

Summary of background material

In this chapter we review some of the basic facts and definitions that we will need later. The reader can browse this chapter to start, referring back only when needed.

1.1. Flatness

For a more detailed discussion of the notion of flatness see [2, Chapter V].

1.1.1. Let R be a ring. Recall that an R -module M is called *flat* if the functor

$$(-) \otimes_R M : \text{Mod}_R \rightarrow \text{Mod}_R$$

is an exact functor, where Mod_R denotes the abelian category of R -modules. The module M is called *faithfully flat* if M is flat, and if for any two R -modules N and N' the natural map

$$\text{Hom}_R(N, N') \rightarrow \text{Hom}_R(N \otimes_R M, N' \otimes_R M)$$

is injective.

PROPOSITION 1.1.2. *Let M be an R -module. The following are equivalent:*

- (i) M is faithfully flat.
- (ii) M is flat and for any R -module N' the map

$$(1.1.2.1) \quad N' \rightarrow \text{Hom}_R(M, N' \otimes M), \quad y \mapsto (m \mapsto y \otimes m)$$

is injective.

- (iii) A sequence of R -modules

$$(1.1.2.2) \quad N' \rightarrow N \rightarrow N''$$

is exact if and only if the sequence

$$(1.1.2.3) \quad N' \otimes_R M \rightarrow N \otimes_R M \rightarrow N'' \otimes_R M$$

is exact.

- (iv) A morphism of R -modules $N' \rightarrow N$ is injective if and only if the morphism $N' \otimes_R M \rightarrow N \otimes_R M$ is injective.
- (v) M is flat and if $N \otimes_R M = 0$ for some R -module N , then $N = 0$.
- (vi) M is flat and for every maximal ideal $\mathfrak{m} \subset R$ we have $M/\mathfrak{m}M \neq 0$.

PROOF. First let us show that (i) is equivalent to (ii). If $F \rightarrow N$ is a surjective morphism of R -modules, then for any R -module N' we have a commutative square

$$\begin{array}{ccc} \text{Hom}_R(N, N') & \longrightarrow & \text{Hom}_R(N \otimes M, N' \otimes M) \\ \downarrow & & \downarrow \\ \text{Hom}_R(F, N') & \longrightarrow & \text{Hom}_R(F \otimes M, N' \otimes M), \end{array}$$

where the vertical maps are injective. In particular, choosing $F = \bigoplus_{i \in I} R$ to be a free module, in which case we have natural isomorphisms

$$\mathrm{Hom}_R(F, N') \simeq \prod_{i \in I} N', \quad \mathrm{Hom}_R(F \otimes M, N' \otimes M) \simeq \prod_{i \in I} \mathrm{Hom}_R(M, N' \otimes M),$$

we see that an R -module M is faithfully flat if and only if for any R -module N' the map

$$N' \rightarrow \mathrm{Hom}_R(M, N' \otimes M), \quad y \mapsto (m \mapsto y \otimes m)$$

is injective, thereby showing the equivalence of (i) and (ii).

Next, let us show that (ii) implies (iv). If M is flat and $N' \rightarrow N$ is injective, then $N' \otimes M \rightarrow N \otimes M$ is also injective by the flatness assumption on M . So to show that (ii) implies (iv) it suffices to show that if (ii) holds and $N' \rightarrow N$ is a morphism of R -modules such that $N' \otimes M \rightarrow N \otimes M$ is an injection, then $N' \rightarrow N$ is also an inclusion. For this, note that we obtain a commutative diagram

$$(1.1.2.4) \quad \begin{array}{ccc} N' & \hookrightarrow & \mathrm{Hom}_R(M, N' \otimes M) \\ \downarrow & & \downarrow \\ N & \hookrightarrow & \mathrm{Hom}_R(M, N \otimes M), \end{array}$$

where the horizontal arrows are inclusions by (ii). If $N' \otimes M \rightarrow N \otimes M$ is an inclusion, then so is the right vertical arrow in the diagram, from which it follows that $N' \rightarrow N$ is also injective.

Statement (v) follows from (iv) applied to the map $N \rightarrow 0$.

Statement (v) implies (vi) by taking $N = R/\mathfrak{m}$.

Also (v) is equivalent to (iii). Indeed (iii) is equivalent to the statement that M is flat and for any sequence (1.1.2.2) for which (1.1.2.3) is exact the sequence (1.1.2.2) is exact. Now observe that if M is flat and (1.1.2.2) is a sequence of R -modules then setting

$$H := \mathrm{Ker}(N \rightarrow N'')/\mathrm{Im}(N' \rightarrow N)$$

we have

$$H \otimes M := \mathrm{Ker}(N \otimes M \rightarrow N'' \otimes M)/\mathrm{Im}(N' \otimes M \rightarrow N \otimes M).$$

From this it follows that (v) and (iii), and the converse direction is immediate (consider the sequence $0 \rightarrow N \rightarrow 0$).

To prove the proposition we are therefore reduced to showing that (vi) implies (ii). For this we show that a counterexample to (ii) yields a counterexample for (vi). So suppose N' is an R -module and $x \in N'$ is a nonzero element mapping to zero under the map (1.1.2.1). Denote by $L \subset N'$ the submodule generated by x , and let $\mathfrak{a} \subset R$ be the kernel of the map $R \rightarrow N'$ sending $f \in R$ to $f \cdot x$, so we have $R/\mathfrak{a} \simeq L$. If M is flat, then the map

$$L \otimes M \rightarrow N' \otimes M$$

is an inclusion, from which we deduce that the map (1.1.2.1) for L ,

$$L \rightarrow \mathrm{Hom}_R(M, L \otimes M),$$

is the zero map. The image of $x \in L$ under this map is via the isomorphism $L \simeq R/\mathfrak{a}$ identified with the projection map

$$M \rightarrow M \otimes R/\mathfrak{a} \simeq M/\mathfrak{a}M,$$

which implies that $M/\mathfrak{a}M = 0$. If \mathfrak{m} is a maximal ideal containing \mathfrak{a} we then get that $M/\mathfrak{m}M = 0$ thereby obtaining a counterexample to (vi). \square

DEFINITION 1.1.3. A morphism of schemes $f : X \rightarrow Y$ is *flat* if for every point $x \in X$ the map

$$\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$$

is flat. The morphism f is *faithfully flat* if f is flat and surjective.

REMARK 1.1.4. If $R \rightarrow R'$ is a ring homomorphism, then

$$\mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R)$$

is flat (resp. faithfully flat) if and only if R' is flat (resp. faithfully flat) as an R -module (this is exercise 1.A).

PROPOSITION 1.1.5. *Let $f : X \rightarrow Y$ be a flat morphism of locally noetherian schemes that is locally of finite type. Then for any open subset $U \subset X$ the image $f(U) \subset Y$ is open.*

PROOF. See [26, IV.2.4.6]. \square

COROLLARY 1.1.6. *Let $f : X \rightarrow Y$ be a faithfully flat morphism of locally noetherian schemes that is locally of finite type, and let $Y = \bigcup_i U_i$ be an open covering, with each U_i affine. Then for each i , there is a Zariski covering $f^{-1}(U_i) = \bigcup_j V_{ij}$ with V_{ij} quasi-compact and $f(V_{ij}) = U_i$.*

PROOF. It suffices to exhibit for every $x \in f^{-1}(U_i)$ a quasi-compact open neighborhood $x \in V \subset f^{-1}(U_i)$ such that $f(V) = U_i$. For this start with an affine neighborhood $x \in V_0 \subset f^{-1}(U_i)$, and then take $V = V_0 \cup V_1 \cup \cdots \cup V_r$, where the $V_j \subset f^{-1}(U_i)$ are affine open subsets such that the open images $f(V_j)$ cover the quasi-compact U_i . \square

REMARK 1.1.7. The condition ‘flat morphism of locally noetherian schemes that is locally of finite type’ in 1.1.5 and 1.1.6 can be replaced by the more general condition ‘flat morphism of schemes that is locally of finite presentation’ discussed in the next section.

1.2. Morphisms locally of finite presentation

The basic reference for the material in this section is [26, IV §8].

1.2.1. If A is a ring and M is an A -module, then M is called of *finite presentation* if there exists an exact sequence

$$A^r \rightarrow A^s \rightarrow M \rightarrow 0$$

for some integers r and s . Note that in the case when A is noetherian, this is equivalent to M being finitely generated (as the kernel of any surjection $A^s \rightarrow M$ is automatically finitely generated), but in general M being of finite presentation is a stronger condition than being finitely generated.

1.2.2. If $A \rightarrow B$ is a ring homomorphism, then we say that B is of *finite presentation over A* , (or that B is a *finitely presented A -algebra*) if there exists a surjection

$$\pi : A[X_1, \dots, X_s] \rightarrow B$$

with kernel $\text{Ker}(\pi)$ a finitely generated ideal in $A[X_1, \dots, X_s]$. If A is noetherian this is equivalent to B being a finitely generated A -algebra, but in general B being of finite presentation is a stronger condition than being finitely generated.

1.2.3. These notions generalize to schemes as follows.

Let X be a scheme. A quasi-coherent sheaf \mathcal{F} on X is called *locally finitely presented* if for every affine open subset $\text{Spec}(B) \subset X$ the module $\Gamma(\text{Spec}(B), \mathcal{F})$ is a finitely presented B -module.

Note that if X is locally noetherian then a quasi-coherent sheaf is locally finitely presented if and only if it is coherent.

A morphism of schemes $f : X \rightarrow Y$ is called *locally of finite presentation* if for every affine open $\text{Spec}(A) \subset Y$ and affine open $\text{Spec}(B) \subset f^{-1}(\text{Spec}(A))$, the A -algebra B is of finite presentation over A .

A morphism $f : X \rightarrow Y$ is said to be of *finite presentation* (or a *finitely presented morphism*) if f is locally of finite presentation and quasi-compact and quasi-separated (recall that by definition a morphism of schemes $f : X \rightarrow Y$ is quasi-separated if the diagonal morphism is quasi-compact).

In the case when Y is noetherian, the morphism f is locally of finite presentation if and only if f is locally of finite type, and finitely presented if and only if of finite type.

EXAMPLE 1.2.4 ([67, Tag 01KH]). The notion of a quasi-separated morphism usually is not studied extensively in a first year course in algebraic geometry, as any reasonable morphism of schemes that one might encounter in such a course will be quasi-separated. However, in the study of algebraic spaces and stacks properties of the diagonal morphism play a central role and notions such as quasi-separated morphisms occur naturally. Here is an example of a morphism of schemes $f : X \rightarrow Y$ which is quasi-compact and locally of finite presentation but not quasi-separated.

Let k be a field and let $Y = \text{Spec}(k[x_1, x_2, \dots])$ be the spectrum of the polynomial ring on infinitely many variables x_i ($i \geq 1$). Let $z \in Y$ denote the closed point obtained by setting all variables $x_i = 0$, and let $P_i \in Y$ ($i \geq 1$) be the closed point obtained by setting $x_i = 1$ and all other variables equal to 0. Let $U \subset Y$ denote the complement of z . Then U is not quasi-compact. Indeed, U is covered by the basic open subsets $D(x_i)$ and each P_i is contained in a unique such basic open, whence no finite subset of the $D(x_i)$ can cover U . Let X be the scheme obtained by taking two copies of Y and gluing them along the open subset U with the identity morphism, and let $X_1, X_2 \subset X$ be the inclusions of the two copies of Y . Let $f : X \rightarrow Y$ be the morphism which restricts to the identity on each X_i . Then f is quasi-compact and locally of finite presentation, but not quasi-separated. Indeed, we have $X_1 \times_Y X_2 \simeq Y$, and the square

$$\begin{array}{ccc} U & \longrightarrow & X_1 \times_Y X_2 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times_Y X \end{array}$$

is cartesian.

1.2.5. The utility of these notions is that they often allow one to reduce questions over arbitrary base schemes to questions over base schemes of finite type over \mathbb{Z} .

To explain, consider a partially ordered set (I, \geq) which we assume to be filtering (that is for every $\lambda, \mu \in I$ there exists $\tau \in I$ such that $\tau \geq \lambda$ and $\tau \geq \mu$). We often think of I as a category with objects the elements of I and

$$\mathrm{Hom}(\lambda, \mu) = \begin{cases} \{*\} & \text{if } \lambda \geq \mu, \\ \emptyset & \text{otherwise.} \end{cases}$$

Fix a scheme B . A *projective system of B -schemes indexed by I* is a functor

$$S_\bullet : I \rightarrow (B\text{-schemes}).$$

More concretely, for every $\lambda \in I$ we specify a B -scheme S_λ and if $\lambda \geq \mu$ then we specify a morphism over B ,

$$\theta_{\lambda, \mu} : S_\lambda \rightarrow S_\mu,$$

and these transition maps have to be compatible for triples $\tau \geq \lambda \geq \mu$.

We say that a projective system S_\bullet of B -schemes indexed by I has *affine transition maps* if for every $\lambda \geq \mu$ the map $\theta_{\lambda, \mu}$ is an affine morphism.

EXAMPLE 1.2.6. Suppose $B = \mathrm{Spec}(\mathbb{Z})$ and A is a ring. Let I be the partially ordered set of finitely generated subrings of A , and for $\lambda \in I$ let A_λ be the corresponding subring. We then get a projective system of affine schemes setting $S_\lambda = \mathrm{Spec}(A_\lambda)$.

LEMMA 1.2.7. *Let S_\bullet be a projective system of B -schemes indexed by I with affine transition maps. Then the inverse limit $\varprojlim_{\lambda \in I} S_\lambda$ exists in the category of B -schemes, and for every $\lambda \in I$ the map $\varprojlim_{\lambda \in I} S_\lambda \rightarrow S_\lambda$ is affine.*

PROOF. Fix any $\lambda \in I$ and for $\mu \geq \lambda$ let \mathcal{A}_μ denote the quasi-coherent sheaf of algebras on S_λ corresponding to the affine morphism $S_\mu \rightarrow S_\lambda$. Setting

$$\mathcal{A} := \varinjlim_{\mu \geq \lambda} \mathcal{A}_\mu$$

we get a scheme $S := \underline{\mathrm{Spec}}_{S_\lambda}(\mathcal{A})$ equipped with a morphism $\rho_\mu : S \rightarrow S_\mu$ over S_λ for each $\mu \geq \lambda$. By the universal property of the relative spectrum of a sheaf of algebras [26, II.1.2.7] the scheme S represents $\varprojlim_{\lambda \in I} S_\lambda$. \square

The following is a useful characterization of a morphism being locally of finite presentation in terms of its functor of points.

PROPOSITION 1.2.8. *A morphism of schemes $f : X \rightarrow Y$ is locally of finite presentation if and only if for every projective system of Y -schemes $\{S_\lambda\}_{\lambda \in I}$ with each S_λ an affine scheme, the natural map*

$$\varinjlim_{\lambda} \mathrm{Hom}_Y(S_\lambda, X) \rightarrow \mathrm{Hom}_Y(\varprojlim_{\lambda} S_\lambda, X)$$

is a bijection.

PROOF. See [26, IV.8.14.2]. \square

EXAMPLE 1.2.9. The assumption that f is locally of finite presentation is necessary as the following simple example shows. Let $Y = \mathrm{Spec}(k)$ be the spectrum of a field, and let X be the spectrum of a k -algebra A which is not finitely generated over k . Let I be the partially ordered set of finite subsets of A , and for $\lambda \in I$ let $A_\lambda \subset A$ be the k -subalgebra generated by the elements of λ . Let $S_\lambda = \mathrm{Spec}(A_\lambda)$.

Then $X = \varprojlim_{\lambda} S_{\lambda}$, and the identity map $X = \varprojlim_{\lambda} S_{\lambda} \rightarrow X$ gives an element of $\mathrm{Hom}_Y(\varprojlim_{\lambda} S_{\lambda}, X)$ not in the image of $\varinjlim_{\lambda} \mathrm{Hom}_Y(S_{\lambda}, X)$.

1.2.10. Fix a scheme B and let S_{\bullet} be a projective system of B -schemes indexed by a partially ordered set I . A *family of quasi-coherent sheaves* on S_{\bullet} is a collection of data $\{(\mathcal{F}_{\lambda})_{\lambda \in I}, u_{\lambda\mu}\}$ consisting of a quasi-coherent sheaf \mathcal{F}_{λ} on S_{λ} for each $\lambda \in I$, and for every $\lambda \geq \mu$ an isomorphism $u_{\lambda\mu} : \theta_{\lambda,\mu}^* \mathcal{F}_{\mu} \rightarrow \mathcal{F}_{\lambda}$ such that the natural cocycle condition holds for triples $\tau \geq \lambda \geq \mu$.

Assume now that the transition maps in S_{\bullet} are all affine so the projective limit $S := \varprojlim_{\lambda \in I} S_{\lambda}$ exists, and let $\rho_{\lambda} : S \rightarrow S_{\lambda}$ denote the projection. For a family $\{(\mathcal{F}_{\lambda})_{\lambda}, u_{\lambda\mu}\}$ set $\mathcal{F} := \rho_{\lambda}^* \mathcal{F}_{\lambda}$ for any choice of λ . The quasi-coherent sheaf \mathcal{F} on S is up to canonical isomorphism independent of the choice of λ .

Next consider two families $\{(\mathcal{F}_{\lambda})_{\lambda}, u_{\lambda\mu}\}$ and $\{(\mathcal{G}_{\lambda})_{\lambda}, v_{\lambda\mu}\}$. For every λ we have a natural map

$$\rho_{\lambda}^* : \mathrm{Hom}_{S_{\lambda}}(\mathcal{F}_{\lambda}, \mathcal{G}_{\lambda}) \rightarrow \mathrm{Hom}_S(\mathcal{F}, \mathcal{G})$$

which is compatible with the transition maps so we get a map

$$(1.2.10.1) \quad \rho^* : \varinjlim_{\lambda} \mathrm{Hom}_{S_{\lambda}}(\mathcal{F}_{\lambda}, \mathcal{G}_{\lambda}) \rightarrow \mathrm{Hom}_S(\mathcal{F}, \mathcal{G}).$$

THEOREM 1.2.11. (i) *Assume that there exists λ such that S_{λ} is quasi-compact and quasi-separated and \mathcal{F}_{λ} is locally finitely presented. Then the map (1.2.10.1) is an isomorphism.*

(ii) *Assume there exists λ such that S_{λ} is quasi-compact and quasi-separated. Then for any locally finitely presented quasi-coherent sheaf \mathcal{F} on S , there exists μ and a finitely presented quasi-coherent sheaf \mathcal{G}_{μ} on S_{μ} such that $\mathcal{F} \simeq \rho_{\mu}^* \mathcal{G}_{\mu}$.*

PROOF. See [26, IV.8.5.2]. □

Many properties of sheaves are also stable under passing to the limit:

THEOREM 1.2.12. *Let S_{\bullet} be a projective system of B -schemes with affine transition maps indexed by a partially ordered set I , and let S denote $\varprojlim_{\lambda} S_{\lambda}$. Assume that for some λ the scheme S_{λ} is quasi-compact. Let $\{(\mathcal{F}_{\lambda})_{\lambda}, u_{\lambda\mu}\}$ be a family of quasi-coherent sheaves on S_{\bullet} with pullback \mathcal{F} on S .*

(i) *Assume that \mathcal{F}_{λ} is finitely presented for some λ and let n be an integer. Then \mathcal{F} is locally free of rank n if and only if there exists μ such that \mathcal{F}_{μ} is locally free of rank n .*

(ii) *Fix $\lambda \in I$ and let*

$$\mathcal{F}_{\lambda} \rightarrow \mathcal{G}_{\lambda} \rightarrow \mathcal{H}_{\lambda} \rightarrow 0$$

be a sequence of quasi-coherent sheaves on S_{λ} . Assume that the sheaves \mathcal{F}_{λ} and \mathcal{G}_{λ} are of finite type and that \mathcal{H}_{λ} is finitely presented. Then the sequence on S ,

$$\mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0,$$

obtained by pullback to S is exact if and only if for some $\mu \geq \lambda$ the pullback of the sequence to S_{μ} is exact.

PROOF. See [26, IV 8.5.5 and 8.5.6]. □

1.2.13. Similarly many properties of morphisms behave well with respect to passing to the limit. Fix as before a projective system S_{\bullet} of B -schemes indexed by

a partially ordered set I , and assume that there exists an element $\lambda_0 \in I$ such that $\lambda \geq \lambda_0$ for all $\lambda \in I$. Let S denote $\varprojlim_{\lambda} S_{\lambda}$.

For a scheme X_{λ_0} over S_{λ_0} , denote by $X_{\lambda} := X_{\lambda_0} \times_{S_{\lambda_0}} S_{\lambda}$, and let X denote $S \times_{S_{\lambda_0}} X_{\lambda_0}$. For $\lambda \geq \mu$ the map $S_{\lambda} \rightarrow S_{\mu}$ induces a morphism $X_{\lambda} \rightarrow X_{\mu}$.

If Y_{λ_0} is a second scheme over S_{λ_0} , then base change induces for every $\lambda \geq \mu$ a map

$$\mathrm{Hom}_{S_{\mu}}(X_{\mu}, Y_{\mu}) \rightarrow \mathrm{Hom}_{S_{\lambda}}(X_{\lambda}, Y_{\lambda}),$$

and by passing to the limit a map

$$(1.2.13.1) \quad e : \varinjlim_{\lambda} \mathrm{Hom}_{S_{\lambda}}(X_{\lambda}, Y_{\lambda}) \rightarrow \mathrm{Hom}_S(X, Y).$$

THEOREM 1.2.14. (i) *Suppose X_{λ_0} is quasi-compact and that Y_{λ_0} is locally of finite presentation over S_{λ_0} . Then the map (1.2.13.1) is an isomorphism.*

(ii) *Assume S_{λ_0} is quasi-compact and quasi-separated. Then for any finitely presented S -scheme X there exists $\lambda \in I$ and a scheme X_{λ} over S_{λ} such that $X \simeq X_{\lambda} \times_{S_{\lambda}} S$.*

PROOF. See [26, IV 8.8.2]. □

THEOREM 1.2.15. *Assume S_{λ_0} is quasi-compact and quasi-separated, and that X_{λ_0} and Y_{λ_0} are finitely presented S_{λ_0} -schemes. Let $f_{\lambda_0} : X_{\lambda_0} \rightarrow Y_{\lambda_0}$ be a morphism, and let P be one of the following properties of morphisms: an isomorphism, a monomorphism, an imbedding, a closed imbedding, an open imbedding, separated, surjective, radicial, affine, quasi-affine, finite, quasi-finite, proper, projective, quasi-projective.*

Then the base change $f : X \rightarrow Y$ has property P if and only if there exists $\lambda \in I$ such that the morphism $f_{\lambda} : X_{\lambda} \rightarrow Y_{\lambda}$ has property P .

PROOF. See [26, IV 8.10.5]. □

1.3. Étale and smooth morphisms

The basic reference for the development of the theory of smooth and étale morphisms, as presented in this section, is [26, IV, §17].

DEFINITION 1.3.1. Let $f : X \rightarrow Y$ be a morphism of schemes. We call f *formally smooth* (resp. *formally unramified*, *formally étale*) if for every affine Y -scheme $Y' \rightarrow Y$ and every closed imbedding $Y'_0 \hookrightarrow Y'$ defined by a nilpotent ideal, the map

$$(1.3.1.1) \quad \mathrm{Hom}_Y(Y', X) \rightarrow \mathrm{Hom}_Y(Y'_0, X)$$

is surjective (resp. injective, bijective). If f is also locally of finite presentation then f is called *smooth* (resp. *unramified*, *étale*).

REMARK 1.3.2. If $Y'_0 \hookrightarrow Y'$ is defined by an ideal $I \subset \mathcal{O}_{Y'}$ with $I^n = 0$ for some n , then setting $Y'_i \subset Y'$ equal to the closed subscheme defined by I^{i+1} we get a sequence of closed imbeddings of Y -schemes

$$Y'_0 \hookrightarrow Y'_1 \hookrightarrow \cdots \hookrightarrow Y'_{n-1} = Y',$$

with Y'_i defined in Y'_{i+1} by a square-zero ideal. This in turn gives a factorization of (1.3.1.1) as

$$\mathrm{Hom}_Y(Y', X) \rightarrow \mathrm{Hom}_Y(Y'_{n-2}, X) \rightarrow \cdots \rightarrow \mathrm{Hom}_Y(Y'_1, X) \rightarrow \mathrm{Hom}_Y(Y'_0, X).$$

From this it follows that a morphism $f : X \rightarrow Y$ is formally smooth (resp. formally unramified, formally étale) if and only if for every closed imbedding of Y -schemes $Y'_0 \hookrightarrow Y'$ defined by a square-zero ideal the map (1.3.1.1) is surjective (resp. injective, bijective).

1.3.3. Recall that if S is a set and G is a group acting on S , then S is called a G -torsor if S is nonempty and the action of G is simply transitive.

One of the key basic properties of differentials is the following. For any commutative diagram of solid arrows of schemes

$$(1.3.3.1) \quad \begin{array}{ccc} Y'_0 & \xrightarrow{y_0} & X \\ \downarrow i & \swarrow \text{dotted} & \downarrow f \\ Y' & \xrightarrow{a} & Y \end{array}$$

where i is a square-zero closed imbedding with ideal $\mathcal{I} \subset \mathcal{O}_{Y'}$, the set of dotted arrows filling in the diagram is either empty or a torsor under the group

$$\mathrm{Hom}_{Y'_0}(y_0^* \Omega_{X/Y}^1, \mathcal{I}).$$

This can be seen as follows (we leave some of the details as exercise 1.I). Let $\Delta : X \rightarrow X \times_Y X$ be the diagonal morphism, and consider the surjection of sheaves of rings on X :

$$\Delta^{-1} \mathcal{O}_{X \times_Y X} \rightarrow \mathcal{O}_X.$$

If J denotes the kernel of this map, then by definition [41, definition on page 175] we have $\Omega_{X/Y}^1 \simeq J \otimes_{\Delta^{-1} \mathcal{O}_{X \times_Y X}} \mathcal{O}_X$ (which we henceforth abbreviate J/J^2). Given two maps $\rho_1, \rho_2 : Y' \rightarrow X$ filling in (1.3.3.1) we obtain a commutative diagram

$$\begin{array}{ccc} Y'_0 & \xrightarrow{i} & Y' \\ \downarrow y_0 & & \downarrow \rho := \rho_1 \times \rho_2 \\ X & \xrightarrow{\Delta} & X \times_Y X \end{array}$$

From this we obtain a commutative diagram of sheaves of rings on $|Y_0|$:

$$\begin{array}{ccc} y_0^{-1} \Delta^{-1} \mathcal{O}_{X \times_Y X} & \longrightarrow & y_0^{-1} \mathcal{O}_X \\ \downarrow \rho & & \downarrow y_0 \\ \mathcal{O}_{Y'} & \longrightarrow & \mathcal{O}_{Y'_0}. \end{array}$$

Since the kernel of $\mathcal{O}_{Y'} \rightarrow \mathcal{O}_{Y'_0}$ is square-zero, the morphism ρ factors through the pullback to Y_0 of

$$\mathcal{P}_{X/Y}^1 := \Delta^{-1} \mathcal{O}_{X \times_Y X} / J^2.$$

The pullback $p_1^* : \mathcal{O}_X \rightarrow \mathcal{P}_{X/Y}^1$ coming from the first projection $X \times_Y X \rightarrow X$ defines a section of the map $\mathcal{P}_{X/Y}^1 \rightarrow \mathcal{O}_X$ so this gives a decomposition

$$\mathcal{P}_{X/Y}^1 \simeq \mathcal{O}_X \oplus J/J^2.$$

The map $y_0^{-1} \mathcal{P}_{X/Y}^1 \rightarrow \mathcal{O}_{Y'}$ therefore induces a map $y_0^{-1} J/J^2 \rightarrow \mathcal{I}$. We define

$$\rho_1 - \rho_2 \in \mathrm{Hom}_{Y'_0}(y_0^* \Omega_{X/Y}^1, \mathcal{I})$$

to be this map. By exercise 1.I this construction defines a torsorial action of $\text{Hom}_{Y'_0}(y_0^*\Omega_{X/Y}^1, \mathcal{I})$ on the set of morphisms $Y' \rightarrow X$ filling in (1.3.3.1) if this set is nonempty.

PROPOSITION 1.3.4. (i) *If $f : X \rightarrow Y$ is smooth (resp. unramified, étale), and $g : Y' \rightarrow Y$ is any morphism then the base change $f' : X \times_Y Y' \rightarrow Y'$ is smooth (resp. unramified, étale).*

(ii) *A composition of smooth (resp. unramified, étale) morphisms is smooth (resp. unramified, étale).*

(iii) *Suppose given a composition of morphisms locally of finite presentation*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

with gf and g smooth, and such that the map $f^*\Omega_{Y/Z}^1 \rightarrow \Omega_{X/Z}^1$ is an isomorphism. Then f is étale.

PROOF. These follow immediately from the definitions and the discussion in 1.3.3, and are left as exercise 1.J. \square

PROPOSITION 1.3.5. *A morphism of schemes $f : X \rightarrow Y$ is smooth if and only if it is locally of finite presentation and the following condition holds: For every commutative diagram of schemes*

$$(1.3.5.1) \quad \begin{array}{ccc} Y'_0 & \xrightarrow{y_0} & X \\ \downarrow i & & \downarrow f \\ Y' & \xrightarrow{a} & Y, \end{array}$$

where i is a closed imbedding defined by a square-zero ideal, there exists an open covering $Y' = \bigcup_i U'_i$ and morphisms $\rho_i : U'_i \rightarrow X$ over Y such that the restriction of ρ_i to $U'_{i,0} := U_i \times_{Y'} Y'_0$ is equal to the restriction of y_0 .

PROOF. The ‘only if’ direction is immediate.

For the ‘if’ direction, fix a diagram (1.3.5.1) with Y' affine and assume given a covering $Y' = \bigcup_i U_i$ with each U_i also affine and morphisms $\rho_i : U_i \rightarrow X$ as in the proposition. We need to show that we can change our choices of ρ_i so that they agree on the overlaps $U_{ij} := U_i \cap U_j$.

For this, define elements

$$\epsilon_{ij} := \rho_i|_{U_{ij}} - \rho_j|_{U_{ij}} \in \text{Hom}_{U_{ij}}(y_0^*\Omega_{X/Y}^1, \mathcal{I}),$$

where $\mathcal{I} \subset \mathcal{O}_{Y'}$ is the ideal of $Y'_0 \subset Y'$. In other words, let ϵ_{ij} be the unique class which under the torsor action sends $\rho_j|_{U_{ij}}$ to $\rho_i|_{U_{ij}}$. For three indices i, j, k it follows immediately from the definition that on U_{ijk} we have

$$\epsilon_{ij} + \epsilon_{jk} = \epsilon_{ik}$$

so the elements ϵ_{ij} define a Čech cocycle with coefficients in the quasi-coherent sheaf $\mathcal{H}om(y_0^*\Omega_{X/Y}^1, \mathcal{I})$. Since Y'_0 is affine we have

$$H^1(Y'_0, \mathcal{H}om(y_0^*\Omega_{X/Y}^1, \mathcal{I})) = 0,$$

so this implies that there exists elements $\lambda_i \in \mathcal{H}om_{U_i}(y_0^*\Omega_{X/Y}^1|_{U_i}, \mathcal{I})$ such that

$$\epsilon_{ij} = \lambda_j|_{U_{ij}} - \lambda_i|_{U_{ij}}.$$

Setting $\rho'_i := \lambda_i * \rho_i$ we see that the maps ρ'_i agree on the overlaps U_{ij} and therefore glue together to give a global map $\rho : Y' \rightarrow X$ over Y filling in (1.3.5.1). \square

PROPOSITION 1.3.6. (i) *Let $f : X \rightarrow Y$ be a smooth morphism of schemes. Then the sheaf $\Omega_{X/Y}^1$ is a locally free sheaf of finite rank on X .*

(ii) *If $f : X \rightarrow Y$ is étale, then $\Omega_{X/Y}^1 = 0$.*

(iii) *If $g : X \rightarrow Y$ is a smooth morphism, and $i : Z \hookrightarrow X$ is a locally finitely presented closed imbedding, then the composition $f := ig : Z \rightarrow Y$ is smooth if and only if the sequence*

$$(1.3.6.1) \quad 0 \rightarrow i^* \mathcal{I}_Z \rightarrow i^* \Omega_{X/Y}^1 \rightarrow \Omega_{Z/Y}^1 \rightarrow 0$$

is exact and locally split, where \mathcal{I}_Z denotes the ideal sheaf of Z in X .

PROOF. To prove (i), we may work locally on X and Y so we may assume that X and Y are affine, say

$$X = \text{Spec}(B), \quad Y = \text{Spec}(A).$$

We then need to show that the B -module $\Omega_{B/A}^1$ is projective.

For a B -module M , let $B[M]$ denote the B -algebra with underlying B -module the direct sum $B \oplus M$, and the multiplication given by

$$(b, m) \cdot (b', m') := (bb', bm' + b'm).$$

There is a surjection $\pi_M : B[M] \rightarrow B$ given by $(b, m) \mapsto b$, whose kernel is a square-zero ideal.

Consider a morphism of A -algebras $f : B \rightarrow B[M]$ such that the composite map

$$B \xrightarrow{f} B[M] \xrightarrow{\pi_M} B$$

is the identity map. We can then write $f(b) = (b, \partial(b))$ for a function $\partial : B \rightarrow M$. From the relations

$$(b+b', \partial(b)+\partial(b')) = (b, \partial(b)) + (b', \partial(b')) = f(b) + f(b') = f(b+b') = (b+b', \partial(b+b')),$$

$$(bb', b\partial(b') + b'\partial(b)) = (b, \partial(b))(b', \partial(b')) = f(b)f(b') = f(bb') = (bb', \partial(bb')),$$

and

$$(a, \partial(a)) = f(a) = (a, 0), \quad a \in A,$$

we conclude that ∂ is an A -derivation $B \rightarrow M$. Conversely, given such a derivation ∂ , we obtain a morphism of A -algebras $B \rightarrow B[M]$ which composes with π_M to the identity.

Suppose we are now given a diagram of B -modules

$$\begin{array}{ccc} & \Omega_{B/A}^1 & \\ & \downarrow \partial & \\ M & \xrightarrow{t} & N, \end{array}$$

where t is surjective. If $f_0 : B \rightarrow B[N]$ denotes the map corresponding to ∂ , then we obtain a commutative diagram of solid arrows

$$\begin{array}{ccc} B[N] & \xleftarrow{f_0} & B \\ \uparrow t & \swarrow \text{---} & \uparrow \\ B[M] & \xleftarrow{\quad} & A, \end{array}$$

where we also write $t : B[M] \rightarrow B[N]$ for the surjection defined by t . By the preceding discussion giving a dotted arrow as in the diagram is equivalent to giving a lifting of ∂ ,

$$\tilde{\partial} : \Omega_{B/A}^1 \rightarrow M.$$

Now if we assume that $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is smooth, then such a lifting exists, and therefore $\Omega_{B/A}^1$ is projective implying (i).

From this discussion (ii) also follows, for if $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is étale, then the preceding discussion implies that for any B -module M we have

$$\text{Hom}_B(\Omega_{B/A}^1, M) = 0,$$

which implies that $\Omega_{B/A}^1 = 0$.

Finally, let us prove (iii). We always have an exact sequence

$$i^* \mathcal{J}_Z \rightarrow i^* \Omega_{X/Y}^1 \rightarrow \Omega_{Z/Y}^1 \rightarrow 0,$$

so it is equivalent to show that $Z \rightarrow Y$ is smooth if and only if the map $i^* \mathcal{J}_Z \rightarrow i^* \Omega_{X/Y}^1$ is injective and locally split. To prove this we may work locally on X and Y , so as above we may assume that $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$ are affine. Let $I \subset B$ be the ideal defining Z , and let R denote B/I so we have $Z = \text{Spec}(R)$. We have to show that the map induced by $d : B \rightarrow \Omega_{B/A}^1$,

$$\bar{d} : I/I^2 \rightarrow \Omega_{B/A}^1 \otimes_B R,$$

is injective and locally split if and only if $Z \rightarrow Y$ is smooth. Now observe that if \bar{d} is injective and locally split, then $\Omega_{R/A}^1$ is a projective R -module and therefore in fact the map \bar{d} is split over R . It therefore suffices to show that $Z \rightarrow Y$ is smooth if and only if \bar{d} identifies I/I^2 with a direct summand of $\Omega_{B/A}^1 \otimes_B R$.

The condition that I/I^2 is a direct summand is equivalent to the condition that for any R -module N the map

$$(1.3.6.2) \quad \text{Hom}_R(\Omega_{B/A}^1 \otimes_B R, N) \rightarrow \text{Hom}_R(I/I^2, N)$$

is surjective (consider $N = I/I^2$). This map can be interpreted as follows. We have

$$\text{Hom}_R(\Omega_{B/A}^1 \otimes_B R, N) \simeq \text{Hom}_B(\Omega_{B/A}^1, N),$$

so this set is in bijection with the set of A -algebra maps

$$B \rightarrow B[N]$$

which compose with the projection to B to the identity. Since N is an R -module, giving such a map is equivalent to giving an A -algebra map

$$f : B \rightarrow R[N]$$

whose composition with the projection to R is the quotient map. Now given such a map we get an induced map

$$I \rightarrow \text{Ker}(R[N] \rightarrow R) \simeq N.$$

This map factors through I/I^2 , and sending f to the resulting map $I/I^2 \rightarrow N$ is the map (1.3.6.2).

Now suppose that $Z \rightarrow Y$ is smooth. Let $j : \tilde{Z} \hookrightarrow X$ denote the closed subscheme of X defined by I^2 . We then have a closed imbedding $Z \hookrightarrow \tilde{Z}$ defined by the square-zero ideal I/I^2 . Now consider the commutative diagram of solid arrows

$$\begin{array}{ccc} Z & \xlongequal{\quad} & Z \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \tilde{Z} & \longrightarrow & Y. \end{array}$$

Since $Z \rightarrow Y$ is smooth there exists a dotted arrow as in the diagram. Choosing one such morphism we get a homomorphism

$$R \rightarrow B/I^2$$

lifting the projection $B/I^2 \rightarrow B/I = R$. This homomorphism defines an isomorphism $R[I/I^2] \simeq B/I^2$. We therefore get a map

$$B \rightarrow R[I/I^2]$$

whose composition with $R[I/I^2] \rightarrow R$ is the quotient map, and such that the composite map

$$I \hookrightarrow B \rightarrow R[I/I^2]$$

is equal to the projection $I \rightarrow I/I^2$ followed by the inclusion $I/I^2 \hookrightarrow R[I/I^2]$.

This map corresponds (1.3.6.2) to a morphism $\Omega_{B/A}^1 \otimes R \rightarrow I/I^2$ whose composition with \bar{d} is the identity. This shows that if $Z \rightarrow Y$ is smooth then \bar{d} is injective and locally split.

For the converse suppose \bar{d} is injective and split, and consider a commutative diagram of solid arrows

$$\begin{array}{ccc} \text{Spec}(C_0) & \xrightarrow{h_0} & Z \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \text{Spec}(C) & \longrightarrow & Y, \end{array}$$

where $C \rightarrow C_0$ is surjective with square-zero kernel J .

Since $X \rightarrow Y$ is smooth, there exists a morphism $\tilde{h} : \text{Spec}(C) \rightarrow X$ such that the composition

$$\text{Spec}(C_0) \hookrightarrow \text{Spec}(C) \rightarrow X$$

is the composition of h_0 with $Z \hookrightarrow X$. We have to show that we can find \tilde{h} that factors through Z . The map \tilde{h} defines a map

$$I \hookrightarrow B \rightarrow C$$

which gives a map $\kappa : I/I^2 \rightarrow J$. For the map \tilde{h} to factor through Z it suffices that this map is zero. Now if this map is not zero, then choose an element in

$$\text{Hom}_R(\Omega_{B/A}^1 \otimes R, J)$$

mapping to κ under (1.3.6.2). Changing our choice of \tilde{h} by the negative of this map κ (under the torsorial action discussed in 1.3.3), we obtain a map \tilde{h} which factors through Z . \square

1.3.7. Part (iii) of the proposition gives a useful way to describe smooth and étale morphisms locally.

Let

$$f : \text{Spec}(B) \rightarrow \text{Spec}(A)$$

be a smooth morphism of affine schemes, and let $x \in \text{Spec}(B)$ be a point. Assume that $\Omega_{B/A}^1$ is free of finite rank r (we can always arrange this by shrinking on $\text{Spec}(B)$). Choose a surjection

$$A[x_1, \dots, x_n] \rightarrow B$$

and let I denote the kernel. The sequence

$$(1.3.7.1) \quad 0 \longrightarrow I/I^2 \xrightarrow{\bar{d}} \Omega_{A[x_1, \dots, x_n]/A}^1 \otimes B \longrightarrow \Omega_{B/A}^1 \longrightarrow 0$$

is exact by 1.3.6 (iii), so after possibly shrinking some more to a neighborhood of x there exists $n - r$ elements $f_1, \dots, f_{n-r} \in I$ such that f_1, \dots, f_{n-r} map to a basis for I/I^2 . We have

$$(1.3.7.2) \quad \bar{d}(f_j) = \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} dx_i.$$

Since the classes of the f_j form a basis for I/I^2 and the differentials dx_i form a basis for $\Omega_{A[x_1, \dots, x_n]/A}^1$, the condition that \bar{d} is a split injection is equivalent to the condition that the $(n - r) \times (n - r)$ minors of the matrix

$$(1.3.7.3) \quad (\partial f_j / \partial x_i) \in \mathbb{M}_{n \times (n-r)}(B)$$

generate the unit ideal in B . By Nakayama's lemma, we can therefore find an element $g \in A[x_1, \dots, x_r]$ which is invertible at x such that the $(n - r) \times (n - r)$ minors of the matrix $(\partial f_j / \partial x_i)$ generate the unit ideal in $A[x_1, \dots, x_r][1/g]$ and such that the ideal in $A[x_1, \dots, x_r][1/g]$ generated by I is equal to the ideal generated by the elements f_1, \dots, f_{n-r} . We have shown the 'only if' direction of the following:

PROPOSITION 1.3.8. *Let $f : X \rightarrow Y$ be a morphism locally of finite presentation. Then f is smooth if and only if for every point $x \in X$ there exists affine neighborhoods*

$$x \in \text{Spec}(B) \subset X, \quad f(x) \in \text{Spec}(A) \subset Y$$

with $f(\text{Spec}(B)) \subset \text{Spec}(A)$, and such that

$$(1.3.8.1) \quad B \simeq (A[x_1, \dots, x_n]/(f_1, \dots, f_s))[1/g]$$

for some $f_1, \dots, f_s, g \in A[x_1, \dots, x_n]$ and $s \leq n$ for which the $s \times s$ -minors of the matrix (1.3.7.3) generate the unit ideal in B . The morphism f is étale if and only if for any point $x \in X$ we can find an affine neighborhood (1.3.8.1) as above with $n = s$.

PROOF. For the converse suppose we have an isomorphism (1.3.8.1). The condition on the minors of (1.3.7.3) insures that the sequence (1.3.7.1) is exact, and locally split. By 1.3.6 (iii) this implies that $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is smooth. Finally, note that from the local description (1.3.8.1), the rank of $\Omega_{X/Y}^1$ is equal to

$n - s$, so $\Omega_{X/Y}^1 = 0$ if and only if for each presentation (1.3.8.1) we have $n = s$. This implies the statement describing étale morphisms. \square

1.3.9. Suppose now that $f : X \rightarrow Y$ is a smooth morphism and let $x \in X$ be a point. We know that the sheaf $\Omega_{X/Y}^1$ is locally free of finite rank. It follows that there exists a neighborhood $x \in U \subset X$ and sections $f_1, \dots, f_n \in \Gamma(U, \mathcal{O}_U)$ such that the differentials df_1, \dots, df_n form a basis for $\Omega_{X/Y}^1|_U$. Let

$$\mathbf{f} : U \rightarrow \mathbb{A}_Y^n$$

be the morphism defined by f_1, \dots, f_n . Since $\mathbf{f}^* \Omega_{\mathbb{A}_Y^n/Y}^1 \rightarrow \Omega_{X/Y}^1|_U$ is an isomorphism by construction, it follows from (1.3.4 (iii)) that \mathbf{f} is étale.

COROLLARY 1.3.10. *Let $f : X \rightarrow Y$ be a smooth morphism, and let $x \in X$ be a point. Then there exists an étale morphism $\pi : Y' \rightarrow Y$ with image containing $f(x)$ and a morphism $s : Y' \rightarrow X$ such that $f \circ s = \pi$.*

PROOF. After shrinking on X we may assume that there exists an étale morphism $\mathbf{f} : X \rightarrow \mathbb{A}_Y^n$ for some n . Let Y' denote the fiber product of the diagram

$$\begin{array}{ccc} & X & \\ & \downarrow \mathbf{f} & \\ Y & \xrightarrow{0} & \mathbb{A}_Y^n. \end{array}$$

Then Y' fits into a diagram

$$\begin{array}{ccc} Y' & \xrightarrow{s} & X \\ \downarrow & & \\ Y & & \end{array}$$

as desired. \square

COROLLARY 1.3.11. *Let A be a ring, and let $f \in A[X]$ be a monic polynomial with derivative f' . Then the finite ring homomorphism*

$$A \rightarrow A[X]/(f)$$

is étale if and only if f' maps to a unit in $A[X]/(f)$.

PROOF. By 1.3.6 (ii) applied to the closed imbedding

$$\mathrm{Spec}(A[X]/(f)) \hookrightarrow \mathrm{Spec}(A[X])$$

over $\mathrm{Spec}(A)$, it follows that $\mathrm{Spec}(A[X]/(f)) \rightarrow \mathrm{Spec}(A)$ is étale if and only if the map

$$\cdot f' : A[X]/(f) \rightarrow A[X]/(f)$$

is an isomorphism. \square

EXAMPLE 1.3.12. If K is a field and $L = K[X]/(f)$ is a field extension defined by an irreducible polynomial f , then L/K is étale if and only if $f' \neq 0$.

1.3.13. An important consequence of 1.3.8, which we now explain, is the so-called invariance of the étale site under infinitesimal thickenings. Let $i : S_0 \hookrightarrow S$ be a closed imbedding defined by a nilpotent ideal. Let $\mathrm{Et}(S)$ (resp. $\mathrm{Et}(S_0)$) denote

the category whose objects are étale S -schemes (resp. étale S_0 -schemes) and whose morphisms are S -morphisms (resp. S_0 -morphisms). There is a functor

$$(1.3.13.1) \quad F : \text{Et}(S) \rightarrow \text{Et}(S_0), \quad (Z \rightarrow S) \mapsto (Z \times_S S_0 \rightarrow S_0).$$

THEOREM 1.3.14 ([**36**, Exposé I, 8.3]). *The functor F is an equivalence of categories.*

PROOF. The full faithfulness of F can be seen as follows. Suppose $Z_1 \rightarrow S$ and $Z_2 \rightarrow S$ are two étale morphisms. Let $Z_{j,0}$ ($j = 1, 2$) denote the reduction to S_0 of Z_j , and suppose given an isomorphism

$$\sigma_0 : Z_{1,0} \rightarrow Z_{2,0}.$$

We then obtain a commutative diagram of solid arrows:

$$\begin{array}{ccccc} Z_{1,0} & \xrightarrow{\sigma_0} & Z_{2,0} & \hookrightarrow & Z_2 \\ \downarrow & & & \nearrow & \downarrow \\ Z_1 & \xrightarrow{\quad} & & & S. \end{array}$$

Now since $Z_2 \rightarrow S$ is étale and $Z_{1,0} \hookrightarrow Z_1$ is defined by a nilpotent ideal, there exists locally on Z_1 a unique dotted arrow filling in the diagram. By the uniqueness these locally defined morphisms agree on overlaps and therefore there exists globally on Z_1 a unique dotted arrow $\sigma : Z_1 \rightarrow Z_2$ filling in the diagram. By the same argument applied to σ_0^{-1} there exists a unique morphism $\tau : Z_2 \rightarrow Z_1$ reducing to σ_0^{-1} . The morphisms $\tau \circ \sigma$ and $\sigma \circ \tau$ must be the identities as they reduce to identity morphisms over S_0 .

It remains to show the essential surjectivity. For this observe first that if $Z \rightarrow S$ is an étale morphism with reduction $Z_0 \rightarrow S_0$, then $Z_0 \hookrightarrow Z$ is defined by a nilpotent ideal so the map on topological spaces $|Z_0| \rightarrow |Z|$ is an isomorphism. Therefore any open subset $U_0 \subset Z_0$ lifts uniquely to an open subset $U \subset Z$.

Now fix an étale morphism $Z_0 \rightarrow S_0$. We then construct a lifting $Z \rightarrow S$ of Z_0 as follows. First note that the problem of finding a lifting of Z_0 is local on Z_0 in the following sense. If $Z_0 = \bigcup_{i \in I} U_0^{(i)}$ is an open covering and for each i we are given an étale lifting $U^{(i)} \rightarrow S$ of $U_0^{(i)}$, then the $U^{(i)}$ glue together to give a lifting Z of Z_0 . Indeed, for any two indices $i, j \in I$ the open subset $U_0^{(ij)} := U_0^{(i)} \cap U_0^{(j)}$ lifts by the preceding remark to unique open subsets

$$U^{(ij)} \subset U^{(i)}, \quad U^{(ij)} \subset U^{(j)}.$$

Moreover, by the uniqueness of liftings already shown, there is a unique isomorphism $\varphi_{ij} : U^{(ij)} \rightarrow U^{(ij)}$ reducing to the identity on $U_0^{(ij)}$. Furthermore, the uniqueness of liftings of morphisms implies that the assumption in the gluing lemma [**41**, Chapter II, exercise 2.12] are satisfied, so we get the desired lifting Z by gluing together the $U^{(i)}$ along the $U^{(ij)}$.

It remains to construct local liftings. By 1.3.8, we may therefore assume that $S = \text{Spec}(A)$ for some ring A , $S_0 = \text{Spec}(A_0)$ with $A_0 = A/I$ for some nilpotent ideal I , and $Z_0 = \text{Spec}(B_0)$, where

$$B_0 = (A_0[X_1, \dots, X_n]/(f_1, \dots, f_n))[1/g],$$

for polynomials $f_1, \dots, f_n, g \in A_0[X_1, \dots, X_n]$ such that the determinant

$$\det(\partial f_i / \partial X_j)$$

is invertible in B_0 . We then get a lifting of B_0 simply by lifting the polynomials f_1, \dots, f_n, g to polynomials $\tilde{f}_1, \dots, \tilde{f}_n, \tilde{g} \in A[X_1, \dots, X_n]$ and setting

$$B := (A[X_1, \dots, X_n]/(\tilde{f}_1, \dots, \tilde{f}_n))_{\tilde{g}},$$

which is an étale A -algebra by 1.3.8. \square

1.3.15. Let X be a scheme. A *geometric point* of X is a morphism $\bar{x} : \text{Spec}(k) \rightarrow X$, where k is a separably closed field. An *étale neighborhood* of a geometric point \bar{x} is a commutative diagram

$$\begin{array}{ccc} & & U \\ & \nearrow u & \downarrow b \\ \text{Spec}(k) & \xrightarrow{\bar{x}} & X, \end{array}$$

where the morphism b is étale. Given two étale neighborhoods ($i = 1, 2$),

$$\begin{array}{ccc} & & U_i \\ & \nearrow u_i & \downarrow b_i \\ \text{Spec}(k) & \xrightarrow{\bar{x}} & X, \end{array}$$

a morphism between them is an X -morphism $g : U_1 \rightarrow U_2$ such that $u_2 = g \circ u_1$ and $b_1 = b_2 \circ g$. We therefore get a category $I_{\bar{x}}$ of étale neighborhoods of \bar{x} , which is filtering.

Define

$$\mathcal{O}_{X, \bar{x}} := \varinjlim_{(U, u) \in I_{\bar{x}}} \Gamma(U, \mathcal{O}_U).$$

This as an $\mathcal{O}_{X, x}$ -algebra, called the *strict henselization of X at \bar{x}* .

PROPOSITION 1.3.16. *The ring $\mathcal{O}_{X, \bar{x}}$ is a local henselian ring with separably closed residue field.*

PROOF. There is a natural map

$$\mathcal{O}_{X, \bar{x}} \rightarrow k$$

induced by the maps

$$u^* : \Gamma(U, \mathcal{O}_U) \rightarrow k$$

by passing to the limit. The kernel of this map is a maximal ideal. Indeed, if $f \in \mathcal{O}_{X, \bar{x}}$ is an element whose image in k is nonzero, then we can find an object $(U, u) \in I_{\bar{x}}$ and an element $f_U \in \Gamma(U, \mathcal{O}_U)$ mapping to f in $\mathcal{O}_{X, \bar{x}}$. Let $V \subset U$ be the open subset where f_U is invertible. Since f_U maps to a nonzero element in k , the map u factors through a morphism $v : \text{Spec}(k) \rightarrow V$ and (V, v) is also an étale neighborhood of \bar{x} . It follows that $f \in \mathcal{O}_{X, \bar{x}}$ is a unit, which implies that the image of $\mathcal{O}_{X, \bar{x}}$ in k is a field.

To show that $\mathcal{O}_{X, \bar{x}}$ is henselian with separably closed residue field, it suffices to show that if $F \in \mathcal{O}_{X, \bar{x}}[T]$ is a polynomial and $b_0 \in k$ is a root of F in k such that $F'(b_0) \neq 0$, then b_0 is the image of a root of F in $\mathcal{O}_{X, \bar{x}}$ (for this characterization of henselian rings see, for example, [28, Chapter 7]). For this let $(U, u) \in I_{\bar{x}}$ be an étale neighborhood such that the finitely many coefficients of F are in the image of $\Gamma(U, \mathcal{O}_U)$, and let $F_U \in \Gamma(U, \mathcal{O}_U)[T]$ be a polynomial mapping to F . Then

$$Z := \underline{\text{Spec}}_U(\mathcal{O}_U[T]/F_U)_{F'_U}$$

is an étale U -scheme and the root b_0 defines a morphism $z : \text{Spec}(k) \rightarrow Z$ over u , making (Z, z) an étale neighborhood of \bar{x} . The image of $T \in \Gamma(Z, \mathcal{O}_Z)$ in $\mathcal{O}_{X, \bar{x}}$ is then a root of F mapping to b_0 as desired. \square

COROLLARY 1.3.17. *Let $f : Y \rightarrow X$ be a finite morphism of schemes, and let $\bar{x} : \text{Spec}(k) \rightarrow X$ be a geometric point. Then the map on sets of connected components*

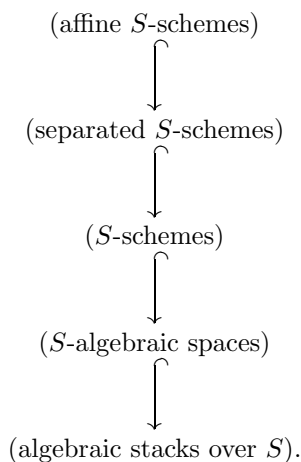
$$\pi_0(Y \times_X \text{Spec}(\mathcal{O}_{X, \bar{x}})) \rightarrow \pi_0(Y \times_{X, \bar{x}} \text{Spec}(k))$$

is a bijection.

PROOF. This follows from 1.3.16 and [28, 7.5]. \square

1.4. Schemes as functors

1.4.1. While not strictly necessary for the rest of the book, we review in this section how to characterize schemes as functors with certain properties (this characterization can for example be found in [29, VI-14]). This point of view is the starting point for the subsequent definitions of algebraic spaces and stacks. The basic idea is to start with the notion of affine scheme, and iteratively define more general objects. We are not advocating this as a preferred alternative to the usual definition of scheme, but the results of this section serve as a warmup for subsequent constructions. The end result is that for an affine base scheme S we have a hierarchy of categories of geometric objects



1.4.2. Let \mathcal{C} be a category. Recall the Yoneda imbedding A.2.2

$$h : \mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \text{Set})$$

which sends an object $X \in \mathcal{C}$ to the functor h_X given by

$$Y \mapsto \text{Hom}_{\mathcal{C}}(Y, X).$$

DEFINITION 1.4.3. (i) A functor $F : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ is *representable* if $F \simeq h_X$ for some $X \in \mathcal{C}$.

(ii) For two functors $F, G : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ a morphism of functors $f : F \rightarrow G$ is called *relatively representable* if for every $X \in \mathcal{C}$, and for every $g : h_X \rightarrow G$, the fiber product $h_X \times_G F : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ is representable.

REMARK 1.4.4. By Yoneda's lemma, for any object $X \in \mathcal{C}$ the map of sets

$$(\text{morphisms of functors } g : h_X \rightarrow G) \rightarrow G(X), \quad g \mapsto g(\text{id}_X : X \rightarrow X)$$

is a bijection, so we will often think of $g : h_X \rightarrow G$ as an element of $G(X)$.

REMARK 1.4.5. For $X \in \mathcal{C}$ and $F : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ we often abusively denote a morphism of functors $h_X \rightarrow F$ simply by $X \rightarrow F$, identifying X with its image under the Yoneda imbedding.

For the rest of this section we take $\mathcal{C} = \text{Aff}_S$ to be the category of affine schemes over some fixed affine scheme S . Any S -scheme X defines a functor

$$h_X : \mathcal{C}^{\text{op}} \rightarrow \text{Set}, \quad \text{Spec}(A) \mapsto \text{Hom}_S(\text{Spec}(A), X).$$

In the case when X is an affine S -scheme this is just the functor associated to X by the Yoneda imbedding.

DEFINITION 1.4.6. A morphism of functors $f : F \rightarrow G$ is an *affine open (resp. closed) imbedding* if the following conditions hold:

- (i) f is relatively representable.
- (ii) For all $X \in \text{Aff}_S$ and $g : h_X \rightarrow G$, the map $F \times_G h_X \rightarrow h_X$ is an open (resp. closed) imbedding.

REMARK 1.4.7. Note that here we are working with the category of affine S -schemes, so the condition in (i) that f is relatively representable means that for every affine S -scheme X and $g : h_X \rightarrow G$ the fiber product $F \times_G h_X$ is isomorphic to h_Y for an affine S -scheme Y .

REMARK 1.4.8. Note that in (ii), both the functors $F \times_G h_X$ and h_X are representable functors, so it makes sense to say that a morphism between them is an open or closed imbedding.

DEFINITION 1.4.9. A *big Zariski sheaf* on Aff_S is a functor

$$F : \text{Aff}_S^{\text{op}} \rightarrow \text{Set}$$

such that for any $U \in \text{Aff}_S$ and open covering $\{U_i\}_{i \in I}$ of U by affine open subschemes $U_i \hookrightarrow U$, the sequence of sets

$$F(U) \rightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$$

is exact.

A morphism of big Zariski sheaves

$$g : F \rightarrow G$$

is called *surjective* if for any $U \in \text{Aff}_S$ and $u \in G(U)$ there exists an open covering $U = \bigcup_{i \in I} U_i$ of U such that $u|_{U_i} \in G(U_i)$ is in the image of $F(U_i)$ for every $i \in I$.

REMARK 1.4.10. A map of Zariski sheaves $g : F \rightarrow G$ is surjective in the above sense, if and only if g is an epimorphism in the category of big Zariski sheaves on Aff_S (this is exercise 1.K).

PROPOSITION 1.4.11. *A functor $F : \text{Aff}_S^{\text{op}} \rightarrow \text{Set}$ is representable by a separated S -scheme if and only if the following hold:*

- (i) F is a big Zariski sheaf.
- (ii) The diagonal morphism $\Delta : F \rightarrow F \times F$ is an affine closed imbedding.

- (iii) *There exists a family of objects $\{X_i\}$ in Aff_S and morphisms $\pi_i : h_{X_i} \rightarrow F$ which are affine open immersions and such that the map of Zariski sheaves $\coprod_i h_{X_i} \rightarrow F$ is surjective.*

Moreover, the functor

(1.4.11.1)

$h_- : (\text{separated } S\text{-schemes}) \rightarrow (\text{functors } F : \text{Aff}_S^{\text{op}} \rightarrow \text{Set} \text{ satisfying (i)-(iii)})$

is an equivalence of categories.

PROOF. We start by showing that if X is a separated S -scheme and F denotes h_X , then F satisfies conditions (i)–(iii).

For (i), let $U \in \text{Aff}_S$ be an object and $U = \bigcup_i U_i$ an open covering. Then statement (i) is equivalent to the statement that giving a morphism of schemes

$$f : U \rightarrow X$$

is equivalent to giving a collection of morphisms

$$\{f_i : U_i \rightarrow X\}$$

such that for every $i, j \in I$ the restriction $f_i|_{U_i \cap U_j}$ and $f_j|_{U_i \cap U_j}$ are equal, which is immediate.

For (ii), note that the Yoneda imbedding commutes with products, so $h_X \times h_X \cong h_{X \times X}$ and the diagonal morphism

$$\Delta : h_X \rightarrow h_X \times h_X$$

is identified with the morphism

$$h_X \rightarrow h_{X \times_S X}$$

induced by the diagonal morphism of schemes

$$X \rightarrow X \times_S X,$$

which is a closed imbedding since X/S is separated.

To verify property (iii), let $X = \bigcup X_i$ be an open covering by affines, and let

$$\pi_i : h_{X_i} \rightarrow h_X$$

be the map induced by the inclusion $X_i \hookrightarrow X$. For any $T \in \text{Aff}_S$ and morphism $h_T \rightarrow h_X$ corresponding to a morphism of schemes $f : T \rightarrow X$ the fiber product

$$h_T \times_{h_X} h_{X_i}$$

is represented by $f^{-1}(X_i)$ since the Yoneda imbedding commutes with fiber products. Since X/S is separated and S is affine, $f^{-1}(X_i)$ is an affine open subset of T , and therefore π_i is an affine open imbedding.

The statement that the map

$$\coprod_i h_{X_i} \rightarrow F$$

is a surjective map of big Zariski sheaves amounts to the statement that given a morphism of schemes $f : T \rightarrow X$ with $T \in \text{Aff}_S$ there exists an open covering $T = \bigcup_{i \in I} T_i$ such that $f|_{T_i}$ factors through X_i for some i . This is clear as we can just take $T_i = f^{-1}(X_i)$.

Next, suppose given a functor F satisfying (i)–(iii), and choose morphisms $\pi_i : h_{X_i} \rightarrow F$ as in (iii). For every i and j , consider the fiber product of the diagram

$$\begin{array}{ccc} & h_{X_i} & \\ & \downarrow \pi_i & \\ h_{X_j} & \xrightarrow{\pi_j} & F. \end{array}$$

Since the maps π_i are representable by open imbeddings this fiber product is representable by an affine scheme V_{ij} , and we get open imbeddings

$$\begin{array}{ccc} V_{ij} & \hookrightarrow & X_i \\ \downarrow & & \\ X_j & & \end{array}$$

The isomorphism of functors

$$h_{X_j} \times_F h_{X_i} \rightarrow h_{X_i} \times_F h_{X_j}$$

obtained by switching the factors induces an isomorphism

$$\varphi_{ij} : V_{ij} \rightarrow V_{ji}.$$

This enables us to glue the schemes X_i together along the subschemes V_{ij} . Indeed, it follows immediately from the construction that the following two conditions hold, so we can apply [41, Chapter II, exercise 2.12]:

- (1) For any two indices i and j we have $\varphi_{ij} = \varphi_{ji}^{-1}$.
- (2) For any three indices i, j, k we have

$$\varphi_{ij}(V_{ij} \cap V_{ik}) = V_{ji} \cap V_{jk},$$

and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $V_{ij} \cap V_{ik}$.

Let X denote the scheme obtained by gluing, and let G denote $\coprod_i h_{X_i}$. Notice that the fiber product

$$G \times_{h_X} G$$

is isomorphic to

$$\coprod_{i,j} h_{V_{ij}}.$$

This implies that the surjection of Zariski sheaves

$$G \twoheadrightarrow F$$

factors through an isomorphism $h_X \simeq F$.

To also prove the last statement of 1.4.11, it remains to see that the functor (1.4.11.1) is fully faithful. Let X and Y be two separated schemes, and let h_X and h_Y denote the corresponding functors on Aff_S . Choose an open covering $X = \bigcup_i U_i$ with each U_i affine, and let U_{ij} denote the intersection $U_i \cap U_j$ which is again an affine scheme since X is S -separated. We then have a coequalizer diagram of big Zariski sheaves

$$\coprod_{i,j} h_{U_{ij}} \rightrightarrows \coprod_i h_{U_i} \rightarrow h_X$$

and therefore we have

$$\mathrm{Hom}(h_X, h_Y) = \mathrm{Eq}\left(\prod_i h_Y(U_i) \rightrightarrows \prod_{i,j} h_Y(U_{ij})\right).$$

Since the map

$$\mathrm{Hom}(X, Y) \rightarrow \mathrm{Eq}\left(\prod_i h_Y(U_i) \rightrightarrows \prod_{i,j} h_Y(U_{ij})\right)$$

is an isomorphism, we obtain the full faithfulness. \square

1.4.12. Now that we have the notion of a separated scheme defined purely functorially we can repeat the argument to get an alternate definition of scheme defined purely in terms of functors.

Let

$$F, G : \mathrm{Aff}_S^{\mathrm{op}} \rightarrow \mathrm{Set}$$

be two functors. We say that a morphism of functors $\epsilon : F \rightarrow G$ is *representable by separated schemes* if for any $U \in \mathrm{Aff}_S$ and morphism $h_U \rightarrow G$ the fiber product

$$F \times_G h_U \rightarrow h_U$$

is a separated U -scheme (which is a notion defined purely functorially using 1.4.11). Repeating the above argument one then obtains the following:

PROPOSITION 1.4.13. *A functor $F : \mathrm{Aff}_S^{\mathrm{op}} \rightarrow \mathrm{Set}$ is representable by a S -scheme if and only if the following hold:*

- (i)' *F is a big Zariski sheaf.*
- (ii)' *The diagonal morphism $\Delta : F \rightarrow F \times F$ is representable by separated schemes.*
- (iii)' *There exists a family of objects $\{X_i\}$ in Aff_S and morphisms $\pi_i : h_{X_i} \rightarrow F$ which are open immersions and such that the map of Zariski sheaves $\coprod_i h_{X_i} \rightarrow F$ is surjective.*

Moreover, the functor

$$h_- : (S\text{-schemes}) \rightarrow (\text{functors } F : \mathrm{Aff}_S^{\mathrm{op}} \rightarrow \mathrm{Set} \text{ satisfying (i)'--(iii)'})$$

is an equivalence of categories.

PROOF. Omitted. \square

EXAMPLE 1.4.14. To get used to the above point of view, let us verify the conditions (i)–(iii) for projective space (over \mathbb{Z}).

Let Aff denote the category of affine schemes, and let

$$\mathcal{P}^n : \mathrm{Aff}^{\mathrm{op}} \rightarrow \mathrm{Set}$$

denote the functor which sends an affine scheme $\mathrm{Spec}(R)$ to the set of isomorphism classes of surjections of R -modules

$$R^{n+1} \xrightarrow{\pi} L,$$

where L is a projective R -module of rank 1. Equivalently, $\mathcal{P}^n(\mathrm{Spec}(R))$ is the set of isomorphism classes of surjections of sheaves

$$\mathcal{O}_{\mathrm{Spec}(R)}^{n+1} \longrightarrow \mathcal{L},$$

with \mathcal{L} locally free of rank 1.

Condition (i) for this functor is immediate (this essentially amounts to the statement that sheaves and morphisms between them can be constructed locally).

To verify condition (ii), let $\text{Spec}(R)$ be an affine scheme, and suppose given two elements of $\mathcal{P}^n(\text{Spec}(R))$ that

$$\mathcal{O}_{\text{Spec}(R)}^{n+1} \xrightarrow{\pi_j} \mathcal{L}_j, \quad j = 1, 2.$$

The fiber product W of the diagram

$$\begin{array}{ccc} & & \text{Spec}(R) \\ & & \downarrow \\ \mathcal{P}^n & \xrightarrow{\Delta} & \mathcal{P}^n \times \mathcal{P}^n \end{array}$$

is the functor on the category of affine R -schemes which to any such scheme $f : \text{Spec}(B) \rightarrow \text{Spec}(R)$ associates the unital set if

$$\text{Ker}(f^* \pi_1) = \text{Ker}(f^* \pi_2),$$

and the empty set otherwise. Let $K_1 \subset \mathcal{O}_{\text{Spec}(R)}^{n+1}$ denote the locally free sheaf $\text{Ker}(\pi_1)$, and let $\gamma : K_1 \rightarrow \mathcal{L}_2$ denote the composite morphism

$$K_1 \hookrightarrow \mathcal{O}_{\text{Spec}(R)}^{n+1} \xrightarrow{\pi_2} \mathcal{L}_2.$$

Let K_1^\vee denote the dual of K_1 so we can also think of γ as a global section of $K_1^\vee \otimes \mathcal{L}_2$. The functor W is then the functor which to $f : \text{Spec}(B) \rightarrow \text{Spec}(R)$ associates the unital set if $f^* \gamma = 0$, and the empty set otherwise. Now the condition that a section of a vector bundle is zero is representable by a closed subscheme (locally if we trivialize the vector bundle and write the section as a vector, then the representing closed subscheme is the closed subscheme defined by the entries of the vector). This verifies (ii).

Finally, let us verify (iii). For $i = 0, \dots, n$ let $\sigma_i : \mathcal{U}_i \subset \mathcal{P}^n$ denote the subfunctor whose value on an affine scheme $\text{Spec}(R)$ is the set of quotients

$$(1.4.14.1) \quad \pi : \mathcal{O}_{\text{Spec}(R)}^{n+1} \rightarrow \mathcal{L}$$

such that if $j_i : \mathcal{O}_{\text{Spec}(R)} \hookrightarrow \mathcal{O}_{\text{Spec}(R)}^{n+1}$ is the inclusion of the i -th component, then the composite morphism

$$(1.4.14.2) \quad \mathcal{O}_{\text{Spec}(R)} \xrightarrow{j_i} \mathcal{O}_{\text{Spec}(R)}^{n+1} \xrightarrow{\pi} \mathcal{L}$$

is surjective. Note that if this is the case then this composite map is actually an isomorphism. Giving an element (1.4.14.1) of $\mathcal{U}_i(\text{Spec}(R))$ is therefore equivalent to giving elements

$$x_s \in R, \quad s = 0, \dots, n, \quad s \neq i,$$

the map π being the map

$$\mathcal{O}_{\text{Spec}(R)}^{n+1} \rightarrow \mathcal{O}_{\text{Spec}(R)}$$

whose composition with $j_s : \mathcal{O}_{\text{Spec}(R)} \hookrightarrow \mathcal{O}_{\text{Spec}(R)}^{n+1}$ is equal to multiplication by x_s if $s \neq i$ and the identity if $s = i$. In particular, we find that \mathcal{U}_i is represented by \mathbb{A}^n .

The map σ_i is representable by open imbeddings. Indeed, given any object (1.4.14.1), the fiber product of the resulting diagram

$$\begin{array}{ccc} & \text{Spec}(R) & \\ & \downarrow & \\ \mathcal{U}_i & \longrightarrow & \mathcal{P}^n \end{array}$$

is represented by the complement in $\text{Spec}(R)$ of the zero locus of the section of \mathcal{L} defined by the composition (1.4.14.2). It is also clear that the map

$$\prod_{i=0}^n \mathcal{U}_i \rightarrow \mathcal{P}^n$$

is a surjective map of big Zariski sheaves and therefore (iii) holds. This is the functorial interpretation of the standard open covering of \mathbb{P}^n .

1.5. Hilbert and Quot schemes

The basic reference for this section is [35, part IV].

1.5.1. Let $f : X \rightarrow S$ be a finitely presented separated morphism of schemes, let L be a relatively ample invertible sheaf on X , let $P \in \mathbb{Q}[z]$ be a polynomial, and let F be a quasi-coherent locally finitely presented sheaf on X .

REMARK 1.5.2. If S is noetherian, then f finitely presented is equivalent to f being of finite type, and F quasi-coherent and locally finitely presented is equivalent to F being coherent.

1.5.3. Define

$$\underline{\text{Quot}}^P(F/X/S) : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$$

to be the functor which to any $S' \rightarrow S$ associates the set of isomorphism classes of quotients of quasi-coherent sheaves

$$F_{S'} \longrightarrow G,$$

where $F_{S'}$ denotes the pullback of F to $X_{S'} := X \times_S S'$, and G is a locally finitely presented quasi-coherent sheaf on $X_{S'}$ whose support is proper over S' and such that for every point $s' \in S'$ the Hilbert polynomial of G restricted to the fiber of $X_{S'}$ over s' is equal to P .

THEOREM 1.5.4. *The functor $\underline{\text{Quot}}^P(F/X/S)$ is representable by a scheme quasi-projective over S (projective if X/S is proper).*

PROOF. See [35, part IV, Théorème 3.2]. \square

REMARK 1.5.5. If $F = \mathcal{O}_X$, then $\underline{\text{Quot}}^P(F/X/S)$ is called the *Hilbert scheme* of X/S and will be denoted $\underline{\text{Hilb}}_{X/S}^P$.

Theorem 1.5.4 has many important consequences. Here we just mention a few which will be needed later. The following is a special case of [65, 5.2.2].

THEOREM 1.5.6. *Let $f : X \rightarrow S$ be a quasi-projective morphism between noetherian schemes with S reduced. Then there exists a projective birational morphism $S' \rightarrow S$ which is an isomorphism over a dense open subset of S such that the strict transform of X in $X_{S'} := X \times_S S'$ is flat over S' .*

We will deduce this from the following variant result.

THEOREM 1.5.7. *Let $f : X \rightarrow S$ be a projective morphism with S integral and noetherian, and let $U \subset S$ be the maximal open subset of S over which f is flat. Then U is dense in S , and there exists a blowup $S' \rightarrow S$ with center in $S - U$ such that the strict transform of X in $X_{S'}$ is flat over S' .*

THEOREM 1.5.7 IMPLIES 1.5.6. Replacing S by the disjoint union of its irreducible components with the reduced structure, we may assume that S is integral. Since f is quasi-projective, we can find a commutative diagram

$$\begin{array}{ccc} X & \hookrightarrow & \overline{X} \\ & \searrow f & \downarrow \bar{f} \\ & & S, \end{array}$$

where j is a dense open imbedding and $\bar{f} : \overline{X} \rightarrow S$ is projective. Now observe that if $S' \rightarrow S$ is a blowup with nowhere dense center such that the strict transform of \overline{X} in $\overline{X}_{S'}$ is flat over S' , then the strict transform of X in $X_{S'}$ is also flat over S' . To prove 1.5.6 it therefore suffices to consider the case when f is projective and S is integral. \square

PROOF OF THEOREM 1.5.7. By [26, IV.11.1.1] (or exercise 9.4 in Chapter III of [41]), the set of points in X where f fails to be flat is closed, and since f is proper the image $Z \subset S$ of this set under f is also closed and $Z = S - U$. Since S is integral, f is flat over the generic point of S , so U is nonempty and dense in S .

Fix a relatively ample line bundle on X . For any point $u \in U$ we can consider the Hilbert polynomial P_u of the fiber X_u , and since f is flat over U all these Hilbert polynomials are equal to a fixed polynomial $P \in \mathbb{Q}[z]$, since Hilbert polynomials are constant in flat families.

Consider the Hilbert scheme

$$\underline{\text{Hilb}}_{X/S}^P \rightarrow S.$$

Over U we have a section (corresponding to X_U/U)

$$s : U \rightarrow \underline{\text{Hilb}}_{X/S}^P.$$

Let S' denote the scheme-theoretic closure of $s(U)$. Then $S' \rightarrow S$ is a projective morphism, whence a blowup by [26, III.2.3.5] (or [41, II.7.17]). Moreover, over S' we have a closed imbedding

$$Z \hookrightarrow X_{S'}$$

such that Z/S' is flat with Hilbert polynomial P , and such that the restriction of Z to U is X_U . It follows that Z is the strict transform of X , which proves the theorem. \square

1.6. Exercises

EXERCISE 1.A. Let $R \rightarrow R'$ be a ring homomorphism. Show that the morphism of schemes

$$\text{Spec}(R') \rightarrow \text{Spec}(R)$$

is flat (resp. faithfully flat) in the sense of 1.1.3 if and only if R' is flat (resp. faithfully flat) as an R -module.

EXERCISE 1.B. Let S be a scheme and let \mathcal{O} be the functor on the category of S -schemes which sends X/S to $\Gamma(X, \mathcal{O}_X)$. Show that \mathcal{O} is represented by \mathbb{A}_S^1 .

EXERCISE 1.C. (a) Let $n \geq 1$ be an integer and let

$$GL_n : (\text{Sch})^{\text{op}} \rightarrow \text{Set}$$

be the functor sending a scheme Y to the set $GL_n(\Gamma(Y, \mathcal{O}_Y))$. Prove that GL_n is representable by an affine scheme.

(b) Let X represent the functor GL_n . Prove that the group structure on $GL_n(\Gamma(Y, \mathcal{O}_Y))$ induces morphisms

$$m : X \times X \rightarrow X, \quad i : X \rightarrow X, \quad e : \text{Spec}(\mathbb{Z}) \rightarrow X$$

such that the following diagrams commute:

(i)

$$\begin{array}{ccc} \text{Spec}(\mathbb{Z}) \times X & \xrightarrow{e \times \text{id}} & X \times X \\ & \searrow & \downarrow m \\ & & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X \times \text{Spec}(\mathbb{Z}) & \xrightarrow{\text{id} \times e} & X \times X \\ & \searrow & \downarrow m \\ & & X \end{array}$$

(ii)

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{m \times \text{id}} & X \times X \\ \downarrow \text{id} \times m & & \downarrow m \\ X \times X & \xrightarrow{m} & X \end{array}$$

(iii)

$$\begin{array}{ccc} X & \xrightarrow{i \times \text{id}} & X \times X \\ \downarrow & & \downarrow m \\ \text{Spec}(\mathbb{Z}) & \xrightarrow{e} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\text{id} \times i} & X \times X \\ \downarrow & & \downarrow m \\ \text{Spec}(\mathbb{Z}) & \xrightarrow{e} & X \end{array}$$

EXERCISE 1.D. (a) Let

$$\mathbb{A}^n - \{0\} : (\text{Sch})^{\text{op}} \rightarrow \text{Set}$$

be the functor sending a scheme Y to the set of n -tuples (y_1, \dots, y_n) of sections $y_i \in \Gamma(Y, \mathcal{O}_Y)$ such that for every point $y \in Y$ the images of the y_i in $k(y)$ are not all zero. Show that $\mathbb{A}^n - \{0\}$ is representable.

(b) Let

$$(\mathbb{A}^n - \{0\})/\mathbb{G}_m : (\text{Sch})^{\text{op}} \rightarrow \text{Set}$$

be the functor sending a scheme Y to the quotient of the set $(\mathbb{A}^n - \{0\})(Y)$ by the equivalence relation

$$(y_1, \dots, y_r) \sim (y'_1, \dots, y'_r)$$

if there exists a unit $u \in \Gamma(Y, \mathcal{O}_Y^*)$ such that $y_j = uy'_j$ for all j . Show that $(\mathbb{A}^n - \{0\})/\mathbb{G}_m$ is not representable. For further discussion of this and other related examples see [71, 2.1.3].

EXERCISE 1.E. Let (Top) be the category of topological spaces with morphisms being continuous maps. Let

$$F : (\text{Top})^{\text{op}} \rightarrow \text{Set}$$

be the functor sending a topological space S to the collection $F(S)$ of all its open subsets.

(a) Endow $\{0, 1\}$ with the coarsest topology in which the subset $\{0\} \subset \{0, 1\}$ is closed. Show that the open subsets in this topology are \emptyset , $\{1\}$, and $\{0, 1\}$.

(b) Show that $\{0, 1\}$ with the above topology represents F .

(c) Let $(\text{HausTop}) \subset (\text{Top})$ denote the full subcategory of Hausdorff topological spaces. Show that the restriction of F ,

$$F|_{(\text{HausTop})} : (\text{HausTop})^{\text{op}} \rightarrow \text{Set},$$

is not representable.

For further discussion of this and other related examples see [71, 2.1.3]

EXERCISE 1.F. Let S be a noetherian scheme and $f : X \rightarrow S$ a finite type morphism of schemes. Let F be a coherent sheaf on X , which is locally free over a dense open subset $U \subset X$. Show that there exists a projective birational morphism $\pi : X' \rightarrow X$ and a locally free sheaf E on X' of finite rank such that for some dense open subset $U' \subset X'$ we have $E|_{U'} \simeq \pi^* F|_{U'}$.

EXERCISE 1.G. Let k be a field and let K/k be a finite extension of fields. Show that $\text{Spec}(K) \rightarrow \text{Spec}(k)$ is étale if and only if K/k is separable.

EXERCISE 1.H. Let R be a ring and let $F \in R[X_0, \dots, X_n]$ be a homogeneous polynomial of degree d . Let

$$Z(F) : (R\text{-schemes})^{\text{op}} \rightarrow \text{Set}$$

be the functor sending an R -scheme T to the set of isomorphism classes of quotients $\pi : \mathcal{O}_T^{n+1} \rightarrow L$, where L is an invertible \mathcal{O}_T -module and such that the image of F , viewed as an element of $\text{Sym}_R^d(R^{n+1})$, under the map

$$\text{Sym}_R^d(R^{n+1}) \longrightarrow \text{Sym}^d(\mathcal{O}_T^{n+1}) \xrightarrow{\pi} L^{\otimes d}$$

is zero. Verify the conditions in 1.4.11 for the functor $Z(F)$ thereby showing it is representable, and show that the representing scheme is isomorphic to

$$\text{Proj}(R[X_0, \dots, X_n]/(F)).$$

EXERCISE 1.I. Verify that the construction in paragraph 1.3.3 defines a simply transitive action of $\text{Hom}_{Y_0}(y_0^* \Omega_{X/Y}^1, \mathcal{S})$ on the set of dotted arrows as in diagram (1.3.3.1), assuming that at least one such arrow exists.

EXERCISE 1.J. Prove Proposition 1.3.4.

EXERCISE 1.K. Verify that a morphism $f : F \rightarrow G$ of big Zariski sheaves on a scheme S is surjective in the sense of 1.3.4 if and only if it is an epimorphism in the category of big Zariski sheaves on Aff_S . Furthermore, show that if $f : F \rightarrow G$ is a surjective morphism of big Zariski sheaves, then G is the coequalizer in the category of big Zariski sheaves of the diagram

$$F \times_G F \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{array} F$$

EXERCISE 1.L. Let $f : X \rightarrow S$ be a projective morphism of schemes, and let L and M be two invertible sheaves on X . Let

$$I : (S\text{-schemes})^{\text{op}} \rightarrow \text{Set}$$

be the functor which to any S -scheme S' associates the set of isomorphisms $L_{S'} \rightarrow M_{S'}$ of invertible sheaves on $X_{S'} := X \times_S S'$. Show that I is representable by a quasi-projective S -scheme. Hint: Let $U \rightarrow X$ be the complement of the zero section of the total space $\mathbb{V}(L^\vee \otimes M)$ of the invertible sheaf $L^\vee \otimes M$ and note that giving an isomorphism of invertible sheaves $L \rightarrow M$ is equivalent to giving a closed subscheme $\Gamma \subset U$ such that the projection to X is an isomorphism. Now consider the Hilbert scheme of U .

EXERCISE 1.M. Let B be a ring. Show that if M is a finitely presented projective B -module, then the associated quasi-coherent sheaf \widetilde{M} on $\text{Spec}(B)$ is a locally free sheaf of finite rank.