Contents

Preface xi

Part 1. Large graphs: an informal introduction 1

Chapter 1. Very large networks 3
1.1. Huge networks everywhere 3
1.2. What to ask about them? 4
1.3. How to obtain information about them? 5
1.4. How to model them? 8
1.5. How to approximate them? 11
1.6. How to run algorithms on them? 18
1.7. Bounded degree graphs 22

Chapter 2. Large graphs in mathematics and physics 25
2.1. Extremal graph theory 25
2.2. Statistical physics 32

Part 2. The algebra of graph homomorphisms 35

Chapter 3. Notation and terminology 37
3.1. Basic notation 37
3.2. Graph theory 38
3.3. Operations on graphs 39

Chapter 4. Graph parameters and connection matrices 41
4.1. Graph parameters and graph properties 41
4.2. Connection matrices 42
4.3. Finite connection rank 45

Chapter 5. Graph homomorphisms 55
5.1. Existence of homomorphisms 55
5.2. Homomorphism numbers 56
5.3. What hom functions can express 62
5.4. Homomorphism and isomorphism 68
5.5. Independence of homomorphism functions 72
5.6. Characterizing homomorphism numbers 75
5.7. The structure of the homomorphism set 79

Chapter 6. Graph algebras and homomorphism functions 83
6.1. Algebras of quantum graphs 83
6.2. Reflection positivity 88

6.3. Contractors and connectors 94
6.4. Algebras for homomorphism functions 101
6.5. Computing parameters with finite connection rank 106
6.6. The polynomial method 108

Part 3. Limits of dense graph sequences 113

Chapter 7. Kernels and graphons 115
7.1. Kernels, graphons and stepfunctions 115
7.2. Generalizing homomorphisms 116
7.3. Weak isomorphism I 121
7.4. Sums and products 122
7.5. Kernel operators 124

Chapter 8. The cut distance 127
8.1. The cut distance of graphs 127
8.2. Cut norm and cut distance of kernels 131
8.3. Weak and L_1-topologies 138

Chapter 9. Szemerédi partitions 141
9.1. Regularity Lemma for graphs 141
9.2. Regularity Lemma for kernels 144
9.3. Compactness of the graphon space 149
9.4. Fractional and integral overlays 151
9.5. Uniqueness of regularity partitions 154

Chapter 10. Sampling 157
10.1. W-random graphs 157
10.2. Sample concentration 158
10.3. Estimating the distance by sampling 160
10.4. The distance of a sample from the original 164
10.5. Counting Lemma 167
10.6. Inverse Counting Lemma 169
10.7. Weak isomorphism II 170

Chapter 11. Convergence of dense graph sequences 173
11.1. Sampling, homomorphism densities and cut distance 173
11.2. Random graphs as limit objects 174
11.3. The limit graphon 180
11.4. Proving convergence 185
11.5. Many disguises of graph limits 193
11.6. Convergence of spectra 194
11.7. Convergence in norm 196
11.8. First applications 197

Chapter 12. Convergence from the right 201
12.1. Homomorphisms to the right and multicuts 201
12.2. The overlay functional 205
12.3. Right-convergent graphon sequences 207
12.4. Right-convergent graph sequences 211

Chapter 13. On the structure of graphons 217
13.1. The general form of a graphon 217
13.2. Weak isomorphism III 220
13.3. Pure kernels 222
13.4. The topology of a graphon 225
13.5. Symmetries of graphons 234

Chapter 14. The space of graphons 239
14.1. Norms defined by graphs 239
14.2. Other norms on the kernel space 242
14.3. Closures of graph properties 247
14.4. Graphon varieties 250
14.5. Random graphons 256
14.6. Exponential random graph models 259

Chapter 15. Algorithms for large graphs and graphons 263
15.1. Parameter estimation 263
15.2. Distinguishing graph properties 266
15.3. Property testing 268
15.4. Computable structures 276

Chapter 16. Extremal theory of dense graphs 281
16.1. Nonnegativity of quantum graphs and reflection positivity 281
16.2. Variational calculus of graphons 283
16.3. Densities of complete graphs 285
16.4. The classical theory of extremal graphs 293
16.5. Local vs. global optima 294
16.6. Deciding inequalities between subgraph densities 299
16.7. Which graphs are extremal? 307

Chapter 17. Multigraphs and decorated graphs 317
17.1. Compact decorated graphs 318
17.2. Multigraphs with unbounded edge multiplicities 325

Part 4. Limits of bounded degree graphs 327

Chapter 18. Graphings 329
18.1. Borel graphs 329
18.2. Measure preserving graphs 332
18.3. Random rooted graphs 338
18.4. Subgraph densities in graphings 344
18.5. Local equivalence 346
18.6. Graphings and groups 349

Chapter 19. Convergence of bounded degree graphs 351
19.1. Local convergence and limit 351
19.2. Local-global convergence 360

Chapter 20. Right convergence of bounded degree graphs 367
20.1. Random homomorphisms to the right 367
20.2. Convergence from the right 375

Chapter 21. On the structure of graphings 383
21.1. Hyperfiniteness 383
21.2. Homogeneous decomposition 393

Chapter 22. Algorithms for bounded degree graphs 397
22.1. Estimable parameters 397
22.2. Testable properties 402
22.3. Computable structures 405

Part 5. Extensions: a brief survey 413

Chapter 23. Other combinatorial structures 415
23.1. Sparse (but not very sparse) graphs 415
23.2. Edge-coloring models 416
23.3. Hypergraphs 421
23.4. Categories 425
23.5. And more... 429

Appendix A. Appendix 433
A.1. Möbius functions 433
A.2. The Tutte polynomial 434
A.3. Some background in probability and measure theory 436
A.4. Moments and the moment problem 441
A.5. Ultraproduct and ultralimit 444
A.6. Vapnik–Chervonenkis dimension 445
A.7. Nonnegative polynomials 446
A.8. Categories 447

Bibliography 451

Author Index 465

Subject Index 469

Notation Index 473