Foreword: A selective overview

This preface contains a summary of the contents of the volume. We start with a rough description of the main theorems. We then give short descriptions of the contents of the various chapters. At the end, we will add a couple of remarks on the overall structure of the proof, notably our use of induction. The preface can serve as an introduction. The beginning of the actual text, in the form of the first two or three sections of Chapter 1, represents a different sort of introduction. It will be our attempt to motivate what follows from a few basic principles. A reader might consider going directly to these sections after reading the first half of the preface here. One could then return to the more technical second half (on the organization of the volume) only as needed later.

Automorphic representations for $GL(N)$ have been important objects of study for many years. We recall that $GL(N)$, the general linear group of invertible $(N \times N)$-matrices, assigns a group $GL(N, R)$ to any commutative ring R with identity. For example, R could be a fixed number field F, or the ring $\mathbb{A} = \mathbb{A}_F$ of adèles over F. *Automorphic representations* of $GL(N)$ are the irreducible representations of $GL(N, \mathbb{A})$ that occur in the decomposition of its regular representation on $L^2\big(GL(N, F)\backslash GL(N, \mathbb{A})\big)$. This informal definition is made precise in [**L6**], and carries over to any connected reductive group G over F.

The primary aim of the volume is to classify the automorphic representations of special orthogonal and symplectic groups G in terms of those of $GL(N)$. Our main tool will be the stable trace formula for G, which until recently was conditional on the fundamental lemma. The fundamental lemma has now been established in complete generality, and in all of its various forms. In particular, the stabilization of the trace formula is now known for any connected group. However, we will also require the stabilization of twisted trace formulas for $GL(N)$ and $SO(2n)$. Since these have yet to be established, our results will still be conditional.

A secondary purpose will be to lay foundations for the endoscopic study of more general groups G. It is reasonable to believe that the methods we introduce here extend to groups that Ramakrishnan has called *quasiclassical*. These would comprise the largest class of groups whose representations could ultimately be tied to those of general linear groups. Our third goal is expository. In adopting a style that is sometimes more discursive than strictly necessary, we have tried to place at least some of the techniques into perspective. We hope that there will be parts of the volume that are accessible to readers who are not experts in the subject.

Automorphic representations are interesting for many reasons, but among the most fundamental is the arithmetic data they carry. Recall that

$$GL(N,\mathbb{A}) = \widetilde{\prod_v} GL(N,F_v)$$

is a restricted direct product, taken over (equivalence classes of) valuations v of F. An automorphic representation of $GL(N)$ is a restricted direct product

$$\pi = \widetilde{\bigotimes_v} \pi_v,$$

where π_v is an irreducible representation of $GL(N,F_v)$ that is *unramified* for almost all v. We recall that π_v is unramified if v is nonarchimedean, and π_v contains the trivial representation of the hyperspecial maximal compact subgroup $GL(N,\mathfrak{o}_v)$ of integral points in $GL(N,F_v)$. The representation is then parametrized by a semisimple conjugacy class

$$c_v(\pi) = c(\pi_v)$$

in the complex dual group

$$GL(N)^\wedge = GL(N,\mathbb{C})$$

of $GL(N)$. (See [**Bo**, (6.4), (6.5)] for the precise assertion, as it applies to a general connected reductive group G.) It is the relations among the semisimple conjugacy classes $c_v(\pi)$ that will contain the fundamental arithmetic information.

There are three basic theorems for the group $GL(N)$ that together give us a pretty clear understanding of its representations. The first is local, while others, which actually predate the first, are global.

The first theorem is the local Langlands *correspondence* for $GL(N)$. It was established for archimedean fields by Langlands, and more recently for p-adic (which is to say nonarchimedean) fields by Harris, Taylor and Henniart. It classifies the irreducible representations of $GL(N,F_v)$ at all places v by (equivalence classes of) semisimple, N-dimensional representations of the local Langlands group

$$L_{F_v} = \begin{cases} W_{F_v}, & v \text{ archimedean}, \\ W_{F_v} \times SU(2), & \text{otherwise}. \end{cases}$$

In particular, an unramified representation of $GL(N,F_v)$ corresponds to an N-dimensional representation of L_{F_v} that is trivial on the product of $SU(2)$ with the inertia subgroup I_{F_v} of the local Weil group W_{F_v}. It therefore corresponds to a semisimple representation of the cyclic quotient

$$L_{F_v}/I_{F_v} \times SU(2) \cong W_{F_v}/I_{F_v} \cong \mathbb{Z},$$

and hence a semisimple conjugacy class in $GL(N,\mathbb{C})$, as above.

The first of the global theorems is due to Jacquet and Shalika. If π is an irreducible (admissible) representation of $GL(N, \mathbb{A})$, one can form the family of semisimple conjugacy classes

$$c(\pi) = \varinjlim_S \{c_v(\pi) = c(\pi_v) : \; v \notin S\}$$

in $GL(N, \mathbb{C})$, defined up to a finite set of valuations S. In other words, $c(\pi)$ is an equivalence class of families, two such families being equivalent if they are equal for almost all v. The theorem of Jacquet and Shalika asserts that if an automorphic representation π of $GL(N)$ is restricted slightly to be *isobaric* [**L7**, §2], it is uniquely determined by $c(\pi)$. This theorem can be regarded as a generalization of the theorem of strong multiplicity one for cuspidal automorphic representations of $GL(N)$.

The other global theorem for $GL(N)$ is due to Moeglin and Waldspurger. It characterizes the automorphic (relatively) discrete spectrum of $GL(N)$ in terms of the set of cuspidal automorphic representations. Since Langlands' general theory of Eisenstein series characterizes the full automorphic spectrum of any group G in terms of discrete spectra, this theorem characterizes the automorphic spectrum for $GL(N)$ in terms of cuspidal automorphic representations. Combined with the first global theorem, it classifies the full automorphic spectrum of $GL(N)$ explicitly in terms of families $c(\pi)$ attached to cuspidal automorphic representations of general linear groups.

Our goal is to generalize these three theorems. As we shall see, there is very little that comes easily. It has been known for many years that the representations of groups other than $GL(N)$ have more structure. In particular, they should separate naturally into L-packets, composed of representations with the same L-functions and ε-factors. This was demonstrated for the group

$$G = SL(2) = Sp(2)$$

by Labesse and Langlands, in a paper [**LL**] that became a model for Langlands's conjectural theory of endoscopy [**L8**], [**L10**].

The simplest and most elegant way to formulate the theory of endoscopy is in terms of the global Langlands group L_F. This is a hypothetical global analogue of the explicit local Langlands groups L_{F_v} defined above. It is thought to be a locally compact extension

$$1 \longrightarrow K_F \longrightarrow L_F \longrightarrow W_F \longrightarrow 1$$

of W_F by a compact connected group K_F. (See [**L7**, §2], [**K3**, §9].) However, its existence is very deep, and could well turn out to be the final theorem in the subject to be proved! One of our first tasks, which we address in §1.4, will be to introduce makeshift objects to be used in place of L_F. For simplicity, however, let us describe our results here in terms of L_F.

Our main results apply to the case that G is a *quasisplit* special orthogonal or symplectic group.* They are stated as three theorems in §1.5. The proof of these theorems will then take up much of the rest of the volume.

Theorem 1.5.1 is the main local result. It contains a local Langlands parametrization of the irreducible representations of $G(F_v)$, for any p-adic valuation v of F, as a disjoint union of finite L-packets Π_{ϕ_v}. These are indexed by local Langlands parameters, namely L-homomorphisms

$$\phi_v : \; L_{F_v} \longrightarrow {}^L G_v$$

from L_{F_v} to the local L-group ${}^L G_v = \widehat{G} \rtimes \mathrm{Gal}(\overline{F}_v/F_v)$ of G. The theorem includes a way to index the representations in an L-packet Π_{ϕ_v} by linear characters on a finite abelian group $\mathcal{S}_{\phi_v}$ attached to ϕ_v. Since similar results for archimedean valuations v are already known from the work of Shelstad, we obtain a classification of the representations of each local group $G(F_v)$.

Theorem 1.5.1 also contains a somewhat less precise description of the representations of $G(F_v)$ that are local components of automorphic representations. These fall naturally into rather different packets Π_{ψ_v}, indexed according to the conjectures of [**A8**] by L-homomorphisms

$$\psi_v : \; L_{F_v} \times SU(2) \longrightarrow {}^L G_v, \tag{1}$$

with bounded image. The theorem includes the assertion, also conjectured in [**A8**], that the representations in these packets are all unitary.

Theorem 1.5.2 is the main global result. As a first approximation, it gives a rough decomposition

$$R_{\mathrm{disc}} = \bigoplus_{\psi} R_{\mathrm{disc},\psi} \tag{2}$$

of the representation R_{disc} of $G(\mathbb{A})$ on the automorphic discrete spectrum $L^2_{\mathrm{disc}}\big(G(F)\backslash G(\mathbb{A})\big)$. The indices can be thought of as L-homomorphisms

$$\psi : \; L_F \times SU(2) \longrightarrow {}^L G \tag{3}$$

of bounded image that do not factor through any proper parabolic subgroup of the global L-group ${}^L G = \widehat{G} \rtimes \mathrm{Gal}(\overline{F}/F)$. They have localizations (1) defined by conjugacy classes of embeddings $L_{F_v} \subset L_F$, or rather the makeshift analogues of such embeddings that we formulate in §1.4. The localizations ψ_v of ψ are unramified at almost all v, and consequently lead to a family

$$c(\psi) = \varinjlim_{S} \big\{ c_v(\psi) = c(\psi_v) : \; v \notin S \big\}$$

of equivalence classes of semisimple elements in ${}^L G$. The rough decomposition (2) is of interest as it stands. It implies that G has no embedded eigenvalues, in the sense of unramified Hecke operators. In other words, the family $c(\psi)$ attached to any global parameter ψ in the decomposition of the

*It is understood here, and in everything that follows, that G is "classical", in the sense that it is not an outer twist of the split group $SO(8)$ by a triality automorphism (of order 3).

automorphic discrete spectrum is distinct from any family obtained from the continuous spectrum. This follows from the nature of the parameters ψ in (3), and the application of the theorem of Jacquet-Shalika to the natural image of $c(\psi)$ in the appropriate complex general linear group.

Theorem 1.5.2 also contains a finer decomposition

$$R_{\mathrm{disc},\psi} = \bigoplus_{\pi} m_\psi(\pi)\, \pi, \tag{4}$$

for any global parameter ψ. The indices π range over representations in the global packet of ψ, defined as a restricted direct product of local packets provided by Theorem 1.5.1. The multiplicities $m_\psi(\pi)$ are given by an explicit reciprocity formula in terms of the finite abelian groups $\mathcal{S}_{\psi_v}$, and their global analogue $\mathcal{S}_\psi$. We thus obtain a decomposition of the automorphic discrete spectrum of G into irreducible representations of $G(\mathbb{A})$. We shall say that a parameter $\phi = \psi$ is *generic* if it is trivial on the factor $SU(2)$. The representations $\pi \in \Pi_\phi$, with ϕ generic and $m_\phi(\pi) \neq 0$, are the constituents of the automorphic discrete spectrum that are expected to satisfy the analogue for G of Ramanujan's conjecture. It ψ is not generic, the formula for $m_\psi(\pi)$ has an extra ingredient. It is a sign character ε_ψ on $\mathcal{S}_\psi$, defined (1.5.6) in terms of symplectic root numbers. That the discrete spectrum should be governed by objects of such immediate arithmetic significance seems quite striking.

Theorem 1.5.2 has application to the question of multiplicity one. Suppose that π is an irreducible constituent of the automorphic discrete spectrum of G that also lies in some generic global packet Π_ϕ. We shall then show that the multiplicity of π in the discrete spectrum equals 1 unless $\widehat{G} = SO(2n, \mathbb{C})$, in which case the multiplicity is either 1 or 2, according to an explicit condition we shall give. In particular, if G equals either $SO(2n+1)$ or $Sp(2n)$, the automorphic representations in the discrete spectrum that are expected to satisfy Ramanujan's conjecture all have multiplicity 1. Local results of Moeglin [**M4**] on nontempered p-adic packets suggest that similar results hold for all automorphic representations in the discrete spectrum.

Theorems 1.5.1 and 1.5.2 are founded on the proof of several cases of Langlands' principle of functoriality. In fact, our basic definitions will be derived from the functorial correspondence from G to $GL(N)$, relative to the standard representation of ${}^LG$ into $GL(N, \mathbb{C})$. Otherwise said, our construction of representations of G will be formulated in terms of representations of $GL(N)$. The integer N of course equals $2n$, $2n+1$ and $2n$ as G ranges over the groups $SO(2n+1)$, $Sp(2n)$ and $SO(2n)$ in the three infinite families B_n, C_n and D_n, with dual groups $\widehat{G}$ being equal to $Sp(2n, \mathbb{C})$, $SO(2n+1, \mathbb{C})$ and $SO(2n, \mathbb{C})$, respectively. The third case $SO(2n)$, which includes quasisplit outer twists, is complicated by the fact that it is really the nonconnected group $O(2n)$ that is directly tied to $GL(N)$. This is what is responsible for the failure of multiplicity one described above. It is also the reason we have not yet specified the implicit equivalence relation for the local and global parameters (1) and (3). Let us now agree that they are to be taken up to

$\widehat{G}$-conjugacy if G equals $SO(2n+1)$ or $Sp(2n)$, and up to conjugacy by $O(2n,\mathbb{C})$, a group whose quotient

$$\widetilde{\mathrm{O}}\mathrm{ut}_N(G) = O(2n,\mathbb{C})/SO(2n,\mathbb{C})$$

acts by outer automorphism on $\widehat{G} = SO(2n,\mathbb{C})$, if G equals $SO(2n)$. This is the understanding on which the decomposition (2) holds.

In the text, we shall write $\widetilde{\Psi}(G_v)$ for the set of equivalence classes of local parameters (1). The packet $\widetilde{\Pi}_{\psi_v}$ attached to any $\psi_v \in \widetilde{\Psi}(G_v)$ will then be composed of $\widetilde{\mathrm{O}}\mathrm{ut}_N(G_v)$-orbits of equivalence classes of irreducible representations (with $\widetilde{\mathrm{O}}\mathrm{ut}_N(G)$ being trivial in case G equals $SO(2n+1)$ or $Sp(2n)$). We will write $\widetilde{\Psi}(G)$ for the set of equivalence classes of general global parameters ψ, and $\widetilde{\Psi}_2(G)$ for the subset of classes with the supplementary condition of (3). The packet of any $\psi \in \widetilde{\Psi}(G)$ will then be a restricted direct product

$$\widetilde{\Pi}_\psi = \widetilde{\bigotimes_v} \widetilde{\Pi}_{\psi_v}$$

of local packets. It is this set that corresponds to an (isobaric) automorphic representation of $GL(N)$. In particular, the global packets $\widetilde{\Pi}_\psi$, rather than the individual (orbits of) representations π in $\widetilde{\Pi}_\psi$, are the objects that retain the property of strong multiplicity one from $GL(N)$. Similarly, the global packets $\widetilde{\Pi}_\psi$ attached to parameters $\psi \in \widetilde{\Psi}_2(G)$ retain the qualitative properties of automorphic discrete spectrum of $GL(N)$. They come with a sort of Jordan decomposition, in which the semisimple packets correspond to the generic global parameters ψ, and contain the automorphic representations that are expected to satisfy the G-analogue of Ramanujan's conjecture. In view of these comments, we see that Theorem 1.5.2 can be regarded as a simultaneous analogue for G of both of the global theorems for $GL(N)$.

Theorem 1.5.3 is a global supplement to Theorem 1.5.2. Its first assertion applies to global parameters $\phi \in \widetilde{\Psi}(G)$ that are both generic and simple, in the sense that they correspond to cuspidal automorphic representations π_ϕ of $GL(N)$. Theorem 1.5.3(a) asserts that the dual group $\widehat{G}$ is orthogonal (resp. symplectic) if and only if the symmetric square L-function $L(s,\pi_\phi,S^2)$ (resp. the skew-symmetric square L-function $L(s,\pi_\phi,\Lambda^2)$) has a pole at $s=1$. Theorem 1.5.3(b) asserts that the Rankin-Selberg ε-factor $\varepsilon\left(\frac{1}{2},\pi_{\phi_1}\times\pi_{\phi_2}\right)$ equals 1 for any pair of generic simple parameters $\phi_i \in \widetilde{\Phi}(G_i)$ such that $\widehat{G}_1$ and $\widehat{G}_2$ are either both orthogonal or both symplectic. These two assertions are automorphic analogues of well known properties of Artin L-functions and ε-factors. They are interesting in their own right. But they are also an essential part of our induction argument. We will need them in Chapter 4 to interpret the terms in the trace formula attached to compound parameters $\psi \in \widetilde{\Psi}(G)$.

This completes our summary of the main theorems. The first two sections of Chapter 1 contain further motivation, for the global Langlands group

L_F in §1.1, and the relations between representations of G and $GL(N)$ in §1.2. In §1.3, we will recall the three basic theorems for $GL(N)$. Section 1.4 is given over to our makeshift substitutes for global Langlands parameters, while §1.5 contains the formal statements of the theorems.

As might be expected, the three theorems will have to be established together. The unified proof will take us down a long road, which starts in Chapter 2, and crosses much diverse territory before coming to an end finally in §8.2. The argument is ultimately founded on harmonic analysis, represented locally by orbital integrals and characters, and globally by the trace formula. This of course is at the heart of the theory of endoscopy. We refer the reader to the introductory remarks of individual sections, where we have tried to offer guidance and motivation. We shall be content here with a minimal outline of the main stages.

Chapter 2 is devoted to local endoscopy. It contains a more precise formulation (Theorem 2.2.1) of the local Theorem 1.5.1. This provides for a canonical construction of the local packets $\widetilde{\Pi}_{\psi_v}$ in terms of twisted characters on $GL(N)$. Chapter 2 also includes the statement of Theorem 2.4.1, which we call the local intertwining relation. This is closely related to Theorem 1.5.1 and its refinement Theorem 2.2.1, and from a technical standpoint, can be regarded as our primary local result. It includes a delicate construction of signs, which will be critical for the interpretation of terms in the trace formula.

Chapter 3 is devoted to global endoscopy. We will recall the discrete part of the trace formula in §3.1, and its stabilization in §3.2. We are speaking here of those spectral terms that are linear combinations of automorphic characters, and to which all of the other terms are ultimately dedicated. They are the only terms from the trace formula that will appear explicitly in this volume. In §3.5, we will establish criteria for the vanishing of coefficients in certain identities (Proposition 3.5.1, Corollary 3.5.3). We shall use these criteria many times throughout the volume in drawing conclusions from the comparison of discrete spectral terms.

In general, we will have to treat three separate cases of endoscopy. They are represented respectively by pairs (G, G'), where G is one of the groups to which Theorems 1.5.1, 1.5.2 and 1.5.3 apply and G' is a corresponding endoscopic datum, pairs $\big(\widetilde{G}(N), G\big)$ in which $\widetilde{G}(N)$ is the twisted general linear group $\widetilde{G}^0(N) = GL(N)$ and G is a corresponding twisted endoscopic datum, and pairs $(\widetilde{G}, \widetilde{G}')$ in which $\widetilde{G}$ is a twisted even orthogonal group $\widetilde{G}^0 = SO(2n)$ and $\widetilde{G}'$ is again a corresponding twisted endoscopic datum. The first two cases will be our main concern. However, the third case $(\widetilde{G}, \widetilde{G}')$ is also a necessary part of the story. Among other things, it is forced on us by the need to specify the signs in the local intertwining relation. For the most part, we will not try to treat the three cases uniformly as cases of the general theory of endoscopy. This might have been difficult, given that we have to deduce many local and global results along the way. At any rate, the separate treatment of the three cases gives our exposition a more concrete flavour, if at the expense of some possible sacrifice of efficiency.

In Chapter 4, we will study the comparison of trace formulas. Specifically, we will compare the contribution (4.1.1) of a parameter ψ to the discrete part of the trace formula with the contribution (4.1.2) of ψ to the corresponding endoscopic decomposition. We begin with the statement of Theorem 4.1.2, which we call the stable multiplicity formula. This is closely related to Theorem 1.5.2, and from a technical standpoint again, is our primary global result. Together with the global intertwining relation (Corollary 4.2.1), which we state as a global corollary of Theorem 2.4.1, it governs how individual terms in trace formulas are related. Chapter 4 represents a standard model, in the sense that if we grant the analogues of the two primary theorems for general groups, it explains how the terms on the right hand sides of (4.1.1) and (4.1.2) match. This is discussed heuristically in Sections 4.7 and 4.8. However, our real aim is in the opposite direction. It is to derive Theorems 2.4.1 and 4.1.2 for our groups G *from* the standard model, and whatever else we can bring to bear on the problem. This is the perspective of Sections 4.3 and 4.4. In §4.5, we combine the analysis of these sections with a general induction hypothesis to deduce the stable multiplicity formula and the global intertwining relation for many ψ. Section 4.6 contains the proof of two critical sign lemmas that are essential ingredients of the parallel Sections 4.3 and 4.4.

Chapter 5 is the center of the volume. It is a bridge between the global discussion of Chapters 3 and 4 and the local discussion of Chapters 6 and 7. It also represents a transition from the general comparisons of Chapter 4 to the study of the remaining parameters needed to complete the induction hypotheses. These exceptional cases are the crux of the matter. In §5.2 and §5.3, we shall extract several identities from the standard model, in which we display the possible failure of Theorems 2.4.1 and 4.1.2 as correction terms. Section 5.3 applies to the critical case of a parameter $\psi \in \widetilde{\Psi}_2(G)$, and calls for the introduction of a supplementary parameter ψ_+. In §5.4, we shall resolve the global problems for families of parameters ψ that are assumed to have certain rather technical local properties.

Chapter 6 applies to generic local parameters. It contains a proof of the local Langlands classification for our groups G (modified by the outer automorphism in the case $G = SO(2n)$). We will first have to embed a given local parameter into a family of global parameters with the local constraints of §5.4. This will be the object of Sections 6.2 and 6.3, which rest ultimately on the simple form of the invariant trace formula for G. We will then have to extract the required local information from the global results obtained in §5.4. In §6.4, we will deduce the generic local intertwining relation from its global counterpart in §5.4. Then in §6.5, we will stabilize the orthogonality relations that are known to hold for elliptic tempered characters. This will allow us to quantify the contributions from the remaining elliptic tempered characters, the ones attached to square integrable representations. We will use the information so obtained in §6.6 and §6.7. In these sections, we shall establish Theorems 2.2.1 and 1.5.1 for the remaining "square integrable" Langlands parameters $\phi \in \widetilde{\Phi}_2(G)$. Finally, in §6.8, we shall resolve the various hypotheses taken on at the beginning of Chapter 6.

Chapter 7 applies to nongeneric local parameters. It contains the proof of the local theorems in general. In §7.2, we shall use the construction of §6.2 to embed a given nongeneric local parameter into a family of global parameters, but with local constraints that differ slightly from those of §5.4. We will then deduce special cases of the local theorems that apply to the places v with local constraints. These will follow from the local theorems for generic parameters, established in Chapter 6, and the duality operator of Aubert and Schneider-Stuhler, which we review in §7.1. We will then exploit our control over the places v to derive the local theorems at the localization $\psi = \dot{\psi}_u$ that represents the original given parameter.

We will finish the proof of the global theorems in the first two sections of Chapter 8. Armed with the local theorems, and the resulting refinements of the lemmas from Chapter 5, we will be able to establish almost all of the global results in §8.1. However, there will still to be one final obstacle. It is the case of a simple parameter $\psi \in \widetilde{\Psi}_{\mathrm{sim}}(G)$, which among other things will be essential for a resolution of our induction hypotheses. An examination of this case leads us to the initial impression that it will be resistant to all of our earlier techniques. Fortunately, with further reflection, we will find that there is a way to treat it after all. We will introduce a second supplementary parameter ψ_{++}, which appears ungainly at first, but which, with the support of two rather intricate lemmas, takes us to a successful conclusion.

Section 8.2 is the climax of our long running induction argument, as well as its most difficult point of application. Its final resolution is what brings us to the end of the proof. We will then be free in §8.3 for some general reflections that will give us some perspective on what has been established. In §8.4, we will sharpen our results for the groups $SO(2n)$ in which the outer automorphism creates some ambiguity. We will use the stabilized trace formula to construct the local and global L-packets for generic parameters predicted for these groups by the conjectural theory of endoscopy. In §8.5, we will describe an approximation of a part of the global Langlands group L_F, which is tailored to the classical groups of this volume. It could potentially be used in place of some of the ad hoc global parameters of §1.4 to streamline the statements of the global theorems.

We shall discuss inner forms of orthogonal and symplectic groups in Chapter 9. The automorphic representation theory for inner twists is in some ways easier for knowing what happens in the case of quasisplit groups. In particular, the stable multiplicity formula is already in place, since it applies only to the quasisplit case. However, there are also new difficulties for inner twists, particularly in the local case. We shall describe some of these in Sections 9.1–9.3. We shall then state analogues for inner twists of the main theorems, with the understanding that their proofs will appear elsewhere.

Having briefly summarized the various chapters, we should add some comment on our use of induction. As we have noted, induction is a central part of the unified argument that will carry us from Chapter 2 to Section 8.2. We will have two kinds of hypotheses, both based on the positive integer N that indexes the underlying general linear group $GL(N)$. The first kind includes various ad hoc assumptions, such as those implicit in some of our definitions. For example, the global parameter sets $\widetilde{\Psi}(G)$ defined in §1.4 are based on the inductive application of two "seed" Theorems 1.4.1 and 1.4.2. The second will be the formal induction hypotheses introduced explicitly at the beginning of §4.3, and in more refined form at the beginning of §5.1. They assert essentially that the stated theorems are all valid for parameters of rank less than N. In particular, they include the informal hypotheses implicit in the definitions.

We do not actually have to regard the earlier, informal assumptions as inductive. They really represent implicit appeals to stated theorems, in support of proofs of what amount to corollaries. In fact, from a logical standpoint, it is simpler to treat them as inductive assumptions *only* after we introduce the formal induction hypotheses in §4.3 and §5.1. For a little more discussion of this point, the reader can consult the two parallel *Remarks* following Corollaries 4.1.3 and 4.2.4.

The induction hypotheses of §5.1 are formulated for an abstract family $\widetilde{\mathcal{F}}$ of global parameters. They pertain to the parameters $\psi \in \widetilde{\mathcal{F}}$ of rank less than N, and are supplemented also by a hypothesis (Assumption 5.1.1) for certain parameters in $\widetilde{\mathcal{F}}$ of rank equal to N. The results of Chapter 5 will be applied three times, to three separate families $\widetilde{\mathcal{F}}$. These are the family of generic parameters with local constraints used to establish the local classification of Chapter 6, the family of nongeneric parameters with local constraints used to deduce the local results for nontempered representations in Chapter 7, and the family of all global parameters used to establish the global theorems in the first two sections of Chapter 8. In each of these cases, the assumptions have to be resolved for the given family $\widetilde{\mathcal{F}}$. In the case of Chapter 6, the induction hypotheses are actually imposed in two stages. The local hypothesis at the beginning of §6.3 is needed to construct the family $\widetilde{\mathcal{F}}$, on which we then impose the global part of the general hypothesis of Chapter 5 at the beginning of §6.4. The earlier induction hypotheses of §4.3 apply to general global parameters of rank less than N. They are used in §4.3–§4.6 to deduce the global theorems for parameters that are highly reducible. Their general resolution comes only after the proof of the global theorems in §8.2.

Our induction assumptions have of course to be distinguished from the general condition (Hypothesis 3.2.1) on which our results rely. This is the stabilization of the twisted trace formula for the two groups $GL(N)$ and $SO(2n)$. As a part of the condition, we implicitly include twisted analogues of the two local results that have a role in the stabilization of the ordinary trace formula. These are the orthogonality relations for elliptic tempered characters of [**A10**, Theorem 6.1], and the weak spectral transfer of tempered p-adic characters given by [**A11**, Theorems 6.1 and 6.2]. The stabilization of orthogonality relations in §6.5 requires twisted orthogonality relations for $GL(N)$, as well as those of [**A10**]. It will be an essential part of the local classification in Chapter 6. The two theorems from [**A11**] can be regarded as a partial generalization of the fundamental lemma for the full spherical Hecke algebra [**Hal**]. (Their global proof of course depends on the basic fundamental lemma for the unit, established by Ngo.) We will use them in combination with their twisted analogues in the proofs of Proposition 2.1.1 and Corollary 6.7.4. The first of these gives the image of the twisted transfer of functions from $GL(N)$, which is needed in the proof of Proposition 3.5.1. The second gives a relation among tempered characters that completes the local classification.

There is one other local theorem whose twisted analogues for $GL(N)$ and $SO(2n)$ we shall also have to take for granted. It is Shelstad's strong spectral transfer of tempered archimedean characters, which is to say, her endoscopic classification of representations of real groups. This of course is a major result. Together with its two twisted analogues, it gives the archimedean cases of the local classification in Theorems 2.2.1 and 2.2.4. We shall combine it with a global argument in Chapter 6 to establish the p-adic form of these theorems. The general twisted form of Shelstad's endoscopic classification appears to be within reach. It is likely to be established soon by some extension of recent work by Mezo [**Me**] and Shelstad [**S8**].

Finally, let me include a comment on notation. Because our main theorems require interlocking proofs, which consume a good part of the volume, there is always the risk of losing one's way. Until the end of §8.2, assertions as *Theorems* are generally stated with the understanding that their proofs will usually be taken up much later (unless of course they are simply quoted from some other source). On the other hand, assertions denoted *Propositions*, *Lemmas* or *Corollaries* represent results along the route, for which the reader can expect a timely proof. The theorems stated in §8.4 are not part of the central induction argument. Their proofs, which are formally labeled as such, follow relatively soon after their statements. The theorems stated in §9.4 and §9.5 apply to inner twists. They will be proved elsewhere.

The actual mathematical notation of the volume might appear unconventional at times. I have tried to structure it so as to reflect implicit symmetries in the various objects it represents. With luck, it might help a reader navigate the arguments without necessarily being aware of such symmetries.

The three main theorems of the volume were described in [**A18**, §30]. I gave lecture courses on them in 1994–1995 at the Institute for Advanced

Study and the University of Paris VII, and later in 2000, again at the Institute for Advanced Study. Parts of Chapter 4 were also treated heuristically in the earlier article [**A9**]. In writing this volume, I have added some topics to my original notes. These include the local Langlands classification for $GL(N)$, the treatment of inner twists in Chapter 9 and the remarks on Whittaker models in §8.3. I have also had to fill unforeseen gaps in the notes. For example, I did not realize that twisted endoscopy for $SO(2n)$ was needed to formulate the local intertwining relation. In retrospect, it is probably for the best that this second case of twisted endoscopy does have a role here, since it forces us to confront a general phenomenon in a concrete situation. I have tried to make this point explicit in §2.4 with the discussion surrounding the short exact sequence (2.4.10). In any case, I hope that I have accounted for most of the recent work on the subject, in the references and the text. There will no doubt be omissions. I most regret not being able to describe the results of Moeglin on the structure of p-adic packets $\widetilde{\Pi}_{\psi_v}$ ([**M1**]–[**M4**]). It is clearly an important problem to establish analogues of her results for archimedean packets.

Acknowledgments: I am greatly indebted to the referee, for a close and careful reading of an earlier draft of the manuscript (which went to the end of §8.2), and for many very helpful comments. I also thank Chung Pang Mok, who read a later draft of the manuscript, and offered further helpful suggestions.