

## Preface

The goal of this book is to present several central topics in geometric group theory, primarily related to the large scale geometry of infinite groups, and of the spaces on which such groups act, and to illustrate them with fundamental theorems such as Gromov's Theorem on groups of polynomial growth, Tits' Alternative Theorem, Mostow's Rigidity Theorem, Stallings' Theorem on ends of groups, theorems of Tukia and Schwartz on quasiisometric rigidity for lattices in real-hyperbolic spaces, etc. We give essentially self-contained proofs of all the above-mentioned results, and we use the opportunity to describe several powerful tools/toolkits of geometric group theory, such as coarse topology, ultralimits and quasiconformal mappings. We also discuss three classes of groups central in geometric group theory: amenable groups, hyperbolic groups, and groups with Property (T).

The key idea in geometric group theory is to study groups by endowing them with a metric and treating them as *geometric objects*. This can be done for groups that are *finitely generated*, i.e. that can be reconstructed from a finite subset, *via* multiplication and inversion. Many groups naturally appearing in topology, geometry and algebra (e.g. fundamental groups of manifolds, groups of matrices with integer coefficients) are finitely generated. Given a finite generating set  $S$  of a group  $G$ , one can define a metric on  $G$  by constructing a connected graph, the *Cayley graph* of  $G$ , with  $G$  serving as a set of vertices, and oriented edges joining elements in  $G$  that differ by a right multiplication with a generator  $s$  from  $S$ , and labeled by  $s$ . A Cayley graph  $\mathcal{G}$ , as any other connected graph, admits a natural metric invariant under automorphisms of  $\mathcal{G}$ : Edges are assumed to be of length one, and the distance between two points is the length of the shortest path in the graph joining these points (see Section 2.3). The restriction of this metric to the vertex set  $G$  is called the *word metric*  $\text{dist}_S$  on the group  $G$ . The first obstacle to “geometrizing” groups in this fashion is the fact that a Cayley graph depends not only on the group but also on a particular choice of finite generating set. Cayley graphs associated with different generating sets are not isometric but merely *quasiisometric*.

Another typical situation in which a group  $G$  is naturally endowed with a (pseudo-)metric is when  $G$  acts on a metric space  $X$ : In this case the group  $G$  maps to  $X$  *via* the *orbit map*  $g \mapsto gx$ . The pull-back of the metric to  $G$  is then a pseudo-metric on  $G$ . If  $G$  acts on  $X$  isometrically, then the resulting pseudo-metric on  $G$  is  $G$ -invariant. If, furthermore, the space  $X$  is proper and geodesic and the action of  $G$  is *geometric* (i.e. properly discontinuous and cocompact), then  $G$  is finitely generated and the resulting (pseudo-)metric is quasiisometric to word metrics on  $G$  (Theorem 8.37). For example, if a group  $G$  is the fundamental group of a closed Riemannian manifold  $M$ , the action of  $G$  on the universal cover  $\widetilde{M}$  of

$M$  satisfies all these properties. The second class of examples of isometric actions (whose origin lies in functional analysis and representation theory) comes from isometric actions of a group  $G$  on a Hilbert space. The square of the corresponding pull-back (pseudo-)metric on  $G$  is known in the literature as a *conditionally negative semidefinite kernel*. In this case, if the group is not virtually abelian, the action cannot be geometric. (Here and in what follows, when we say that a group has a certain property *virtually* we mean that it has a finite index subgroup with that property.) On the other hand, the mere existence of a proper action of a group  $G$  on a Hilbert space  $H$  (i.e. an action so that, as  $g \in G$  escapes every compact, the norm  $\|gv\|$  diverges to infinity, where  $v$  is any vector in  $H$ ), equivalently the mere existence of a conditionally negative semidefinite kernel on  $G$  that is proper as a topological map has many interesting implications, detailed in Chapter 19.

In the setting of the geometric view of groups, the following questions become fundamental:

- QUESTIONS. (A) *If  $G$  and  $G'$  are quasiisometric groups, to what extent do  $G$  and  $G'$  share the same algebraic properties?*
- (B) *If a group  $G$  is quasiisometric to a metric space  $X$ , what geometric properties (or structures) on  $X$  translate to interesting algebraic properties of  $G$ ?*

Addressing these questions is the primary focus of this book. Several striking results (like Gromov’s Polynomial Growth Theorem) state that certain algebraic properties of a group can be reconstructed from its loose geometric features and in particular must be shared by quasiisometric groups.

Closely connected to these considerations are two foundational problems which appeared in different contexts, but both render the same sense of existence of a “demarcation line” dividing the class of infinite groups into “abelian-like” groups and “free-like” groups. The invariants used to draw the line are quite different (existence of a finitely additive invariant measure in one case and behavior of the growth function in the other); nevertheless, the two problems/questions and the classification results that grew out of these questions have much in common.

The first of these questions was inspired by work investigating the existence of various types of group-invariant measures that originally appeared in the context of Euclidean spaces. Namely, the *Banach–Tarski paradox* (see Chapter 17), while denying the existence of such measures on the Euclidean space, inspired John von Neumann to formulate two important concepts: that of *amenable groups* and that of *paradoxical decompositions and groups* [vN28]. In an attempt to connect amenability to the algebraic properties of a group, von Neumann made the observation, in the same paper, that the existence of a free subgroup excludes amenability. Mahlon Day (in [Day50] and [Day57]) extended von Neumann’s work, introduced the terminology *amenable groups*, defined the class of *elementary amenable groups* and proved several foundational results about amenable and elementary amenable groups. In [Day57, p. 520] he also noted:<sup>1</sup>

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<sup>1</sup>Contrary to the common belief, Day neither formulated a conjecture about this issue nor attributed the problem to von Neumann.

- It is not known whether the class of elementary amenable groups equals the class of amenable groups and whether the class of amenable groups coincides with the class of groups containing no free non-abelian subgroups.

This observation later became commonly known as the von Neumann–Day problem (or conjecture):

QUESTION (The von Neumann–Day problem). *Is non-amenability of a group equivalent to the existence of a free non-abelian subgroup?*

The second problem appeared in the context of Riemannian geometry, in connection to attempts to relate, for a compact Riemannian manifold  $M$ , the geometric features of its universal cover  $\widetilde{M}$  to the behavior of its fundamental group  $G = \pi_1(M)$ . Two of the most basic objects in Riemannian geometry are the *volume* and the *volume growth rate*. The notion of volume growth extends naturally to discrete metric spaces, such as finitely generated groups. The *growth function* of a finitely generated group  $G$  (with a fixed finite generating set  $S$ ) is the cardinality  $\mathfrak{G}(n)$  of the ball of radius  $n$  in the metric space  $(G, \text{dist}_S)$ . While the function  $\mathfrak{G}(n)$  depends on the choice of the finite generating set  $S$ , the *growth rate* of  $\mathfrak{G}(n)$  is independent of  $S$ . In particular, one can speak of groups of linear, polynomial, exponential growth, etc. More importantly, the growth rate is preserved by quasi-isometries, which allows one to establish a close connection between the Riemannian growth of a manifold  $\widetilde{M}$  as above and the growth of  $G = \pi_1(M)$ .

One can easily see that every abelian group has polynomial growth. It is a more difficult theorem (proven independently by Hyman Bass [Bas72] and Yves Guivarc’h [Gui70, Gui73]) that all nilpotent groups also have polynomial growth. We provide a proof of this result in Section 14.2. In this context, John Milnor [Mil68c] and Joe Wolf [Wol68] asked the following question:

QUESTION. *Is it true that the growth of each finitely generated group is either polynomial (i.e.  $\mathfrak{G}(n) \leq Cn^d$  for some fixed  $C$  and  $d$ ) or exponential (i.e.  $\mathfrak{G}(n) \geq Ca^n$  for some fixed  $a > 1$  and  $C > 0$ )?*

Note that Milnor stated the problem in the form of a question, not a conjecture; however, he conjectured in [Mil68c] that each group of polynomial growth is virtually nilpotent.

The answer to the question is positive for *solvable groups*: This is the *Milnor–Wolf Theorem*, which moreover states that *solvable groups of polynomial growth are virtually nilpotent*; see Theorem 14.37 in this book (the theorem is a combination of results due to Milnor and Wolf). This theorem still holds for the larger class of *elementary amenable groups* (see Theorem 18.58); moreover, such groups with non-polynomial growth must contain a free non-abelian subsemigroup.

The proof of the Milnor–Wolf Theorem essentially consists of a careful examination of increasing/decreasing sequences of subgroups in nilpotent and solvable groups. Along the way, one discovers other features that nilpotent groups share with abelian groups, but not with solvable groups. For instance, *in a nilpotent group all finite subgroups are contained in a maximal finite subgroup, while solvable groups may contain infinite strictly increasing sequences of finite subgroups*. Furthermore, *all subgroups of a nilpotent group are finitely generated, but this is no longer true for solvable groups*. One step further into the study of a finitely generated subgroup  $H$  in a group  $G$  is to compare a word metric  $\text{dist}_H$  on the subgroup

$H$  to the restriction to  $H$  of a word metric  $\text{dist}_G$  on the ambient group  $G$ . With an appropriate choice of generating sets, the inequality  $\text{dist}_G \leq \text{dist}_H$  is immediate: All the paths in  $H$  joining  $h, h' \in H$  are also paths in  $G$ , but there might be some other, shorter, paths in  $G$  joining  $h, h'$ . The problem is to find an upper bound on  $\text{dist}_H$  in terms of  $\text{dist}_G$ . If  $G$  is abelian, the upper bound is linear as a function of  $\text{dist}_G$ . If  $\text{dist}_H$  is bounded by a polynomial in  $\text{dist}_G$ , then the subgroup  $H$  is said to be *polynomially distorted* in  $G$ , while if  $\text{dist}_H$  is approximately  $\exp(\lambda \text{dist}_G)$  for some  $\lambda > 0$ , the subgroup  $H$  is said to be *exponentially distorted*. It turns out that *all subgroups in a nilpotent group are polynomially distorted, while some solvable groups contain finitely generated subgroups with exponential distortion*.

Both the von Neumann–Day and the Milnor–Wolf questions were answered in the affirmative for linear groups by Jacques Tits:

**THEOREM (Tits’ Alternative).** *Let  $F$  be a field of zero characteristic and let  $\Gamma$  be a subgroup of  $GL(n, F)$ . Then either  $\Gamma$  is virtually solvable or  $\Gamma$  contains a free non-abelian subgroup.*

We prove Tits’ Alternative in Chapter 15. Note that this alternative also holds for fields of positive characteristic, provided that  $\Gamma$  is finitely generated.

There are other classes of groups in which both the von Neumann–Day and the Milnor–Wolf questions have positive answers; they include subgroups of Gromov-hyperbolic groups ([Gro87, §8.2.F], [GdlH90, Chapter 8]), fundamental groups of closed Riemannian manifolds of non-positive curvature [Bal95], subgroups of mapping class groups of surfaces [Iva92] and of groups of outer automorphisms of free groups [BFH00, BFH05].

The von Neumann–Day question in general has a negative answer: The first counterexamples were given by A. Ol’shanskiĭ in [Ol’80]. In [Ady82] it was shown that the free Burnside groups  $B(n, m)$  with  $n \geq 2$  and  $m \geq 665$ ,  $m$  odd, are also counterexamples. The first finitely presented counterexamples were constructed by A. Ol’shanskiĭ and M. Sapir in [OS02]. Y. Lodha and J. T. Moore later provided in [LM16] another finitely presented counterexample, a subgroup of the group of piecewise-projective homeomorphisms of the real projective line, which is torsion-free (unlike the previous counterexamples, based precisely on the existence of a large torsion) and has an explicit presentation with three generators and nine relators. These papers have led to the development of certain techniques of constructing “infinite finitely generated monsters”. While the negation of amenability (i.e. the paradoxical behavior) is, thus, still not completely understood algebraically, several stronger properties implying non-amenability were introduced, among which are various fixed-point properties, most importantly *Kazhdan’s Property (T)* (Chapter 19). Remarkably, amenability (hence paradoxical behavior) is a quasiisometry invariant, while Property (T) is not.

The Milnor–Wolf question, in full generality, likewise has a negative answer: The first groups of *intermediate growth*, i.e. growth which is superpolynomial but subexponential, were constructed by Rostislav Grigorchuk. Moreover, he proved the following:

**THEOREM (Grigorchuk’s Subexponential Growth Theorem).** *Let  $f(n)$  be an arbitrary subexponential function larger than  $2^{\sqrt{n}}$ . Then there exists a finitely*

generated group  $\Gamma$  with subexponential growth function  $\mathfrak{G}(n)$  such that

$$f(n) \leq \mathfrak{G}(n)$$

for infinitely many  $n \in \mathbb{N}$ .

Later on, Anna Erschler [Ers04] adapted Grigorchuk's arguments to improve the above result with the inequality  $f(n) \leq \mathfrak{G}(n)$  for all but finitely many  $n$ . In the above examples, the exact growth function was unknown. However, Laurent Bartholdi and Anna Erschler [BE12] constructed examples of groups of intermediate growth, where they actually compute  $\mathfrak{G}(n)$ , up to an appropriate equivalence relation. Note, however, that the Milnor–Wolf Problem is still open for finitely presented groups.

On the other hand, Mikhael Gromov proved an even more striking result:

**THEOREM** (Gromov's Polynomial Growth Theorem, [Gro81a]). *Every finitely generated group of polynomial growth is virtually nilpotent.*

This is a typical example of an algebraic property that may be recognized *via*, seemingly, weak geometric information. A corollary of Gromov's Theorem is *quasiisometric rigidity* for virtually nilpotent groups:

**COROLLARY.** *Suppose that  $G$  is a group quasiisometric to a nilpotent group. Then  $G$  itself is virtually nilpotent.*

Gromov's Theorem and its corollary will be proven in Chapter 16. Since the first version of this book was written, Bruce Kleiner [Kle10] and, later, Narutaka Ozawa [Oza15] gave completely different (and much shorter) proofs of Gromov's Polynomial Growth Theorem, using harmonic functions on graphs (Kleiner) and functional-analytic tools (Ozawa). Both proofs still require the Tits Alternative. Kleiner's techniques provided the starting point for Y. Shalom and T. Tao, who proved the following effective version of Gromov's Theorem [ST10]:

**THEOREM** (Shalom–Tao Effective Polynomial Growth Theorem). *There exists a constant  $C$  such that for any finitely generated group  $G$  and  $d > 0$ , if for some  $R \geq \exp(\exp(Cd^C))$ , the ball of radius  $R$  in  $G$  has at most  $R^d$  elements, then  $G$  has a finite index nilpotent subgroup of class less than  $C^d$ .*

It is also possible to prove Gromov's Theorem without using the Tits Alternative. Indeed, the proofs of either Gromov, Kleiner or Ozawa allow us to restrict to the case of linear groups, and from there two different approaches are possible.

The first one is to use the well-known remark that groups with subexponential growth are amenable (see Proposition 18.6), and the direct proof of Shalom [Sha98] of the fact that linear amenable groups are virtually solvable. The main ingredient in Shalom's proof is a version of the Furstenberg lemma stating that, for any local field  $\mathbb{F}$ , the stabilizer in  $PGL(n, \mathbb{F})$  of a probability measure on the projective space  $\mathbb{P}^{n-1}$  whose support is not included in a finite number of hyperplanes is a compact subgroup of  $PGL(n, \mathbb{F})$ . See also [Bre14].

The second approach is *via* simple additive combinatorics. E. Breuillard and B. Green have shown in [BG12] that if a finite subset  $A$  of the unitary group  $U(n)$  satisfies  $|A^3| < K|A|$ , then  $A$  is contained in at most  $K^C$  cosets of an abelian subgroup of  $U(n)$ , where  $K > 1$  is an arbitrary constant and  $C = C(K)$  is independent of  $K$  and  $A$ . From this, it can be easily deduced that finitely generated subgroups

of  $U(n)$  that have polynomial growth are virtually abelian; see [BG12, Proposition 5.1]. As Kleiner’s proof allows to restrict to the case of subgroups of the unitary group  $U(n)$ , this concludes the proof of Gromov’s Theorem. The advantage of this approach is that it is elementary: It relies on simple properties of compact Lie groups and uses neither proximality nor amenability. The result of Breuillard–Green has been further generalized in their joint work with T. Tao [BGT11] to subsets  $A$  in Lie groups that are not compact. This improved result can be combined with either of the arguments of Gromov, Kleiner or Ozawa, reducing the problem to linear groups, to provide yet another proof of the Polynomial Growth Theorem avoiding the Tits Alternative, less elementary though. Both additive combinatorics proofs have the further advantage that, unlike when using the Tits Alternative or the proof of Shalom, one does not need to change field: The entire argument can be carried out in the setting of the real numbers.

We decided to retain, however, Gromov’s original proof since it contains a wealth of ideas that generated in their turn new areas of research. Remarkably, the same piece of logic (a weak version of the Axiom of Choice) that makes the Banach–Tarski paradox possible also allows one to construct *ultralimits*, a powerful tool in the proof of Gromov’s Theorem and that of many rigidity theorems (e.g., quasiisometric rigidity theorems of Kapovich, Kleiner and Leeb) as well as in the investigation of fixed-point properties.

Regarding Questions (A) and (B), the best one can hope for is that the geometry of a group (up to quasiisometric equivalence) allows one to recover not just some of its algebraic features, but the group itself, *up to virtual isomorphism*. Two groups  $G_1$  and  $G_2$  are said to be *virtually isomorphic* if there exist subgroups

$$F_i \triangleleft H_i \leq G_i, \quad i = 1, 2,$$

so that  $H_i$  has finite index in  $G_i$ ,  $F_i$  is a finite normal subgroup in  $H_i$ ,  $i = 1, 2$ , and  $H_1/F_1$  is isomorphic to  $H_2/F_2$ . Virtual isomorphism implies quasiisometry but, in general, the converse is false; see Example 8.48. In the situation when the converse implication also holds, one says that the group  $G_1$  is *quasiisometrically rigid*.

An example of quasiisometric rigidity is given by the following theorem proven by Richard Schwartz [Sch96b]:

**THEOREM (Schwartz QI Rigidity Theorem).** *Suppose that  $\Gamma$  is a non-uniform lattice of isometries of the hyperbolic space  $\mathbb{H}^n$ ,  $n \geq 3$ . Then each group quasiisometric to  $\Gamma$  must be virtually isomorphic to  $\Gamma$ .*

We will present a proof of this theorem in Chapter 24. In the same chapter we will use similar “zooming” arguments to prove the following special case of Mostow’s Rigidity Theorem:

**THEOREM (The Mostow Rigidity Theorem).** *Let  $\Gamma_1$  and  $\Gamma_2$  be lattices of isometries of  $\mathbb{H}^n$ ,  $n \geq 3$ , and let  $\varphi : \Gamma_1 \rightarrow \Gamma_2$  be a group isomorphism. Then  $\varphi$  is given by a conjugation via an isometry of  $\mathbb{H}^n$ .*

Note that Schwartz’ Theorem no longer holds for  $n = 2$ , where non-uniform lattices are virtually free. However, in this case, quasiisometric rigidity still holds as a corollary of Stallings’ Theorem on ends of groups:

**THEOREM.** *Let  $\Gamma$  be a group quasiisometric to a free group of finite rank. Then  $\Gamma$  is itself virtually free.*

This theorem will be proven in Chapter 20. We also prove:

**THEOREM** (Stallings “Ends of Groups” Theorem). *If  $G$  is a finitely generated group with infinitely many ends, then  $G$  splits as a graph of groups with finite edge-groups.*

In this book we give two proofs of the above theorem, which, while quite different, are both inspired by the original argument of Stallings. In Chapter 20 we prove Stallings’ Theorem for *almost finitely presented* groups. This proof follows the ideas of Dunwoody, Jaco and Rubinstein: We will be using *minimal Dunwoody tracks*, where minimality is defined with respect to a certain hyperbolic metric on the presentation complex (unlike the combinatorial minimality used by Dunwoody). In Chapter 21, we will give another proof, which works for all finitely generated groups and follows a proof sketched by Gromov in [Gro87], using least energy harmonic functions. We decided to present both proofs, since they use different machinery (the first is more geometric and the second more analytical) and different (although related) geometric ideas.

In Chapter 20 we also prove the following:

**THEOREM** (Dunwoody’s Accessibility Theorem). *Let  $G$  be an almost finitely presented group. Then  $G$  is accessible, i.e. the decomposition process of  $G$  as a graph of groups with finite edge-groups eventually terminates.*

In Chapter 23 we prove Tukia’s Theorem, which establishes quasiisometric rigidity of the class of fundamental groups of compact hyperbolic  $n$ -manifolds and, thus, complements Schwartz’ Theorem above:

**THEOREM** (Tukia’s QI Rigidity Theorem). *If a group  $\Gamma$  is quasiisometric to the hyperbolic  $n$ -space, then  $\Gamma$  is virtually isomorphic to the fundamental group of a compact hyperbolic  $n$ -manifold.*

Note that the proofs of the theorems of Mostow, Schwartz and Tukia all rely upon the same analytical tool: quasiconformal mappings of Euclidean spaces. In contrast, the analytical proofs of Stallings’ Theorem presented in the book are mostly motivated by another branch of geometric analysis, namely, the theory of minimal submanifolds and harmonic functions.

In regard to Question (B), we investigate two closely related classes of groups: hyperbolic and relatively hyperbolic groups. These classes generalize fundamental groups of compact negatively curved Riemannian manifolds and, respectively, complete negatively curved Riemannian manifolds of finite volume. To this end, in Chapters 4 and 11 we cover the basics of hyperbolic geometry and of the theory of hyperbolic and relatively hyperbolic groups.

**Other sources.** Our choice of topics in geometric group theory is far from exhaustive. We refer the reader to [Aea91], [Bal95], [Bow91], [VSCC92], [Bow06a], [BH99], [CDP90], [Dav08], [Geo08], [GdlH90], [dlH00], [NY11], [Pap03], [Roe03], [Sap14], [Väi05] for the discussion of other parts of the theory.

Work on this book started in 2002 and the material which we cover mostly concerns developments in geometric group theory from the 1960s through the 1990s. In the meantime, while we were working on the book, some major exciting developments in the field have occurred which we did not have a chance to discuss. To name

a few, these developments are subgroup separability and its connections with 3-dimensional topology [Ago13, KM12, HW12, Bes14], applications of geometric group theory to higher-dimensional and coarse topology [Yu00, MY02, BLW10, BL12], the theory of Kleinian groups [Min10, BCM12, Mj14b, Mj14a], quasiconformal analysis on boundaries of hyperbolic groups and the Cannon Conjecture [BK02a, BK05, Bon11, BK13, Mar13, Hai15], the theory of approximate groups [BG08a, Tao08, BGT12, Hru12], the first-order logic of free groups (see [Sel01, Sel03, Sel05a, Sel04, Sel05b, Sel06a, Sel06b, Sel09, Sel13] and [KM98a, KM98b, KM98c, KM05]), the theory of systolic groups [JŚ03, JŚ06, HŚ08, Osa13], probabilistic aspects of geometric group theory [Gro03, Ghy04, Oll04, Oll05, KSS06, Oll07, KS08, OW11, AŁŚ15, DM16].

**Requirements.** The book is intended as a reference for graduate students and more experienced researchers; it can be used as a basis for a graduate course and as a first reading for a researcher wishing to learn more about geometric group theory. This book is partly based on lectures which we gave at Oxford University (C.D.) and University of Utah and University of California, Davis (M.K.). We expect the reader to be familiar with the basics of group theory, algebraic topology (fundamental groups, covering spaces, (co)homology, Poincaré duality) and elements of differential topology and Riemannian geometry. Some of the background material is covered in Chapters 1, 3 and 5. We tried to make the book as self-contained as possible, but some theorems are stated without a proof and they are indicated as **Theorem**.

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