Introduction

The algebraic theory of quadratic forms, i.e., the study of quadratic forms over arbitrary fields, really began with the pioneering work of Witt. In his paper [**139**], Witt considered the totality of nondegenerate symmetric bilinear forms over an arbitrary field F of characteristic different from 2. Under this assumption, the theory of symmetric bilinear forms and the theory of quadratic forms are essentially the same. His work allowed him to form a ring $W(F)$, now called the Witt ring, arising from the isometry classes of such forms. This work set the stage for further study. From the viewpoint of ring theory, Witt gave a presentation of this ring as a quotient of the integral group ring where the group consists of the nonzero square classes of the field F. Three methods of study arise: ring theoretic, field theoretic, i.e., the relationship of $W(F)$ and $W(K)$ where K is a field extension of F, and algebraic geometric. In this book, we will develop all three methods. Historically, the powerful approach using algebraic geometry has been the last to be developed. This volume attempts to show its usefulness.

The theory of quadratic forms lay dormant until the work of Cassels and then of Pfister in the 1960's when it was still under the assumption of the field being of characteristic different from 2. Pfister employed the first two methods, ring theoretic and field theoretic, as well as a nascent algebraic geometric approach. In his postdoctoral thesis [**110**] Pfister determined many properties of the Witt ring. His study bifurcated into two cases: formally real fields, i.e., fields in which -1 is not a sum of squares and nonformally real fields. In particular, the Krull dimension of the Witt ring is one in the formally real case and zero otherwise. This makes the study of the interaction of bilinear forms and orderings an imperative, hence the importance of looking at real closures of the base field resulting in extensions of Sylvester's work and Artin-Schreier theory. Pfister determined the radical, zero-divisors, and spectrum of the Witt ring. Even earlier, in [**108**], he discovered remarkable forms, now called Pfister forms. These are forms that are tensor products of binary forms that represent one. Pfister showed that scalar multiples of these were precisely the forms that become hyperbolic over their function field. In addition, the nonzero value set of a Pfister form is a group and in fact the group of similarity factors of the form. As an example, this applies to the quadratic form that is a sum of 2^n squares. Pfister also used it to show that in a nonformally real field, the least number $s(F)$ so that -1 is a sum of $s(F)$ squares is always a power of 2 (cf. [**109**]). Interest in and problems about other arithmetic field invariants have also played a role in the development of the theory.

The nondegenerate even-dimensional symmetric bilinear forms determine an ideal $I(F)$ in the Witt ring of F, called the fundamental ideal. Its powers $I^n(F) := \big(I(F)\big)^n$, each generated by appropriate Pfister forms, give an important filtration of $W(F)$. The problem then arises: What ring theoretic properties respect this filtration? From $W(F)$ one also forms the graded ring $GW(F)$ associated to $I(F)$ and asks the same question.

Using Matsumoto's presentation of $K_2(F)$ of a field (cf. [**98**]), Milnor gave an *ad hoc* definition of a graded ring $K_*(F) := \bigoplus_{n\geq 0} K_n(F)$ of a field in [**106**]. From the viewpoint of Galois cohomology, this was of great interest as there is a natural map, called the norm residue map, from $K_n(F)$ to the Galois cohomology group $H^n(\Gamma_F, \mu_m^{\otimes n})$ where Γ_F is the absolute Galois group of F and m is relatively prime to the characteristic of F. For the case $m = 2$, Milnor conjectured this map to be an epimorphism with kernel $2K_n(F)$ for all n. Voevodsky proved this conjecture in [**136**]. Milnor also related his algebraic K-ring of a field to quadratic form theory by asking if $GW(F)$ and $K_*(F)/2K_*(F)$ are isomorphic. This was solved in the affirmative by Orlov, Vishik, and Voevodsky in [**107**]. Assuming these results, one can answer some of the questions that have arisen about the filtration of $W(F)$ induced by the fundamental ideal.

In this book, we do not restrict ourselves to fields of characteristic different from 2. Historically the cases of fields of characteristic different from 2 and 2 have been studied separately. Usually the case of characteristic different from 2 is investigated first. In this book, we shall give characteristic free proofs whenever possible. This means that the study of symmetric bilinear forms and the study of quadratic forms must be done separately, then interrelated. We not only present the classical theory characteristic free but we also include many results not proven in any text as well as some previously unpublished results to bring the classical theory up to date.

We shall also take a more algebraic geometric viewpoint than has historically been done. Indeed, the final two parts of the book will be based on such a viewpoint. In our characteristic free approach, this means a firmer focus on quadratic forms which have nice geometric objects attached to them rather than on bilinear forms. We do this for a variety of reasons.

First, one can associate to a quadratic form a number of algebraic varieties: the quadric of isotropic lines in a projective space and, more generally, for an integer $i > 0$, the variety of isotropic subspaces of dimension i. More importantly, basic properties of quadratic forms can be reformulated in terms of the associated varieties: a quadratic form is isotropic if and only if the corresponding quadric has a rational point. A nondegenerate quadratic form is hyperbolic if and only if the variety of maximal totally isotropic subspaces has a rational point.

Not only are the associated varieties important but also the morphisms between them. Indeed, if φ is a quadratic form over F and L/F a finitely generated field extension, then there is a variety Y over F with function field L, and the form φ is isotropic over L if and only if there is a rational morphism from Y to the quadric of φ.

Working with correspondences rather than just rational morphisms adds further depth to our study, where we identify morphisms with their graphs. Working with these leads to the category of Chow correspondences. This provides greater flexibility because we can view correspondences as elements of Chow groups and apply the rich machinery of that theory: pull-back and push-forward homomorphisms, Chern classes of vector bundles, and Steenrod operations. For example, suppose we wish to prove that a property A of quadratic forms implies a property B. We translate the properties A and B to "geometric" properties A' and B' for the existence of certain cycles on certain varieties. Starting with cycles satisfying

A' we can then attempt to apply the operations over the cycles as above to produce cycles satisfying B'.

All the varieties listed above are projective homogeneous varieties under the action of the orthogonal group or special orthogonal group of φ, i.e., the orthogonal group acts transitively on the varieties. It is not surprising that the properties of quadratic forms are reflected in the properties of the special orthogonal groups. For example, if φ is of dimension $2n$ (with $n \geq 2$) or $2n+1$ (with $n \geq 1$), then the special orthogonal group is a semisimple group of type D_n or B_n. The classification of semisimple groups is characteristic free. This explains why most important properties of quadratic forms hold in all characteristics.

Unfortunately, bilinear forms are not "geometric". We can associate varieties to a bilinear form, but it would be a variety of the associated quadratic form. Moreover, in characteristic 2 the automorphism group of a bilinear form is not semisimple.

In this book we sometimes give several proofs of the same results — one is classical, another is geometric. (This can be the same proof, but written in geometric language.) For example, this is done for Springer's theorem and the Separation Theorem.

The first part of the text will derive classical results under this new setting. It is self-contained, needing minimal prerequisites except for Chapter VII. In this chapter we shall assume the results of Voevodsky in [**136**] and Orlov-Vishik-Voevodsky in [**107**] for fields of characteristic not 2, and Kato in [**78**] for fields of characteristic 2 on the solution for the analog of the Milnor Conjecture in algebraic K-theory. We do give new proofs for the case $n = 2$.

Prerequisites for the second two parts of the text will be more formidable. A reasonable background in algebraic geometry will be assumed. For the convenience of the reader appendices have been included as an aid. Unfortunately, we cannot give details of [**136**] or [**107**] as it would lead us away from the methods at hand.

The first part of this book covers the "classical" theory of quadratic forms, i.e., without heavy use of algebraic geometry, bringing it up to date. As the characteristic of a field is not deemed to be different from 2, this necessitates a bifurcation of the theory into the theory of symmetric bilinear forms and the theory of quadratic forms. The introduction of these subjects is given in the first two chapters.

Chapter I investigates the foundations of the theory of symmetric bilinear forms over a field F. Two major consequences of dealing with arbitrary characteristic are that such forms may not be diagonalizable and that nondegenerate isotropic planes need not be hyperbolic. With this taken into account, standard Witt theory, to the extent possible, is developed. In particular, Witt decomposition still holds, so that the Witt ring can be constructed in the usual way as well as the classical group presentation of the Witt ring $W(F)$. This presentation is generalized to the fundamental ideal $I(F)$ of even-dimensional forms in $W(F)$ and then to the the second power $I^2(F)$ of $I(F)$, a theme returned to in Chapter VII. The Stiefel-Whitney invariants of bilinear forms are introduced along with their relationship with the invariants $\bar{e}_n : I^n(F)/I^{n+1}(F) \to K_n(F)/2K_{n+1}(F)$ for $n = 1, 2$. The theory of bilinear Pfister forms is introduced and some basic properties developed. Following [**32**], we introduce chain p-equivalence and linkage of Pfister forms as well as introducing annihilators of Pfister forms in the Witt ring.

Chapter II investigates the foundations of the theory of quadratic forms over a field F. Because of the arbitrary characteristic assumption on the field F, the definition of nondegenerate must be made more carefully, and quadratic forms are far from having orthogonal bases in general. Much of Witt theory, however, goes through as the Witt Extension Theorem holds for quadratic forms under fairly weak assumptions, hence Witt Decomposition. The Witt group $I_q(F)$ of nondegenerate even-dimensional quadratic forms is defined and shown to be a $W(F)$-module. The theory of quadratic Pfister forms is introduced and some results analogous to that of the bilinear case are introduced. Moreover, cohomological invariants of quadratic Pfister forms are introduced and some preliminary results about them and their extension to the appropriate filtrant of the Witt group of quadratic forms are discussed. In addition, the classical quadratic form invariants, discriminant and Clifford invariant, are defined.

Chapter III begins the utilization of function field techniques in the study of quadratic forms, all done without restriction of characteristic. The classical Cassels-Pfister theorem is established. Values of anisotropic quadratic and bilinear forms over a polynomial ring are investigated, special cases being the representation of one form as a subform of another and various norm principles due to Knebusch (cf. [**82**]). To investigate norm principles of similarity factors due to Scharlau (cf. [**119**]), quadratic forms over valuation rings and transfer maps are introduced.

Chapter IV introduces algebraic geometric methods, i.e., looking at the theory under the base extension of the function field of a fixed quadratic form. In particular, the notion of domination of one form by another is introduced where an anisotropic quadratic form φ is said to dominate an anisotropic quadratic form ψ (both of dimension of at least two) if $\varphi_{F(\psi)}$ is isotropic. The geometric properties of Pfister forms are developed, leading to the Arason-Pfister Hauptsatz that nonzero anisotropic quadratic (respectively, symmetric bilinear) forms in $I_q^n(F)$ (respectively, $I^n(F)$) are of dimension at least 2^n and its application to linkage of Pfister forms. Knebusch's generic tower of an anisotropic quadratic form is introduced and the $W(F)$-submodules $J_n(F)$ of $I_q(F)$ are defined by the notion of degree. These submodules turn out to be precisely the corresponding $I_q^n(F)$ (to be shown in Chapter VII). Hoffmann's Separation Theorem that if φ and ψ are two anisotropic quadratic forms over F with $\dim\varphi \leq 2^n < \dim\psi$ for some $n \geq 0$, then $\varphi_{F(\psi)}$ is anisotropic is proven as well as Fitzgerald's theorem characterizing quadratic Pfister forms. In addition, excellent forms and extensions are discussed. In particular, Arason's result that the extension of a field by the function field of a nondegenerate 3-dimensional quadratic form is excellent is proven. The chapter ends with a discussion of central simple algebras over the function field of a quadric.

Chapter V studies symmetric bilinear and quadratic forms under field extensions. The chapter begins with the study of the structure of the Witt ring of a field F based on the work of Pfister. After dispensing with the nonformally real F, we turn to the study over a formally real field utilizing the theory of pythagorean fields and the pythagorean closure of a field, leading to the Local-Global Theorem of Pfister and its consequences for structure of the Witt ring over a formally real field. The total signature map from the Witt ring to the ring of continuous functions from the order space of a field to the integers is then carefully studied, in particular, the approximation of elements in this ring of functions by quadratic forms. The behavior of quadratic and bilinear forms under quadratic extensions (both separable and inseparable) is then investigated. The special case of the torsion of the Witt ring under such extensions is studied. A detailed investigation of torsion Pfister forms is begun, leading to the theorem of Krüskemper which implies that if K/F is a quadratic field extension with $I^n(K) = 0$, then $I^n(F)$ is torsion-free.

Chapter VI studies u-invariants, their behavior under field extensions, and values that they can take. Special attention is given to the case of formally real fields.

Chapter VII establishes consequences of the result of Orlov-Vishik-Voevodsky in [**107**] which we assume in this chapter. In particular, answers and generalizations of results from the previous chapters are established. For fields of characteristic not 2, the ideals $I^n(F)$ and $J_n(F)$ are shown to be identical. The annihilators of Pfister forms in the Witt ring are shown to filter through the $I^n(F)$, i.e., the intersection of such annihilators and $I^n(F)$ are generated by Pfister forms in the intersection. A consequence is that torsion in $I^n(F)$ is generated by torsion n-fold Pfister forms, solving a conjecture of Lam. A presentation for the group structure of the $I^n(F)$'s is determined, generalizing that given for $I^2(F)$ in Chapter I. Finally, it is shown that if K/F is a finitely generated field extension of transcendence degree m, then $I^n(K)$ torsion-free implies the same for $I^{n-m}(F)$.

In Chapter VIII, we give a new elementary proof of the theorem in [**100**] that the second cohomological invariant is an isomorphism in the case that the characteristic of the field is different from 2 (the case of characteristic 2 having been done in Chapter II). This is equivalent to the degree two case of the Milnor Conjecture in [**106**] stating that the *norm residue homomorphism*

$$h_F^n : K_n(F)/2K_n(F) \to H^n(F, \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism for every integer n. The Milnor Conjecture was proven in full by V. Voevodsky in [**136**]. Unfortunately, the scope of this book does not allow us to prove this beautiful result as the proof requires motivic cohomology and Steenrod operations developed by Voevodsky. In Chapter VIII, we give an "elementary" proof of the degree two case of the Milnor Conjecture that does not rely either on a specialization argument or on higher K-theory as did the original proof of this case in [**100**].

In the second part of the book, we develop the needed tools in algebraic geometry that will be applied in the third part. The main object studied in Part Two is the Chow group of algebraic cycles modulo rational equivalence on an algebraic scheme. Using algebraic cycles, we introduce the category of correspondences.

In Chapter IX (following the approach of [**117**] given by Rost), we develop the K-homology and K-cohomology theories of schemes over a field. This generalizes the Chow groups. We establish functorial properties of these theories (pull-back, push-forward, deformation and Gysin homomorphisms), introduce Euler and Chern classes of vector bundles, and prove basic results such as the Homotopy Invariance and Projective Bundle Theorems. We apply these results to Chow groups in the next chapter.

Chapter XI is devoted to the study of Steenrod operations on Chow groups modulo 2 over fields of characteristic not 2. Steenrod operations for motivic cohomology modulo a prime integer p of a scheme X were originally constructed by

Voevodsky in [**137**]. The reduced power operations (but not the Bockstein operation) restrict to the Chow groups of X. An "elementary" construction of the reduced power operations modulo p on Chow groups (requiring equivariant Chow groups) was given by Brosnan in [**20**].

In Chapter XII, we introduce the notion of a Chow motive that is due to Grothendieck. Many (co)homology theories defined on the category of smooth complete varieties, such as Chow groups and more generally the K-(co)homology groups, take values in the category of abelian groups. But the category of smooth complete varieties itself does not have the structure of an additive category as we cannot add morphisms of varieties. The category of Chow motives, however, is an additive tensor category. This additional structure gives more flexibility when working with regular and rational morphisms.

In the third part of the book we apply algebraic geometric methods to the further study of quadratic forms. In Chapter XIII, we prove preliminary facts about algebraic cycles on quadrics and their powers. We also introduce shell triangles and diagrams of cycles, the basic combinatorial objects associated to a quadratic form. The corresponding pictures of these shell triangles simplify visualization of algebraic cycles and operations over the cycles.

In Chapter XIV, we study the Izhboldin dimension of smooth projective quadrics. It is defined as the integer

$$\dim_{\mathrm{Izh}}(X) := \dim X - \mathfrak{i}_1(X) + 1,$$

where $\mathfrak{i}_1(X)$ is the first Witt index of the quadric X. The Izhboldin dimension behaves better than the classical dimension with respect to splitting properties. For example, if X and Y are anisotropic smooth projective quadrics and Y is isotropic over the function field $F(X)$, then $\dim_{\mathrm{Izh}} X \leq \dim_{\mathrm{Izh}} Y$ but $\dim X$ may be bigger than $\dim Y$.

Chapter XV is devoted to applications of the Steenrod operations. The following problems are solved:

(1) All possible values of the first Witt index of quadratic forms are determined.
(2) All possible values of dimensions of anisotropic quadratic forms in $I^n(F)$ are determined.
(3) It is shown that excellent forms have the smallest height among all quadratic forms of a given dimension.

In Chapter XVI, we study the variety of maximal isotropic subspaces of a quadratic forms. A discrete invariant $J(\varphi)$ of a quadratic form φ is introduced. We also introduce the notion of canonical dimension and compute it for projective quadrics and varieties of totally isotropic subspaces.

In the last chapter we study motives of smooth projective quadrics in the category of correspondences and motives.

This book could not have been written without the help and encouragement of many of our friends, collaborators, and students nor the many researchers whose work was essential to producing this volume. These are too numerous to be listed here individually. We do wish to mention those who gave us valuable advice in the preparation and writing of this tome and those who helped proofread the manuscript: Jón Arason, Ricardo Baeza, Alex Boisvert, Detlev Hoffmann, Bryant Mathews, Claus Schubert, Jean-Pierre Tignol, Alexander Vishik, Maksim Zhykhovich. In addition, we would like to thank the referees who made valuable suggestions to improve the manuscript and Sergei Gelfand and the American Mathematical Society for its encouragement. It also gives us great pleasure to thank the National Science Foundation (grant DMS 0652316), as well as the Collaborative Research Centre 701 of the Bielefeld University, École Polytechnique Fédérale de Lausanne, Institute for Advanced Study in Princeton (The James D. Wolfensohn Fund and The Ellentuck Fund), Institut des Hautes Études Scientifiques, Institut Universitaire de France, and Max-Planck-Institut für Mathematik in Bonn, for their generous support. Finally, we wish to thank our families who put up with us through the long process of bringing this volume to fruition.