

Preface

Modular forms are central objects in contemporary mathematics. They are meromorphic functions $f: \mathbb{H} \rightarrow \mathbb{C}$ which satisfy

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in \mathbb{H}$, where Γ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and the weight k is generally in $\frac{1}{2}\mathbb{Z}$. There are various types of modular forms which arise naturally in mathematics. *Modular functions* have weight $k = 0$. *Cusp forms* are those holomorphic modular forms which vanish at the cusps of Γ . *Weakly holomorphic forms* are permitted to have poles provided that they are supported at cusps.

There are many facets of these functions which are of importance in mathematics. The study of their Fourier expansions has driven research in the “Langlands Program” via the development of the theory of Galois representations and progress on the Ramanujan-Petersson Conjecture. The values of these functions appear in explicit class field theory. Their L -functions are devices which bridge analysis and arithmetic geometry.

The “web of modularity” is breathtaking. Indeed, modular forms play central roles in algebraic number theory, algebraic topology, arithmetic geometry, combinatorics, number theory, representation theory, and mathematical physics. In the last few decades, modular forms have been featured in fantastic achievements such as progress on the Birch and Swinnerton-Dyer Conjecture, mirror symmetry, Monstrous Moonshine, and the proof of Fermat’s Last Theorem. These works are dramatic examples which illustrate the evolution of mathematics. It would have been nearly impossible to prophesize them fifty years ago.

This book is about a generalization of the theory of modular forms and the corresponding extension of their web of applications. This is the theory of harmonic Maass forms and mock modular forms. Instead of traveling back in time to the 1960s, the first glimpses of harmonic Maass forms and mock modular forms can be found in much older work, namely the enigmatic “deathbed” letter that Ramanujan wrote to G. H. Hardy in 1920 (cf. pages 220-224 of [54]):

“I am extremely sorry for not writing you a single letter up to now...I discovered very interesting functions recently which I call “Mock” ϑ -functions. Unlike the “False” ϑ -functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.”

The letter contained 17 examples including:

$$\begin{aligned}
 f(q) &:= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}, \\
 \omega(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2 \cdots (1-q^{2n+1})^2}, \\
 \lambda(q) &:= \sum_{n=1}^{\infty} \frac{(-1)^n(1-q)(1-q^3) \cdots (1-q^{2n-1})q^n}{(1+q)(1+q^2) \cdots (1+q^{n-1})}.
 \end{aligned}$$

For eight decades, very little was understood about Ramanujan’s mock theta functions. Despite dozens of papers on them, a comprehensive theory which explained them and their role in mathematics remained elusive. Finally, Zwegers [528, 529] recognized that Ramanujan had discovered glimpses of special families of nonholomorphic modular forms. More precisely, Ramanujan’s mock theta functions turned out to be holomorphic parts of these modular forms. For this reason, mathematicians now refer to the holomorphic parts of such modular forms as *mock modular forms*.

Zwegers’ work fit Ramanujan’s mock theta functions beautifully into a theory which involves basic hypergeometric series, indefinite theta functions, and an extension of the theory of Jacobi forms as developed by Eichler and Zagier in their seminal monograph [191].

At almost the same time, Bruinier and Funke [121] wrote an important paper on the theory of geometric theta lifts. In their work they defined the notion of a harmonic Maass form. The nonholomorphic modular forms constructed by Zwegers turned out to be weight 1/2 harmonic Maass forms. This coincidental development ignited research on harmonic Maass forms and mock modular forms. This book represents a survey of this research. This work includes the development of general theory about harmonic Maass forms and mock modular forms, as well as the applications of this theory within the context of the web of modularity.

There have been a number of expository survey articles on mock modular forms by two of the authors, Duke, and Zagier [166, 195, 198, 407, 408, 520]. Furthermore, the books by Bruinier [119] and M. R. Murty and V. K. Murty [392] include nice treatments of some aspects of the theory of harmonic Maass forms and mock modular forms. This book is intended to serve as a uniform and somewhat comprehensive introduction to the subject for graduate students and research mathematicians. We assume that readers are familiar with the classical theory of modular forms which is contained in books such as [162, 282, 316, 388, 405, 451, 455]. There have also been a number of conferences, schools, and workshops devoted to the subject. The reader is encouraged to view the exercises [197] assembled for the 2013 Arizona Winter School, and notes which accompanied the 2016 “School on mock modular forms and related topics” at Kyushu University [438].

We conclude with a brief description of the contents of this book. For the convenience of the reader, we begin in Part 1 by recalling much of the standard theory of elliptic functions, theta functions, Jacobi forms, and classical Maass forms. The idea is to provide a comprehensive and self-contained treatment of these subjects in order to provide a suitable foundation for learning the theory of harmonic Maass forms. Part 2 contains the framework of the theory of harmonic Maass forms,

including a treatment of Zwegers' celebrated Ph.D. thesis which has not been published elsewhere. Part 3 includes a sampling of some of the most interesting and exciting applications of the theory of harmonic Maass forms. These applications include a discussion of Ramanujan's original mock theta functions, the theory of partitions, the theory of singular moduli, Borcherds products, the arithmetic of elliptic curves, the representation theory of infinite dimensional affine Kac-Moody Lie algebras, and generalized Moonshine.

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