Preface

The unifying theme, which the reader will encounter in different guises throughout the book, is the interplay between noncommutative geometry and number theory, the latter especially in its manifestation through the theory of motives. For us, this interwoven texture of noncommutative spaces and motives will become a tool in the exploration of two spaces, whose role is central to many developments of modern mathematics and physics:

- Space-time
- The set of prime numbers

One may be tempted to think that, looking from the vantage point of those who sit atop the vast edifice of our accumulated knowledge of such topics as space and numbers, we ought to know a great deal about these two spaces. However, there are two fundamental problems whose difficulty is a clear reminder of our limited knowledge, and whose solution would require a more sophisticated understanding than the one currently within our immediate grasp:

- The construction of a theory of quantum gravity (QG)
- The Riemann hypothesis (RH)

The purpose of this book is to explain the relevance of noncommutative geometry (NCG) in dealing with these two problems. Quite surprisingly, in so doing we shall discover that there are deep analogies between these two problems which, if properly exploited, are likely to enhance our grasp of both of them.

Although the book is perhaps more aimed at mathematicians than at physicists, or perhaps precisely for that reason, we choose to begin our account in Chapter 1 squarely on the physics side. The chapter is dedicated to discussing two main topics:

- Renormalization
- The Standard Model of high energy physics

We try to introduce the material as much as possible in a self-contained way, taking into consideration the fact that a significant number of mathematicians do not necessarily have quantum field theory and particle physics as part of their cultural background. Thus, the first half of the chapter is dedicated to giving a detailed account of perturbative quantum field theory, presented in a manner that, we hope, is palatable to the mathematician's taste. In particular, we discuss basic tools, such as the effective action and the perturbative expansion in Feynman graphs, as well as the regularization procedures used to evaluate the corresponding Feynman integrals. In particular, we concentrate on the procedure known as "dimensional regularization", both because of its being the one most commonly used in the actual calculations of particle physics, and because of the fact that it admits a very nice and conceptually simple interpretation in terms of noncommutative geometry, as we will come to see towards the end of the chapter. In this first half of Chapter 1 we give a new perspective on perturbative quantum field theory, which gives a clear mathematical interpretation to the *renormalization* procedure used by physicists to extract finite values from the divergent expressions obtained from the evaluations of the integrals associated to Feynman diagrams. This viewpoint is based on the Connes–Kreimer theory and then on more recent results by the authors.

Throughout this discussion, we always assume that we work with the procedure known in physics as "dimensional regularization and minimal substraction". The basic result of the Connes–Kreimer theory is then to show that the renormalization procedure corresponds exactly to the Birkhoff factorization of a loop $\gamma(z) \in G$ associated to the unrenormalized theory evaluated in complex dimension $D - z$, where D is the dimension of space-time and $z \neq 0$ is the complex parameter used in dimensional regularization. The group G is defined through its Hopf algebra of coordinates, which is the Hopf algebra of Feynman graphs of the theory. The Birkhoff factorization of the loop gives a canonical way of removing the singularity at $z = 0$ and obtaining the required finite result for the physical observables. This gives renormalization a clear and well defined conceptual meaning.

The Birkhoff factorization of loops is a central tool in the construction of solutions to the "Riemann-Hilbert problem", which consists of finding a differential equation with prescribed monodromy. With time, out of this original problem a whole area of mathematics developed, under the name of "Riemann–Hilbert correspondence". Broadly speaking, this denotes a way of encoding objects of differential geometric nature, such as differential systems with specified types of singularities, in terms of group representations. In its most general form, the Riemann–Hilbert correspondence is formulated as an equivalence of categories between the two sides. It relies on the "Tannakian formalism" to reconstruct the group from its category of representations. We give a general overview of all these notions, including the formalism of Tannakian categories and its application to differential systems and differential Galois theory.

The main new result of the first part of Chapter 1 is the explicit identification of the Riemann–Hilbert correspondence secretly present in perturbative renormalization.

At the geometric level, the relevant category is that of *equisingular flat vector bundles*. These are vector bundles over a base space B which is a principal $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$-bundle

$$\mathbb{G}_m(\mathbb{C}) \to B \xrightarrow{\pi} \Delta$$

over an infinitesimal disk Δ. From the physical point of view, the complex number $z \neq 0$ in the base space Δ is the parameter of dimensional regularization, while the parameter in the fiber is of the form $\hbar\mu^z$, where $\hbar$ is the Planck constant and μ is a unit of mass. These vector bundles are endowed with a flat connection in the complement of the fiber over $0 \in \Delta$. The fiberwise action of $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ is given by $\hbar\frac{\partial}{\partial\hbar}$. The equisingularity of the flat connection is a mathematical translation of the independence (in the minimal subtraction scheme) of the counterterms on the unit of mass μ. It means that the singularity of the connection, restricted to a section $z \in \Delta \mapsto \sigma(z) \in B$ of the bundle B, only depends upon the value $\sigma(0)$ of the section.

We show that the category of equisingular flat vector bundles is a Tannakian category and we identify explicitly the corresponding group (more precisely, affine group scheme) that encodes, through its category of finite dimensional linear representations, the Riemann-Hilbert correspondence underlying perturbative renormalization. This is a very specific proalgebraic group of the form $\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$, whose unipotent part $\mathbb{U}$ is associated to the free graded Lie algebra

$$\mathcal{F}(1, 2, 3, \cdots)_\bullet$$

with one generator in each degree. We show that this group acts as a universal symmetry group of all renormalizable theories and has the properties of the "Cosmic Galois group" conjectured by Cartier. In many ways this group should be considered as the proper mathematical incarnation of the renormalization group whose role, as a group encoding the ambiguity inherent to the renormalization process in quantum field theory, is similar to that of the Galois group in number theory.

We conclude the first part of Chapter 1 with a very brief introduction to the theory of *motives* initiated by Grothendieck. We draw some parallels between the Tannakian formalism used in differential Galois theory and in particular in our formulation of perturbative renormalization and the same formalism in the context of motivic Galois groups. In particular we signal the fact that the group $\mathbb{U}^*$ also appears (albeit via a non-canonical identification) as a motivic Galois group in the theory of mixed Tate motives. This "motivic nature" of the renormalization group remains to be fully understood.

While the discussion in the first part of Chapter 1 applies to arbitrary renormalizable theories, the second part of this chapter is concerned with the theory which, as of the writing of this book, represents the best of our current knowledge of particle physics: the *Standard Model*. This part is based on joint work of the authors with Ali Chamseddine.

Our main purpose in the second part of Chapter 1 is to show that the intricate Lagrangian of the Standard Model minimally coupled to gravity, where we incorporate the terms that account for recent findings in neutrino physics, can be completely derived from very simple mathematical data. The procedure involves a modification of the usual notion of space-time geometry using the formalism of noncommutative geometry.

Again we do not assume that the reader has any familiarity with particle physics, so we begin this second part of Chapter 1 by reviewing the fundamental facts about the physics of the standard model and its coupling with gravity, in a formulation which is as close as possible to that of the physics literature. A main point that it is important to stress here is the fact that the standard model, in all its complexity, was built over the years as a result of a continuing dialogue between theory and experiment. The result is striking in its depth and complexity: even just the typesetting of the Lagrangian is in itself a time-consuming task.

After this introductory part, we proceed to give a brief description of the main tools of noncommutative geometry that will be relevant to our approach. They include cyclic and Hochschild cohomologies and the basic paradigm of *spectral triples* $(\mathcal{A}, \mathcal{H}, D)$. An important new feature of such geometries, which is absent in the commutative case, is the existence of inner fluctuations of the metric. At the level of symmetries, these correspond to the subgroup of inner automorphisms, a normal subgroup of the group of automorphisms which is non-trivial precisely in the noncommutative case.

We then begin the discussion of our model. This can be thought of as a form of unification, based on the symplectic unitary group in Hilbert space, rather than on finite dimensional Lie groups. The internal symmetries are unified with the gravitational ones. They all arise as automorphisms of the noncommutative algebra of coordinates on a product of an ordinary Riemannian spin manifold M by a finite noncommutative space F. One striking feature that emerges from the computations is the fact that, while the metric dimension of F is zero, its K-theoretic dimension (in real K-theory) is equal to 6 modulo 8.

A long detailed computation then shows how the Lagrangian of the Standard Model minimally coupled with gravity is obtained naturally (in Euclidean form) from spectral invariants of the inner fluctuations of the product metric on $M \times F$.

This model provides specific values of some of the parameters of the Standard Model at unification scale, and one obtains physical predictions by running them down to ordinary scales through the renormalization group, using the Wilsonian approach. In particular, we find that the arbitrary parameters of the Standard Model, as well as those of gravity, acquire a clear geometric meaning in this model, in terms of moduli spaces of Dirac operators on the noncommutative geometry and of the asymptotic expansion of the corresponding *spectral action functional*. Among the physical predictions are relations between some of the parameters of the Standard Model, such as the merging of the coupling constants and a relation between the fermion and boson masses at unification.

Finally, in the last section of Chapter 1, we come to another application of noncommutative geometry to quantum field theory, which brings us back to the initial discussion of perturbative renormalization and dimensional regularization. We construct natural noncommutative spaces X_z of dimension a complex number z, where the dimension here is meant in the sense of the *dimension spectrum* of spectral triples. In this way, we find a concrete geometric meaning for the dimensional regularization procedure.

We show that the algebraic rules due to 't Hooft–Veltman and Breitenlohner–Maison on how to handle chiral anomalies using the dimensional regularization procedure are obtained, as far as one loop fermionic graphs are concerned, using the inner fluctuations of the metric in the product by the spaces X_z. This fits with the similar procedure used to produce the Standard Model Lagrangian from a product of an ordinary geometry by the finite geometry F and establishes a relation between chiral anomalies, computed using dimensional regularization, and the local index formula in NCG.

Towards the end of Chapter 1, one is also offered a first glance at the problem posed by a functional integral formulation of quantum gravity. We return only at the very end of the book to the problem of constructing a meaningful theory of quantum gravity, building on the experience we gain along the way through the analysis of quantum statistical mechanical systems arising from number theory, in relation to the statistics of primes and the Riemann zeta function. These topics form the second part of the book, to which we now turn.

The theme of Chapter 2 is the Riemann zeta function and its zeros. Our main purpose in this part of the book is to describe a spectral realization of the zeros as an absorption spectrum and to give an interpretation as a trace formula of the Riemann–Weil explicit formula relating the statistics of primes to the zeros of zeta. The role of noncommutative geometry in this chapter is twofold.

In the first place, the space on which the trace formula takes place is a noncommutative space. It is obtained as the quotient of the adèles $\mathbb{A}_{\mathbb{Q}}$ by the action of non-zero rational numbers by multiplication. Even though the resulting space $X = \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*$ is well defined set-theoretically, it should be thought of as a noncommutative space, because the ergodicity of the action of $\mathbb{Q}^*$ on $\mathbb{A}_{\mathbb{Q}}$ prevents one from constructing measurable functions on the quotient X, as we show in Chapter 3. In particular, the construction of function spaces on X is done by homological methods using coinvariants. This will only acquire a full conceptual meaning in Chapter 4, using cyclic cohomology and the natural noncommutative algebra of coordinates on X.

The space X can be approximated by simpler spaces X_S obtained by restriction to finite sets S of places of $\mathbb{Q}$. We use this simplified setup to obtain the relation with the Riemann–Weil explicit formula. The main point is that, even though the space X_S is in essence a *product* of terms corresponding to the various places, the trace of the action of the group of idèle classes becomes a *sum* of such contributions. It is in the proof of this key additivity property that we use another tool of noncommutative geometry: the quantized calculus.

In the simplest instance, the interpretation of the Riemann–Weil explicit formula as a trace formula gives an interpretation as symplectic volume in phase space for the main term of the Riemann counting function for the asymptotic expansion of the number of non-trivial zeros of zeta of imaginary part less than E. We show that a full quantum mechanical computation then gives the complete formula.

We end Chapter 2 by showing how this general picture and methods extend to the zeta functions of arithmetic varieties, leading to a Lefschetz formula for the local L-factors associated by Serre to the Archimedean places of a number field. The Serre formula describes the Archimedean factors as products of shifted Gamma functions with the shifts and the exponents depending on Hodge numbers. We derive this formula directly from a Lefschetz trace formula for the action of the Weil group on a bundle with base the complex line or the quaternions (for a real place) and with fiber the Hodge realization of the variety.

The origin of the relation described above between the Riemann zeta function and noncommutative geometry can be traced to the work of Bost–Connes. This consists of the construction, using Hecke algebras, of a quantum statistical mechanical system whose partition function is the Riemann zeta function and which exhibits a surprising relation with the class field theory of the field $\mathbb{Q}$. Namely, the system admits as a natural symmetry group the group of idèle classes of $\mathbb{Q}$ modulo the connected component of the identity. This symmetry of the system is spontaneously broken at the critical temperature given by the pole of the partition function. Below this temperature, the various phases of the system are parameterized by embeddings $\mathbb{Q}^{cycl} \to \mathbb{C}$ of the cyclotomic extension $\mathbb{Q}^{cycl}$ of $\mathbb{Q}$. These different phases are described in terms of extremal KMS_β states, where $\beta = \frac{1}{T}$ is the inverse temperature. Moreover, another important aspect of this construction is the existence of a natural algebra of "rational observables" of this quantum statistical mechanical system. This allows one to define in a conceptual manner an action of the Galois group $\mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q})$ on the phases of the system at zero temperature, merely by acting on the values of the states on the rational observables, values which turn out to provide a set of generators for $\mathbb{Q}^{cycl}$, the maximal abelian extension of $\mathbb{Q}$.

Our main purpose in Chapter 3 is to present extensions of this relation between number theory and quantum statistical mechanics to more involved examples than the case of rational numbers. In particular we focus on two cases. The first corresponds to replacing the role of the group GL_1 in the Bost–Connes (BC) system with GL_2. This yields an interesting nonabelian case, which is related to the Galois theory of the field of modular functions. The second is a closely related case of abelian class field theory, where the field $\mathbb{Q}$ is replaced by an imaginary quadratic extension. The results concerning these two quantum statistical mechanical systems are based, respectively, on work of the authors and on a collaboration of the authors with Niranjan Ramachandran.

We approach these topics by first providing a reinterpretation of the BC system in terms of geometric objects. These are the $\mathbb{Q}$-lattices, i.e. pairs (Λ, ϕ) of a lattice $\Lambda \subset \mathbb{R}^n$ (a cocompact free abelian subgroup of $\mathbb{R}^n$ of rank n) together with a homomorphism of abelian groups

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda.$$

Two $\mathbb{Q}$-lattices are commensurable if and only if

$$\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2 \quad \text{and} \quad \phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2.$$

Let $\mathcal{L}_n$ denote the set of commensurability classes of n-dimensional $\mathbb{Q}$-lattices. Even in the simplest one-dimensional case ($n = 1$) the space $\mathcal{L}_n$ is a noncommutative space. In fact in the one-dimensional case it is closely related to the adèle class space $X = \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*$ discussed in Chapter 2.

We first construct a canonical isomorphism of the algebra of the BC system with the algebra of noncommutative coordinates on the quotient of $\mathcal{L}_1$ by the scaling action of $\mathbb{R}_+^*$. Following Weil's analogy between trigonometric and elliptic functions, we then show that the trigonometric analogue of the Eisenstein series generate, together with the commensurability with division points, the arithmetic subalgebra of "rational observables" of the BC system. This opens the way to the higher dimensional case and much of Chapter 3 is devoted to the extension of these results to the two-dimensional case.

The system for the GL_2 case is more involved, both at the quantum statistical level, where there are two phase transitions and no equilibrium state above a certain temperature, and at the number theoretic level, where the cyclotomic field $\mathbb{Q}^{cycl}$ is replaced by the modular field.

We end Chapter 3 with the description of our joint results with Ramachandran on the extension of the BC system to imaginary quadratic fields. This is based on replacing the notion of $\mathbb{Q}$-lattices with an analogous notion of 1-dimensional $\mathbb{K}$-lattices, with $\mathbb{K}$ the imaginary quadratic extension of $\mathbb{Q}$. The relation between commensurability of 1-dimensional $\mathbb{K}$-lattices and of the underlying 2-dimensional $\mathbb{Q}$-lattices gives the relation between the quantum statistical mechanical system for imaginary quadratic fields and the GL_2-system. This yields the relation between the quantum statistical mechanics of $\mathbb{K}$-lattices and the explicit class field theory of imaginary quadratic fields.

Underlying our presentation of the main topics of Chapter 3 there is a unifying theme. Namely, the three different cases of quantum statistical mechanical systems that we present in detail all fit into a similar general picture, where an ordinary moduli space is recovered as the set of classical points (zero temperature states) of a noncommutative space with a natural time evolution. In the setting of Chapter 3 the classical spaces are Shimura varieties, which can be thought of as moduli spaces of motives. This general picture will provide a motivating analogy for our discussion of the quantum gravity problem at the end of the book.

The spectral realization of zeros of zeta and L-functions described in Chapter 2 is based on the action of the idèle class group on the noncommutative space X of adèles classes. Nevertheless, the construction, as we describe it in Chapter 2, makes little use of the formalism of noncommutative geometry and no direct use of the crossed product algebra $\mathcal{A}$ describing the quotient of adèles by the multiplicative group $\mathbb{Q}^*$.

In Chapter 4, the last chapter of the book, we return to this theme. Our main purpose is to show that the spectral realization described in Chapter 2 acquires cohomological meaning, provided that one reinterprets the construction in terms of the crossed product algebra $\mathcal{A}$ and cyclic cohomology. This chapter is based on our joint work with Caterina Consani.

We begin the chapter by explaining how to reinterpret the entire construction of Chapter 2 in "motivic" terms using

- An extension of the notion of Artin motives to suitable projective limits, which we call *endomotives*.
- The category of cyclic modules as a linearization of the category of noncommutative algebras and correspondences.
- An analogue of the action of the Frobenius on ℓ-adic cohomology, based on a thermodynamical procedure, which we call *distillation*.

The construction of an appropriate "motivic cohomology" with a "Frobenius" action of $\mathbb{R}_+^*$ for endomotives is obtained through a very general procedure. It consists of three basic steps, starting from the data of a noncommutative algebra $\mathcal{A}$ and a state φ. One considers the time evolution $\sigma_t \in \operatorname{Aut}\mathcal{A}$, for $t \in \mathbb{R}$, naturally associated to the state φ.

The first step is what we refer to as *cooling*. One considers the space $\mathcal{E}_\beta$ of extremal KMS_β states, for β greater than critical. Assuming these states are of type I, one obtains a morphism

$$\pi : \mathcal{A} \rtimes_\sigma \mathbb{R} \to \mathcal{S}(\mathcal{E}_\beta \times \mathbb{R}_+^*) \otimes \mathcal{L}^1,$$

where $\mathcal{A}$ is a dense subalgebra of a C^*-algebra $\bar{\mathcal{A}}$, and where $\mathcal{L}^1$ denotes the ideal of trace class operators. In fact, one considers this morphism restricted to the kernel of the canonical trace τ on $\bar{\mathcal{A}} \rtimes_\sigma \mathbb{R}$.

The second step is *distillation*, by which we mean the following. One constructs a cyclic module $D(\mathcal{A}, \varphi)$ which consists of the cokernel of the cyclic morphism given by the composition of π with the trace $\mathrm{Tr} : \mathcal{L}^1 \to \mathbb{C}$.

The third step is then the *dual action*. Namely, one looks at the spectrum of the canonical action of $\mathbb{R}_+^*$ on the cyclic homology

$$HC_0(D(\mathcal{A}, \varphi)).$$

This procedure is quite general and it applies to a large class of data $(\mathcal{A}, \varphi)$, producing spectral realizations of zeros of L-functions. It gives a representation of the multiplicative group $\mathbb{R}_+^*$ which combines with the natural representation of the Galois group G when applied to the noncommutative space (analytic endomotive) associated to an (algebraic) endomotive.

In the particular case of the endomotive associated to the BC system, the resulting representation of $G \times \mathbb{R}_+^*$ gives the spectral realization of the zeros of the Riemann zeta function and of the Artin L-functions for abelian characters of G. One sees in this example that this construction plays a role analogous to the action of the Weil group on the ℓ-adic cohomology. It gives a functor from the category of endomotives to the category of representations of the group $G \times \mathbb{R}_+^*$. Here we think of the action of $\mathbb{R}_+^*$ as a "Frobenius in characteristic zero", hence of $G \times \mathbb{R}_+^*$ as the corresponding Weil group.

We also show that the "dualization" step, i.e. the transition from $\mathcal{A}$ to $\mathcal{A} \rtimes_\sigma \mathbb{R}$, is a very good analog in the case of number fields of what happens for a function field K in passing to the extension $K \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. In fact, in the case of positive characteristic, the unramified extensions $K \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, combined with the notion of places, yield the points $C(\bar{\mathbb{F}}_q)$ over $\bar{\mathbb{F}}_q$ of the underlying curve. This has a good parallel in the theory of factors and this analogy plays an important role in developing a setting in noncommutative geometry that parallels the algebro-geometric framework that Weil used in his proof of RH for function fields.

We end the number theoretic part of the book by a dictionary between Weil's proof and the framework of noncommutative geometry, leaving open the problem of completing the translation and understanding the noncommutative geometry of the "arithmetic site".

We end the book by coming back to the construction of a theory of quantum gravity. Our approach here starts by developing an analogy between the electroweak phase transition in the Standard Model and the phase transitions in the quantum statistical mechanical systems described in Chapter 3. Through this analogy a consistent picture emerges which makes it possible to define a natural candidate for the algebra of observables of quantum gravity and to conjecture an extension of the electroweak phase transition to the full gravitational sector, in which the geometry of space-time emerges through a symmetry breaking mechanism and a cooling process. As a witness to the unity of the book, it is the construction of the correct category of correspondences which, as in Grothendieck's theory of motives, remains the main challenge for further progress on both QG and RH.

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