$ with the fermions of the Standard Model (after fixing the number of generations $N = 3$) and we show in §13.3 that this identification is dictated by the fact that it reproduces the correct values of the hypercharges.

In §§13.4 and 13.5 we give a complete classification for the possible Dirac operators for this finite geometry and we describe their moduli space, which gives a geometric interpretation for the Yukawa parameters of the Standard Model. The intersection pairing of the finite geometry is analyzed in §13.6, using the fact that the metric and KO-dimensions are not the same, the first being zero and the second being equal to 6 modulo 8.

We then consider in §14 the product $M \times F$ of an ordinary compact spin 4-manifold with the finite noncommutative geometry introduced previously, described as a cup product of spectral triples. We identify the real part of the product geometry in §14.1.

The bosons of the Standard Model, including the Higgs field, are obtained as inner fluctuations of the metric on the product geometry in §15, with the discrete part giving the Higgs analyzed in §15.2 and the gauge bosons in §15.4.

The main computation that shows how to recover the Standard Model Lagrangian, including mixing and Majorana mass terms for neutrinos, minimally coupled to gravity, is carried out in §16 by breaking down the Lagrangian in several steps and relating the resulting terms to the terms in the asymptotic expansion of the spectral action functional

$$\mathrm{Tr}(f(D/\Lambda)) + \frac{1}{2}\langle J\psi, D\psi\rangle,$$

with the additional fermionic term $\langle J\psi, D\psi\rangle$. In particular, we explain in §§16.2 and 16.3 how the fermionic term gives rise to a Pfaffian which takes care of the "fermion doubling problem" of [**210**] by taking the square root of a determinant. Among the physical consequences of deriving the Lagrangian from the spectral action, and making the "big desert" hypothesis, we find in § 17.2 the merging of the coupling constants at unification, in the form of the relation $g_2 = g_3 = \sqrt{5/3}\, g_1$ typical of the grand unified theories. We also find in §§17.4 and 17.6 a simple quadratic relation between the masses of quarks and leptons and the W-mass at unification, compatible with the known physics at ordinary energies. In §17.5 and §17.10 we show that this model also provides a *see-saw mechanism* that accounts for the observed smallness of neutrino masses, and a prediction of a heavy Higgs mass at around 168 GeV. The gravitational terms are discussed in §17.11 and the geometric interpretation of the free parameters of the Standard Model is summarized in §17.12.

In §18 we outline a possible functional integral formulation in the context of spectral geometry. We briefly explain a more conceptual path to the algebra $\mathcal{A}_F$ and its representation given in [**49**], [**50**] based on the classification of irreducible finite geometries of KO-dimension 6 modulo 8. This quickly leads to the algebra $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ with a non-trivial grading on $M_2(\mathbb{H})$ and to its natural representation, playing the role of the above odd-spin representation. The same mechanism, coming from the order one condition, then reduces to the subalgebra

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \subset M_2(\mathbb{H}) \oplus M_4(\mathbb{C}).$$

We also show that the use of the larger algebra restores Poincaré duality.

In the remaining part of Chapter 1, in §19 we return to the dimensional regularization procedure described in the first part of the chapter in the context of the Connes–Kreimer theory. Here we give the construction, using spectral triples, of a noncommutative space X_z whose dimension (in the sense of the dimension spectrum of a spectral triple) consists of a complex number z, and we reinterpret geometrically the DimReg procedure, as far as one-loop fermionic graphs are concerned, as taking the cup product of the spectral triple associated to an ordinary manifold of integer dimension D with this noncommutative space. In Chapter 2 we return to discuss this construction and we give an arithmetic model of a noncommutative space X_z of dimension z. We show in §19.1 and following that the construction of X_z is compatible with the Breitenlohner–Maison prescription for treating γ_5 in the context of DimReg. We continue with the discussion of anomalies, by introducing chiral gauge transformations in §19.3, and discussing the finiteness of the anomalous graphs in §19.4. We treat explicitly the simplest cases of anomalous graphs in §19.5 and we relate anomalous graphs in dimension 2 and the local index cocycle in §19.6.

2. Basics of perturbative QFT

Quantum field theory is the most accurate source of predictions about the world of elementary particles. At the theoretical level, it is full of subtleties and ingenious procedures that extract finite and experimentally testable values from formal series of divergent integrals. The development of this theory, which achieved the unification of two fundamental revolutions of early twentieth century physics, special relativity and quantum mechanics, traces its origins to two crucial developments that took place in the late 1920s. The quantization of the electromagnetic field by Born, Heisenberg,

CHAPTER 2

The Riemann zeta function and noncommutative geometry

1. Introduction

This chapter describes, following the results of [**71**], a spectral realization of the zeros of the Riemann zeta function and an interpretation of Weil's explicit formulae of number theory as a trace formula. The chapter also includes an application of the same techniques to the Archimedean local factors of L-functions of varieties defined over number fields, following [**74**].

We begin the chapter by recalling some basic facts about the Riemann zeta function. In §2 we recall the fundamental relation between primes and zeros of the Riemann zeta function expressed by Riemann in terms of an explicit formula for the prime counting function. We also recall Riemann's estimate for the counting of the nontrivial zeros of the zeta function.

In §3 we describe a classical Hamiltonian system (the "scaling Hamiltonian") and a corresponding quantum mechanical system that recover Riemann's estimate of the counting of zeros of zeta as a counting of modes of the physical system. More precisely, in §3.1 we show a striking similarity between the behavior of the oscillatory part in Riemann's formula for the counting function and the corresponding oscillatory part in the semi-classical formula for the number of eigenvalues of the Hamiltonian obtained by quantizing a classical Hamiltonian system. The two formulae are easily matched up to a sign, which suggests the important point of interpreting the spectrum of our scaling system as an "absorption" rather than an "emission" spectrum. This distinction will play a crucial role both in the results described in this chapter as well as in the formulation given in Chapter 4, where the spectral realization and the trace formula we discuss will live naturally on a space that is a cokernel (or more precisely, as will become evident in Chapter 4) a *motive* in an abelian category of noncommutative spaces. We proceed in §3.2 to introduce a classical Hamiltonian system associated to the group of scaling transformations of the real line. We show that the symplectic volume with infrared and ultraviolet cutoff gives the average part of the Riemann counting function. In §3.3 we describe the quantization of this classical system. A delicate point in obtaining a counting of the modes (the energy levels) of the resulting quantum system lies in the implementation of both an ultraviolet and an infrared cutoff, for the reason that one cannot impose a cutoff on both a function and its Fourier transform, or to say it more precisely, the two projectors associated to the two types of cutoff do not commute, hence one cannot implement them simultaneously by intersecting their ranges. The problem can be solved using a technique that was developed in electrical engineering and laser technology to deal precisely with similar sorts of mathematical problems. This is based on the existence of a differential operator commuting with both cutoff projections, whose eigenfunctions, the *prolate spheroidal wave functions*, can be used to approximate both cutoffs by restricting to a subspace spanned by a number of them, depending on the energy range allowed by the cutoff. We then consider the spectral projections of the scaling action, so that the problem of the counting of quantum states becomes the computation of the trace of the product of these spectral projections N_E with the projection Q_Λ on the span of the spheroidal wave functions implementing the cutoff. The trace can be computed (as a special case of the more general trace formula proved in §7.2 of this same chapter). The resulting trace $\mathrm{Tr}(Q_\Lambda N_E)$ is expressed in a distributional form in terms of a principal value.

The definition and properties of such principal values are described in detail in the following section, §4, following [**71**]. We begin by giving the formal computation of the distributional trace of operators of the form

$$\vartheta_{\mathrm{a}}(h) = \int_{K^*} h(\lambda)\vartheta_{\mathrm{a}}(\lambda)\, d^*\lambda$$

with h a test function and ϑ_{a} the scaling action of $\mathbb{G}_m(K)$ on L^2 functions on $\mathbb{G}_a(K)$ by

$$(\vartheta_{\mathrm{a}}(\lambda)\,\xi)(q) = \xi(\lambda^{-1}q),$$

which covers the case of the operators we used in §3.3 to describe the spectral projections of the scaling action. We give in §4.1 a quick introduction to *modulated groups* and the corresponding Haar measures, while in §4.2 we use these notions to give the general form of principal values as distributions extending from $\mathbb{G}_m(K)$ to $\mathbb{G}_a(K)$ the integration with respect to the normalized Haar measure. In particular we give a description in terms of a variant of the "minimal subtraction" method of regularization described in Chapter 1 (see Definition 1.10).

In §5 we come to the actual counting of the modes of the quantum scaling Hamiltonian. We show in Theorem 2.18 that the trace $\mathrm{Tr}(Q_\Lambda N_E)$ gives the right expression for the counting that matches Riemann's formula $N(E) \sim \frac{E}{2\pi}\log\frac{E}{2\pi} - \frac{E}{2\pi}$. The proof of this theorem is given in §5.2 using the noncommutative geometry method known as *quantized calculus*, which we briefly review in §5.1.

We then come in §6 to the mathematical implementation of the "absorption spectrum" idea. We introduce the map $\mathfrak{E}$ considered in [**71**], which acts as

$$\mathfrak{E}(f)(\lambda) = \lambda^{1/2}\sum_{n\in\mathbb{Z}} f(n\lambda)$$

for $\lambda \in \mathbb{R}_+^*$. We show in §6.1 that, after approximating spheroidal wave functions by Hermite–Weber functions, the Fourier transform $\widehat{\mathfrak{E}(f)}$ is the product of an entire function by the Riemann ξ function.

In §7 we make an important extension of the quantum mechanical system described by the Hamiltonian associated to scaling transformations on the real line to an *adelic* version where one considers all the places of $\mathbb{Q}$ and not only the Archimedean place, thus replacing $\mathbb{R}$ by the adèles $\mathbb{A}_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q},f} \times \mathbb{R}$ and $\mathbb{R}^*$ acting by scaling by $\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})$ acting by multiplication. Thus, the Hilbert space of the quantum statistical mechanical system becomes $L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*)$ with the action of the idèle class group $C_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*$. The quotient space $X_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*$ is called the *adèle class space* and, as we see in more detail in Chapter 3 and Chapter 4, it should be regarded as a noncommutative space. While in this chapter we mostly treat $X_{\mathbb{Q}}$ as an ordinary space, we explain in §7.1 the reason why it is more natural to regard it as a noncommutative space, due to the ergodic nature of the action of $C_{\mathbb{Q}}$. We explain this in the case of the restriction to a set S of finitely many places (semi-local case $X_{\mathbb{Q},S} = \mathbb{A}_{\mathbb{Q},S}/\mathbb{Q}_S^*$). Illustrative examples of two or three places are discussed explicitly in §§7.1.1 and 7.1.2. We explain in §7.2 how to define on rapidly decaying functions on $\mathbb{A}_{\mathbb{Q},S}$ an inner product that descends to the coinvariants of the action of $\mathbb{Q}_S^*$ and we use it to define the Hilbert space $L^2(X_{\mathbb{Q},S})$ with the corresponding action of $C_{\mathbb{Q},S}$. We also extend the proof of the trace formula given via quantized calculus from the case of the single Archimedean place to the case of finitely many places, obtaining in this way the semi-local trace formula of [**71**]. A main feature of the trace formula is its *additivity*: even though the adèle class space is essentially a product over the set $\Sigma_{\mathbb{K}}$ of places of the global field $\mathbb{K}$ the trace computation delivers a *sum* over the set of places $S \subset \Sigma_{\mathbb{K}}$.

Throughout this chapter, we only deal with the semi-local version of the trace formula in the Hilbert space context, as in [**71**]. In Chapter 4 we return to discuss, at length, the trace formula and the spectral realization, but in a different setting like that of [**74**], where we give a cohomological interpretation of the trace formula. In that setting we will have all the zeros appearing (not just those on the critical line), while the Riemann Hypothesis will become a question equivalent to a positivity statement that resembles more closely the positivity argument in Weil's proof for function fields.

The terms that appear in the *semi-local* trace formula are compared in §8 with Weil's distributional formulation of Riemann's explicit formula relating primes and zeros of the zeta function. We begin by recalling in §§8.1 and 8.2 the definition of L-functions with Grössencharakter for a global field $\mathbb{K}$ (the Riemann zeta function being a special case) and the distribution $\mathbf{D}_{\mathbb{K}}$ on $C_{\mathbb{K}}$ and the Weil principal values that enter in the explicit formula. In §8.3 we recall the properties of Fourier transform on $C_{\mathbb{K}}$. In §8.4 we show the explicit computation of the principal values comparing the Weil principal values with those considered earlier in the semi-local trace formula of §7.2.

In particular, we discuss separately in §§8.4.1, 8.4.2, and 8.4.3 the case of a non-Archimedean place and the cases of a real and a complex Archimedean place. Finally, in §8.5, we give a reformulation of the Weil explicit formula in the form

$$\hat{h}(0) + \hat{h}(1) - \sum_{\chi \in \widehat{C_{\mathbb{K},1}}} \sum_{Z_{\tilde{\chi}}} \hat{h}(\tilde{\chi}, \rho) = \sum_v \int'_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|} \, d^*u,$$

where the principal values match the corresponding terms that appear in the semi-local trace formula of §7.2.

We then discuss the spectral realization of [**71**] of the zeros of the Riemann zeta function and of L-functions with Grössencharakter. We consider the cokernel of the map $\mathfrak{E}$, extended to this adelic setting, as an isometry $\mathfrak{E} : L^2_\delta(X_{\mathbb{K}})_0 \to L^2_\delta(C_{\mathbb{K}})$ of the form

$$\mathfrak{E}(f)\,(g) = |g|^{1/2} \sum_{q \in \mathbb{K}^*} f(qg)$$

for $g \in C_{\mathbb{K}}$, where the domain $L^2_\delta(X_{\mathbb{K}})_0$ is the subspace defined by the conditions $f(0) = \hat{f}(0) = 0$ in the Hilbert space of square-integrable functions on $X_{\mathbb{Q}}$ with a weight δ controlling the decay condition. The cokernel $\mathcal{H}$ of $\mathfrak{E}$ carries an induced representation $\underline{\vartheta}_{\mathrm{m}}(g)$, $g \in C_{\mathbb{K}}$ and can be decomposed as a sum $\bigoplus_\chi \mathcal{H}_\chi$ along characters. The infinitesimal generator of the induced scaling action of $\mathbb{R}^*_+$ on $\mathcal{H}_\chi$ has as spectrum the set of zeros on the critical line of the corresponding L-function with Grössencharakter. The trace is then given by

$$\mathrm{Tr}\, \underline{\vartheta}_{\mathrm{m}}(h) = \sum_{\substack{L\left(\tilde{\chi}, \frac{1}{2}+\rho\right)=0 \\ \rho \in i\,\mathbb{R}/N^\perp}} \hat{h}(\tilde{\chi}, \rho),$$

for $\underline{\vartheta}_{\mathrm{m}}(h) = \int \underline{\vartheta}_{\mathrm{m}}(\gamma)\, h(\gamma)\, d^*\gamma$ with $h \in \mathcal{S}(C_{\mathbb{K}})$. The proof of [**71**] of the theorem on the spectral realization (Theorem 2.47) is given then in §9.3, after discussing in §9.1 a way to express L-functions as normalization factors of certain homogeneous distributions on $\mathbb{A}_{\mathbb{K}}$ and, in §9.2, the use of approximate units f_n in the Sobolev space $L^2_\delta(C_{\mathbb{K}})$.

In §10 we consider instead L-functions of varieties defined over number fields as in [**74**]. We concentrate on the Archimedean local factors of the completed L-function. These are suitable products of gamma functions with shifts and exponents that depend upon the Hodge numbers of the complex algebraic variety determined by the embedding $\mathbb{K} \hookrightarrow \mathbb{C}$ defining an Archimedean place. By analogy with the Riemann zeta function and L-functions with Grössencharakter discussed in the previous sections, we consider the counting function of zeros for L-functions of varieties. Conjecturally, the L-functions $L(H^m(X), s)$ satisfy a functional equation and have zeros located on the critical line $\Re(s) = (1+m)/2$. The average part of the counting function $N(E) = \langle N(E) \rangle + N_{\mathrm{osc}}(E)$ should then be expressed in terms of the Archimedean factors

$$\langle N(E)\rangle = \frac{1}{\pi}\sum_{v|\infty} \Im \log L_v \left(H^m(X), \frac{1+m}{2} + iE \right).$$

We show that this expression can be formulated in terms of a Lefschetz trace formula, much as in the case of the Riemann zeta function. After recalling in §10.1 the definition of the Archimedean local L-factors and the counting function in §10.2, we proceed by expressing their logarithmic derivatives in distributional form in terms of principal values in §10.3. We show in §§10.4 and 10.5 that this produces, respectively in the case of a complex or a real place, a trace formula for the logarithmic derivative of the corresponding local factor. In §§10.6 and 10.7 we formulate some more general questions such as the problem of a trace formula involving several places and the related spectral realization, and the problem of finding a global space (the analog in this higher-dimensional setting of the adèle class space) on which the geometric side of the expected trace formula lives. We conclude the chapter with §10.8, where we draw an analogy between the real mixed Hodge structures involved in the definition of the Archimedean local L-factors and their motivic Galois group and the category of flat equisingular connections used in Chapter 1 in the context of perturbative renormalization, and their differential Galois group $\mathbb{U}^*$.

2. Counting primes and the zeta function

We begin by recalling some very well known facts about the Riemann zeta function. We do not prove these classical statements here, but the interested reader can find plenty of material on this subject, for instance by looking at the very pleasant book by Edwards [**126**] in which Riemann's original paper, which we essentially follow below, is reproduced.

Riemann's seminal paper "On the number of primes less than a given magnitude" (in [**255**], cf. also [**126**]) established a remarkable relation between the distribution of prime numbers and the zeros of the zeta function given by the Euler product

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = \prod_p (1-p^{-s})^{-1}, \quad \Re(s) > 1, \tag{2.1}$$

where p ranges over the prime numbers.

The function defined by (2.1) has analytic continuation to the complex plane and is regular everywhere except for a simple pole at $s=1$, where

$$\lim_{s\to 1} \zeta(s) - \frac{1}{s-1} = \gamma,$$

with γ the Euler constant. The analytic continuation is obtained in [**255**] using the Γ function

$$\Gamma(s)\, n^{-s} = \int_0^\infty e^{-nx} x^{s-1}\, dx,$$

CHAPTER 3

Quantum statistical mechanics and Galois symmetries

1. Overview: three systems

We have seen in the previous chapter how the adèles $\mathbb{A}_{\mathbb{Q}}$ and the noncommutative adèle class space $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*$ provide a natural geometric setting for a spectral realization of the zeros of the Riemann zeta function and an interpretation as a (semi-local) Lefschetz trace formula of the Weil explicit formula. In this chapter we discuss more in detail the geometry of the adèle class space, in terms of a simple geometric notion: the commensurability relation on $\mathbb{Q}$-lattices.

This formulation leads us to consider more general types of noncommutative adelic quotients and their relation to Galois theory. We follow [**30**], [**86**], [**88**], [**90**], [**91**]. All of the cases discussed in this Chapter are quantum statistical mechanical systems, with nontrivial phase transition phenomena, and with thermodynamical equilibrium states that, at sufficiently low-temperature, recover the points of a classical algebro-geometric moduli space.

The first system we analyze is the Bost–Connes (BC) system [**30**], which is closely related to the adèle class space and is described geometrically in terms of 1-dimensional $\mathbb{Q}$-lattices. As illustrated in the table below, its partition function is the Riemann zeta function, the extremal equilibrium states (KMS states) at sufficiently low-temperature are parameterized by the points of a very simple classical moduli space, the zero-dimensional Shimura variety $Sh(\mathrm{GL}_1, \{\pm 1\})$. The symmetries of the system are given by the group $\mathrm{GL}_1(\hat{\mathbb{Z}})$. The zero-temperature KMS states evaluated on a natural arithmetic subalgebra of the algebra of observables of the system take values that are algebraic numbers and generate the maximal abelian extension $\mathbb{Q}^{\mathrm{cycl}}$ of $\mathbb{Q}$. The class field theory isomorphism intertwines the action of the symmetries and the Galois action on the values of states, thus providing a quantum statistical mechanical reinterpretation of the explicit class field theory of $\mathbb{Q}$.

The second system we present in this chapter is a generalization of the BC system, where instead of considering 1-dimensional $\mathbb{Q}$-lattices, one works with 2-dimensional $\mathbb{Q}$-lattices and their commensurability classes. The corresponding quantum statistical mechanical system was introduced and studied in [**86**] (cf. also [**88**] for a brief summary of the main results of [**86**]).

Passing from 1-dimensional to 2-dimensional $\mathbb{Q}$-lattices corresponds, at the level of the corresponding classical algebro-geometric objects, to passing from the Shimura variety of GL_1 to that of GL_2. However, at the quantum statistical mechanical level and in terms of the noncommutative spaces, the GL_2-system exhibits many new properties that were not present in the case of the BC system. One such property is the fact that the symmetries of the system now involve not only automorphisms, but also endomorphisms. Symmetries given by endomorphisms are of crucial importance in order to relate again the quantum statistical mechanical system and its low-temperature KMS states to a rich Galois theory. In this case, this will be the Galois theory of the modular field, which we recall in this chapter. The partition function of the GL_2 system is again related to the Riemann zeta function $\zeta(s)$, this time in the form of a product $Z(\beta) = \zeta(\beta)\zeta(\beta - 1)$. The extremal low-temperature KMS states are parameterized by the points of the classical Shimura variety $Sh(\mathrm{GL}_2, \mathbb{H}^{\pm})$. There is an arithmetic algebra, which is here no longer a subalgebra but an algebra of unbounded multipliers of the algebra of observables, which naturally involves modular functions. For a generic choice of an extremal zero-temperature KMS state, the evaluation on arithmetic elements (multipliers) of the algebra intertwines symmetries of the system, given by the adelic group $\mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$, and the Galois action of the Galois group of the modular field, realized (via the choice of the state) as an embedded subfield of $\mathbb{C}$.

In both the GL_1- and the GL_2-system, the arithmetic algebra plays a fundamental role. The arithmetic algebras of these two systems can be seen from the point of view of Weil's analogy between trigonometric and elliptic functions developed in [**298**]. The generators of the arithmetic algebra of the BC system can be built out of functions of 1-dimensional lattices and the arithmetic algebra of the GL_2-system contains similar elements based on Eisenstein series and is defined by abstracting the basic properties of these elements. It naturally involves modular functions and Hecke correspondences.

Finally, we discuss another quantum statistical mechanical system, introduced in our joint work with N. Ramachandran [**90**], [**91**]. This system exhibits properties in between the BC and the GL_2 system. This is closely related to the adèle class space $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$ for $\mathbb{K}$ an imaginary quadratic extension of $\mathbb{Q}$. As the original BC system provides a reformulation of the explicit class field theory of $\mathbb{Q}$, this system will serve the same purpose with respect to the explicit class field theory of $\mathbb{K}$. As is well known, the latter involves in a fundamental way the arithmetic of elliptic curves with complex multiplication. This appears, in our quantum statistical mechanical setting, through the relation to the GL_2-system. Namely, we will see that the system for complex multiplication, based on commensurability of 1-dimensional $\mathbb{K}$-lattices, appears as a subsystem of the GL_2-system, by identifying 1-dimensional $\mathbb{K}$-lattices with a special class of 2-dimensional $\mathbb{Q}$-lattices through the choice of a basis $\{1, \tau\}$ for $\mathbb{K}$ as a vector space over $\mathbb{Q}$. The partition function of this system is the Dedekind zeta function of $\mathbb{K}$. The arithmetic subalgebra is then inherited from the GL_2-system and the values of zero-temperature KMS states and the Galois action are exactly those of class field theory. In this system, as in the GL_2 case, the symmetries are not only given by automorphisms. The presence of symmetries that are given by endomorphisms corresponds to the fact that the field $\mathbb{K}$ may have non-trivial class number.

We also discuss, towards the end of this chapter, some further generalizations of these systems to the case of Shimura varieties [**159**] and to function fields [**172**], [**104**], and the relation of the GL_2-system to the modular Hecke algebras of Connes and Moscovici [**93**], [**94**], [**95**].

The comparative properties of the three systems are illustrated in the table below and will be explained in detail in the rest of this chapter.

System	GL_1	GL_2	$\mathbb{K} = \mathbb{Q}(\sqrt{-d})$
Partition function	$\zeta(\beta)$	$\zeta(\beta)\zeta(\beta-1)$	$\zeta_{\mathbb{K}}(\beta)$
Symmetries	$\mathbb{A}^*_{\mathbb{Q},f}/\mathbb{Q}^*$	$GL_2(\mathbb{A}_{\mathbb{Q},f})/\mathbb{Q}^*$	$\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$
Symmetry group	Compact	Locally compact	Compact
Automorphisms	$\hat{\mathbb{Z}}^*$	$GL_2(\hat{\mathbb{Z}})$	$\hat{\mathcal{O}}^*/\mathcal{O}^*$
Endomorphisms		$GL_2^+(\mathbb{Q})$	$Cl(\mathcal{O})$
Galois group	$Gal(\mathbb{Q}^{ab}/\mathbb{Q})$	$Aut(F)$	$Gal(\mathbb{K}^{ab}/\mathbb{K})$
Extremal KMS_∞	$Sh(GL_1, \pm 1)$	$Sh(GL_2, \mathbb{H}^{\pm})$	$\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$

More precisely, in §2 we give a brief introduction to quantum statistical mechanics. We recall the notion of KMS states and its origin motivated by the quantum mechanical analog of the Gibbs measure. We review in §2.1 the general notions of observables, time evolution and Hamiltonian of a quantum statistical mechanical system. We also recall the notion of unbounded multipliers on a C^*-algebra, which will be useful in §§5, 6, 7. We introduce states in §2.2 and the KMS condition, including the case of zero temperature where we discuss the different notions of ground states and KMS_∞ states. We show that extremal KMS states at nonzero temperature form a Choquet simplex. We discuss the extension of states to the multiplier algebra. In §2.3 we discuss symmetries of quantum statistical mechanical systems, induced symmetries on states, both given by automorphisms and by endomorphisms, and the phenomenon of spontaneous symmetry breaking. The induced action of symmetries on zero-temperature KMS states is described in §2.4. The pushforward of KMS states is described in §2.5.

In §3 we introduce the geometric notion of $\mathbb{Q}$-lattices and the relation of commensurability. These are the fundamental notions used in the rest of the chapter to construct the relevant quantum statistical mechanical systems.

We begin in §4 to discuss the geometry of the space of 1-dimensional $\mathbb{Q}$-lattices modulo commensurability. We introduce the groupoid $\mathcal{G}_1$ of the equivalence relation and the one that describes commensurability on $\mathbb{Q}$-lattices up to scaling by $\mathbb{R}_+^*$. We introduce the corresponding C^*-algebras and recall the explicit presentation of the BC algebra $\mathcal{A}_1 = C^*(\mathcal{G}_1/\mathbb{R}_+^*)$. We describe in §4.1 the time evolution induced by the ratio of covolumes of commensurable lattices. In §4.2 we recall the description of the BC system in terms of Hecke algebras originally given in [**30**]. In §4.3 we show that $\hat{\mathbb{Z}}^*$ acts as symmetries of the BC system. In §4.4 we introduce the arithmetic subalgebra $\mathcal{A}_{1,\mathbb{Q}}$ of $\mathcal{A}_1$, by describing the generators in terms of functions of lattices, their interpretation in terms of trigonometric functions and the relation via Cayley transform to the exponential generators of the BC algebra. We derive explicit division relations between the generators. In §4.5 we give a quick reminder of the Kronecker–Weber theorem and the explicit class field theory of $\mathbb{Q}$. We then formulate the problem of the quantum statistical mechanical approach to explicit class field theory in §4.6, where we also recall the main result of [**30**] on the classification of KMS states for the BC system and its relation to the Galois theory of the cyclotomic field $\mathbb{Q}^{\rm cycl}$. We discuss in §4.7 the reason why it is not an unreasonable expectation that the approach via algebras and states may prove useful in the explicit class field theory problem. We show in particular how strong is the intertwining property of KMS states between symmetries and Galois action on values of states, by showing how the Galois automorphisms are generally badly behaved from the topological standpoint. We give in §4.8 a reinterpretation of the geometry of the BC system in terms of the Shimura variety of GL_1. We discuss briefly in §4.9 the relation between the algebra of the BC system and the algebra of the "dual system" describing commensurability classes of $\mathbb{Q}$-lattices not up to scaling. This relation will be fully developed in Chapter 4.

We move on to discuss the case of 2-dimensional $\mathbb{Q}$-lattices starting in §5. We describe the reformulation in terms of Tate modules of elliptic curves in §5.1. In §5.2 we introduce the relevant groupoids. Unlike in the 1-dimensional case, here one finds that dividing by the scaling action of $\mathbb{C}^*$ does not preserve the groupoid structure, but one can still define a corresponding convolution algebra $\mathcal{A}_2$. The time evolution induced by the ratio of covolumes is introduced in §5.3, where we also introduce the regular representation of $\mathcal{A}_2$, the associated von Neumann algebra, and the linear functional that gives the KMS_2 state. We discuss the symmetries of the system in §5.4, where we show that the group $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ acts by symmetry, with $\mathrm{GL}_2(\hat{\mathbb{Z}})$ acting by automorphisms and $\mathrm{GL}_2^+(\mathbb{Q})$ by endomorphisms. These combine to give an induced action of $\mathbb{Q}^*\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ on KMS states. We give the explicit form of the action by endomorphisms and of the induced action on KMS states.

In §6 we recall many known facts about the field of modular functions which we need to use in discussing the arithmetic properties of the GL_2-system. We recall explicitly the cases of level one and then more generally level N in §§6.1, 6.2, in particular the Weierstrass $\wp$-function, the j-function, the fact that the field $F_N(\mathbb{C})$ is a finite Galois extension of $\mathbb{C}(j)$, the explicit generators given by the Fricke functions, Eisenstein series, θ-functions, the role of torsion points of elliptic curves and the extensions $F_N(j)$ over $\mathbb{Q}(j)$. General facts about modular functions and modular forms are discussed in §6.3 and some explicit computations for the cases $N = 2$ and $N = 4$ are given in §6.4. Starting with §6.5 we relate the modular field to the geometry of 2-dimensional $\mathbb{Q}$-lattices. We show that an invertible $\mathbb{Q}$-lattice with transcendental j-invariant determines an embedding of the modular field in $\mathbb{C}$ and that all embeddings arise in this way and the same embeddings occur for $\mathbb{Q}$-lattices in the same orbit of the right $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ action. We also discuss the relation between $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ and the automorphisms of the modular field, and the cyclotomic action on the coefficients of the q-expansion.

Section 7 deals with the arithmetic properties of the GL_2-system. We begin in §7.1 by describing some explicit elements of what will be the arithmetic algebra of the GL_2-system. These elements are obtained from Eisenstein series and Hecke correspondences. We then abstract the general properties that we want to require and use them to define the arithmetic algebra $\mathcal{A}_{2,\mathbb{Q}}$ in §7.2. We show that the resulting algebra is a subalgebra of the unbounded multipliers of the C^*-algebra $\mathcal{A}_2$ and is globally invariant under the symmetries of the system described in §5.4. In §7.3 we describe explicit division relations satisfied by the special elements of the arithmetic subalgebra introduced in §7.1, which generalize the division relations for elliptic functions. A consequence of these relations is the fact that the subalgebra of $\mathcal{A}_{2,\mathbb{Q}}$ generated by these elements is finite-dimensional and reduced over $\mathbb{Q}(j)$; hence it defines an endomotive in the sense discussed in Chapter 4. We begin the classification of KMS states for the GL_2-system in §7.4, where we give a characterization of KMS states as measures on the space of 2-dimensional $\mathbb{Q}$-lattices up to scaling. We show that at low-temperature $\beta > 2$ these measures are supported on the commensurability classes of the invertible $\mathbb{Q}$-lattices. This gives the classification for low-temperature. We compute the partition function of the system and give an explicit formula for the extremal low-temperature KMS states. In §7.5 we describe explicitly the action of symmetries on low-temperature KMS states and in §7.6 we show that the zero-temperature KMS states associated to invertible $\mathbb{Q}$-lattices with transcendental j-invariant intertwine the action of symmetries of the system and the action of the automorphism group of the modular field on the values of states on arithmetic elements. The classification of KMS states then continues in §7.7, by first showing that there are no KMS states in the range $\beta < 1$ and then by the result of Laca–Larsen–Neshveyev on the uniqueness of the KMS states in the intermediate range $1 < \beta < 2$ and the existence of KMS states at $\beta = 1$. As in the case of the BC system, we give in §7.8 a reinterpretation of the geometry of this system in terms of the Shimura variety of GL_2 and we discuss in §7.9 the relation to the noncommutative boundary of modular curves. The compatibility between the BC system and the GL_2-system is described in §7.10.

The final part of the chapter starts in §8 and is dedicated to our joint work with Ramachandran on the quantum statistical mechanics of 1-dimensional $\mathbb{K}$-lattices, for $\mathbb{K}$ an imaginary quadratic field. The main geometric objects, 1-dimensional $\mathbb{K}$-lattices, and the commensurability relation are introduced in §8.1. The relation to 2-dimensional $\mathbb{Q}$-lattices is explained in §8.2. In §8.3 these notions are rewritten is a suitable adelic form, which provides an explicit description of the space of commensurability classes of 1-dimensional $\mathbb{K}$-lattices up to scaling as an adelic quotient. In §8.4 we introduce the corresponding groupoid and C^*-algebra and the time evolution induced by restriction from the GL_2-system. Describing 1-dimensional $\mathbb{K}$-lattices in terms of ideals one finds in §8.5 that the partition function is the Dedekind zeta function. The arithmetic subalgebra is obtained in §8.6 by restriction from the GL_2-system and the symmetries are described explicitly in §8.7 and give the correct group $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ acting on KMS states. The classification of KMS states of the system is obtained in §8.8, where it is shown that the zero-temperature states have the intertwining property between the symmetries and the Galois action of the Galois group of the maximal abelian extension of $\mathbb{K}$. The low-temperature extremal KMS states, for $\beta > 1$, are parameterized by invertible 1-dimensional $\mathbb{K}$-lattices. For high temperature $\beta \leq 1$ there is a unique KMS state. This is shown in §8.9. Sections 8.9 and 8.10 compare the system for $\mathbb{K}$ with other known systems and quickly describe generalizations of these systems to arbitrary Shimura varieties constructed by Ha and Paugam [**159**].

2. Quantum statistical mechanics

Consider a classical system with Hamiltonian $H(q,p)$ and phase space a symplectic manifold X with local coordinates (q,p) in which the symplectic form ω has the standard Darboux form. In classical statistical mechanics a state for such a Hamiltonian system consists of a probability measure μ on the phase space X, which assigns to each observable f an expectation value, in the form of an average

$$\int_X f\, d\mu. \tag{3.1}$$

In particular, the Hamiltonian $H(q,p)$ and the symplectic structure on the phase space X determine a state, called the *Gibbs canonical ensemble*. It is

CHAPTER 4

Endomotives, thermodynamics, and the Weil explicit formula

The spectral realization of the zeros of L-functions described in Chapter 2 made little explicit use of the formalism of noncommutative geometry, except for the use of the quantized calculus in proving the semi-local trace formula.

One of our goals in this chapter is to clarify the conceptual meaning of the spectral realization and in particular of the map $\mathfrak{E}$ of Chapter 2 §6 in terms of noncommutative geometry and cyclic cohomology. This chapter is based on our joint work with Consani [**74**], [**75**].

There are three essential ingredients in the conceptual understanding of the spectral realization:

(1) Geometry
(2) Thermodynamics
(3) Cohomology and motives

In the first step we extend the notion of zero-dimensional motive, i.e. of Artin motive, to the noncommutative set-up. The main examples of a noncommutative Artin motive we are interested in arise from endomorphisms of algebraic varieties. This is in fact the origin of the terminology "endomotive" we use for these zero-dimensional noncommutative spaces. The reason for thinking of this class of noncommutative spaces as motives lies in the fact that we define morphisms as correspondences, generalizing the correspondences by algebraic cycles used in the theory of motives.

We work over an algebraic number field $\mathbb{K}$ and the absolute Galois group, i.e. the Galois group G of $\bar{\mathbb{K}}/\mathbb{K}$, will act on the various objects we construct (cf. §2.3). Given an abelian semigroup S of endomorphisms of an algebraic variety Y and a point $y_0 \in Y$ fixed under S, the projective limit X of the preimages of y_0 under elements of S admits a natural action of S and this gives rise to an endomotive (X, S). Such datum is best encoded as an algebraic groupoid $\mathcal{G} = X \rtimes S$ and the associated convolution algebra $\mathcal{A}_{\mathbb{K}}$.

We first recall in §1 the basic notions of morphisms for noncommutative spaces provided by Kasparov's KK-theory in the context of C^*-algebras (§1.1) and by the abelian category of cyclic modules in the general algebraic context (§§1.2, 1.3). We describe the relation of the cyclic category to cyclic (co)homology in §1.4.

We then present in §2 the construction of a category of *endomotives* that extends the notion of morphism given by geometric correspondences from the case of Artin motives to the noncommutative endomotives. We construct this category first at the purely algebraic level, i.e. dealing with zero-dimensional pro-varieties over $\mathbb{K}$, in §2.1. We then give a construction at the analytic level in §2.2, working with totally disconnected locally compact spaces. We show in §2.4 that under a "uniform" condition on the projective system that determines an endomotive, this comes endowed with a probability measure, which induces a state on the corresponding C^*-algebra.

We then show in Theorem 4.34 how the functor $\mathcal{P}$ that replaces a variety over $\mathbb{K}$ by its set of $\bar{\mathbb{K}}$ points gives a natural relation between the categories of algebraic and analytic endomotives, cf. §2.5. Moreover, the Galois group G acts by natural transformations of $\mathcal{P}$.

In §2.6 we describe the construction of the endomotive associated to an abelian semigroup S of endomorphisms of an algebraic variety Y. We then show in §2.7 that the simplest example, i.e. the semi-group of endomorphisms of the multiplicative group $\mathbb{G}_m$ for $\mathbb{K} = \mathbb{Q}$, gives an endomotive that coincides at the analytic level with the Bost-Connes system analyzed in Chapter 3.

We discuss in §3 the problem of extending correspondences from endomotives to noncommutative spaces in higher dimension, in a way that is compatible with the definition of correspondences in the algebro-geometric context in terms of K-theory. In §3.1 we recall the setting of geometric correspondences in KK-theory. In §3.2 we compare this with the cycle map defined using algebraic K-theory and we comment in §3.2.1 on the relation between algebraic and topological K-theory and cyclic and Hochschild (co)homology. We also comment on the relation between motives and noncommutative spaces in the context of noncommutative tori and abelian varieties in §3.2.2.

As a conclusion of this first step of the construction, we obtain from the data of a uniform endomotive a noncommutative geometric datum given by a pair $(\mathcal{A}, \varphi)$ of an involutive algebra and a state, together with an action by automorphisms of the Galois group G.

The second step then involves thermodynamics, which we develop in §4. As will become apparent in this chapter, a basic new feature of noncommutative spaces plays a dominant role in our interpretation: these spaces have thermodynamical properties and in particular they can be analyzed at different temperatures by classifying the KMS_β states for the modular automorphism group σ_t^φ, as we already did in in Chapter 3 for three basic examples discussed there.

What we saw in Chapter 3 is that, by lowering the temperature, a given noncommutative space tends to become more and more commutative or *classical*, so that in good situations the extremal KMS_β states play the role of *classical points* of the space. The key feature of extremal KMS_β states that can play the role of points is that the corresponding factor (obtained from the state by the GNS construction) is Morita equivalent to a point, i.e. it is of type I.

After recalling in §4.1 Tomita's theory of the modular automorphism group σ_t^φ, we analyze the type I extremal KMS_β states in §4.2 and we show in particular (cf. Lemma 4.56) that such states persist at lower temperatures. Thus, cooling down the system by lowering the temperature has the effect of adding more and more classical points of the noncommutative space.

The space Ω of classical points comes naturally equipped with a principal $\mathbb{R}_+^*$-bundle $\tilde{\Omega}$. In Theorem 4.85 we obtain a conceptual understanding of the map $\mathfrak{E}$ of Chapter 2 §6 as the natural morphism π of restriction from the dual system defined in §4.3 to the $\mathbb{R}_+^*$-bundle $\tilde{\Omega}$ over the space of classical points, cf. §§4.5, 4.6, 4.7. We first deal with general systems in §4.8. We then specialize to the BC system in §4.9.

We also explain in §4.4 why passing to the dual system is the analogue in characteristic zero of the unramified extension $\mathbb{K} \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ for a global field of positive characteristic. This gives a first indication for the interpretation of the scaling action on the dual system as a characteristic zero analog of the action of Frobenius that we go on to develop in the subsequent sections.

The third step, in fact, provides a replacement in characteristic zero for the action of Frobenius on étale cohomology, given in terms of the scaling action on the cyclic homology of the cokernel of the above restriction map π. This is described in §§4.8, 4.9, 4.10. The latter subsection relates the scaling action on the cyclic homology of the cokernel of the restriction map of the Bost–Connes endomotive to the spectral realization of zeros of the Riemann zeta function.

It is for this analysis of the scaling action and the correct definition of the cokernel of the restriction map that one needs to work in an abelian category of motivic nature, so that one can make sense of the cokernel of a morphism of algebras. This abelian category is the category of Λ-modules, the central tool of cyclic cohomology, which was introduced in §1.2. Since traces Tr define cyclic morphisms $\mathrm{Tr}^\natural$ one can compose the restriction morphism π with the trace on trace class operators and obtain, using the results of §4.5, a cyclic morphism $(\mathrm{Tr} \circ \pi)^\natural$ whose range is contained in the cyclic module of the *commutative* algebra $C(\tilde{\Omega})$. The cyclic homology of the cokernel produces a representation of the product $G \times \mathbb{R}_+^*$ of the Galois group by the multiplicative group $\mathbb{R}_+^*$. The main result is then to apply this general framework to a specific object in the category of endomotives, which corresponds to the Bost–Connes system, and obtain a cohomological version of the Weil explicit formula as a trace formula.

These results are then extended to an arbitrary global field $\mathbb{K}$ in §5. In §5.1 we describe in this generality the adèle class space of a global field and its algebra of coordinates as a noncommutative space. In §5.2 we describe the corresponding cyclic module in the abelian category of Λ-modules and in §5.3

we give the general form of the restriction map corresponding to the inclusion of the idèle class space as "classical points" of the noncommutative space of adèle classes, as a cyclic morphism. We describe this more explicitly in the case of $\mathbb{K} = \mathbb{Q}$ in §5.4. In §5.5 we return to the general case of global fields and we show how the cokernel of the restriction map can be thought of as a motivic H^1 with an induced action of the idèle class group $C_{\mathbb{K}}$. We show that this gives the spectral realization of the zeros of L-functions with Grössencharakter. We show then in §5.6 that the action on this H^1 of functions in the "strong Schwartz space" of $C_{\mathbb{K}}$ that are in the range of the restriction map vanishes identically. Finally, in §5.7 we give the formulation of the Weil explicit formula as a Lefschetz trace formula for the action on the H^1 of §5.5 of elements in the "strong Schwartz space" of $C_{\mathbb{K}}$. We then show in §5.8 that, in terms of this Lefschetz trace formula, the Riemann Hypothesis becomes a statement equivalent to positivity of the induced trace pairing.

We then begin with §6 and §7 a comparative analysis of the Weil proof of the Riemann Hypothesis for function fields and the noncommutative geometry of the adèle class space, as in [**75**]. The aim is to develop a sufficiently rich dictionary that will eventually provide good noncommutative geometry analogs for all the main algebro-geometric notions involved in the Weil proof, such as algebraic curves, divisors, linear equivalence, Riemann–Roch theorem, and Weil positivity.

We begin in §6 by recalling the essential steps in the Weil proof of the Riemann Hypothesis for function fields.

We continue the chapter by drawing some compelling analogies between the Weil proof of the Riemann Hypothesis for function fields and the setting of noncommutative geometry.

In §7 we begin to build the corresponding noncommutative geometry notion, that will be summarized in the dictionary in §7.7. We begin by recalling in §7.1 the computation of the distributional trace of a flow on a smooth manifold. We then describe in §7.2 the periodic orbits of the action of $C_{\mathbb{K}}$ on the adèle class space. This gives a natural definition of the scaling correspondence that parallels the role of the Frobenius correspondence in the Weil proof (cf. §7.3). We then show in §7.4 that the step in the Weil proof of adjusting the degree of a correspondence by trivial correspondences is achieved by correspondences coming from the range of the restriction map as in §5.6. A subtle failure of Fubini's theorem is what makes it possible to use these correspondences to adjust the degree in our case, as shown in §7.4. We then show in §7.5 that, in the case of function fields, one can recover the algebraic points of the curve inside the adèle class space as classical points of the periodic orbits of the dual system. This leads us to refine in §7.5.1 the notion described in §4.4 of passing to the dual system as an analog in characteristic zero of the unramified extension $\mathbb{K} \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. In fact, we now show that there is a natural way to associate to a noncommutative space X not only its set of "classical points", but in a more subtle manner the set of all its classical points that can be defined over an unramified extension. This is obtained through the following basic steps:

$$X \stackrel{\text{Dual System}}{\longrightarrow} \hat{X} \stackrel{\text{Periodic Orbits}}{\longrightarrow} \cup \hat{X}_v \stackrel{\text{Classical Points}}{\longrightarrow} \cup \Xi_v = \Xi.$$

We also describe in §7.5.2, in the case of $\mathbb{K} = \mathbb{Q}$, the quantum statistical mechanical systems associated to the different valuations and how these classical points of the periodic orbits appear as low temperature KMS states for these systems. We describe the cyclic covering of the set of classical points of the periodic orbits in §7.5.3 and in §7.5.4 we obtain a global Morita equivalence that gives the valuation systems globally from the groupoid of the adèle class space (in fact a groupoid that differs from it only in the part that belongs to $C_{\mathbb{K}}$ which disappears upon "taking the complement" via the reduction map). We describe then in §7.5.5 an arithmetic subalgebra for the locus of classical points of the periodic orbits in terms of operators that behave like the Frobenius and local monodromy of the Weil–Deligne group at arithmetic infinity.

We then use in §7.6 the geometry of vortex equations and moduli spaces to justify thinking of the step in the Weil proof that makes a correspondence effective by modifying it via linear equations using the Riemann–Roch theorem in terms of achieving transversality (surjectivity) for a morphism of C^*-modules via a compact perturbation. We then assemble our tentative dictionary between the Weil proof and noncommutative geometry in §7.7.

We conclude this last chapter of the book by drawing in §8 a broad analogy between our approach to the Riemann zeta function and L-functions via the noncommutative geometry of $\mathbb{Q}$-lattices and its thermodynamical properties and the question of a good setup for quantum gravity in physics. Our starting point is a comparison in §8.1 between the role of spontaneous symmetry breaking in the origin of masses from the electroweak phase transition in physics and the phase transitions that occur in our general framework of interaction between quantum statistical mechanics and number theory described in Chapter 3. We propose in §8.2 and §8.4 the existence of a noncommutative algebra of coordinates on a space of "spectral correspondences" between product geometries of the type used in Chapter 1 in modelling particle physics, and a corresponding time evolution giving rise to KMS states that yield, at low temperature, the usual notion of geometry. This possible approach shows that gravity described by classical (pseudo)Riemannian geometry may be a low temperature phenomenon while geometry may disappear entirely in the high temperature regime, just as in the case of 2-dimensional $\mathbb{Q}$-lattices there are no KMS states above a certain temperature. This last section is more speculative in nature and we hope it will open the way to some future developments of the material collected in this book.