CHAPTER 3

# Classical Maass Forms

In 1949, Maass introduced nonholomorphic modular functions, his so-called *Maass forms* (see Maass' works [**366, 367, 368, 369, 370**]). This book is primarily about mock modular forms and harmonic Maass forms, functions which are another type of nonholomorphic modular form. For completeness, here we recall some of the main features of the theory of classical Maass forms. We refer the reader to [**134, 227, 283, 284**], fundamental texts on the subject of Maass forms.

## 3.1. Definitions

The theory of Maass forms is assembled from special nonholomorphic functions on $\mathbb{H}$ which are eigenfunctions of the hyperbolic Laplacian (or Laplace) operator.

Definition 3.1. Let $\tau = u + iv \in \mathbb{H}$, where $u, v \in \mathbb{R}$. The **hyperbolic Laplacian operator** on $\mathbb{H}$ is defined by

$$\Delta := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right).$$

Of particular interest are eigenfunctions of this operator, that is, those functions $f$ for which

$$\Delta(f) = \lambda f$$

for some $\lambda \in \mathbb{C}$. Typically, eigenvalues are normalized to be of the form $\lambda = r(1-r)$, in terms of a parameter $r$. If we additionally require that such eigenfunctions $f$ are invariant under the action of some *Fuchsian group* $\Gamma$ *of the first kind* (discrete, with finite covolume in $\mathrm{SL}_2(\mathbb{Z})$) on $\mathbb{H}$, then we are led to the notion of a *Maass form*. In order to refine this notion even further, we consider Fourier expansions of such functions. For example, for any $f$ which satisfies $f(\tau+1) = f(\tau)$, we have a Fourier expansion at infinity of the form

$$f(\tau) = \sum_{n\in\mathbb{Z}} c_f(v;n)e(nu), \tag{3.1}$$

where

$$c_f(v;n) := \int_0^1 f(u+iv)e(-nu)du.$$

Moreover, such an $f$ has a Fourier expansion at any cusp $\mathfrak{a}$ of $\Gamma$, whose coefficients we denote by $c_{f,\mathfrak{a}}(v;n)$.

In what follows, we let $d\mu(\tau) := v^{-2}dudv$ denote the invariant measure in the Poincaré metric and we let $\mathcal{F}(\Gamma\backslash\mathbb{H})$ be a fundamental domain for the action of $\Gamma$ on $\mathbb{H}$.

Definition 3.2. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a Fuchsian group of the first kind. A **Maass cusp form** $f$ on $\Gamma$ with eigenvalue $\lambda = r(1-r) \in \mathbb{C}$ is any $f \colon \mathbb{H} \to \mathbb{C}$ satisfying

i) $f(\gamma\tau) = f(\tau)$ for all $\gamma \in \Gamma$,

ii) $c_{f,\mathfrak{a}}(0;0) = 0$ for each cusp $\mathfrak{a}$ of $\Gamma$,

iii) $\Delta(f) = r(1-r)f$,

iv) $\displaystyle\int_{\mathcal{F}(\Gamma\backslash\mathbb{H})} |f(\tau)|^2 d\mu(\tau) < \infty$.

We set

$$\mathcal{A}_r^0(\Gamma\backslash\mathbb{H}) := \{\text{Maass cusp forms on } \Gamma \text{ with eigenvalue } \lambda = r(1-r)\}.$$

Remark. More generally, we call functions satisfying i) in Definition 3.2 *automorphic functions*; functions satisfying i) and iii) comprise the space of *automorphic forms*, denoted by $\mathcal{A}_r(\Gamma\backslash\mathbb{H})$.

## 3.2. Fourier expansions

Here we consider the Fourier expansions of forms in $\mathcal{A}_r(\Gamma\backslash\mathbb{H})$, which, as discussed above, includes the space of Maass cusp forms. The following key lemma gives the general shape of these expansions.

Lemma 3.3. *Suppose $f \in \mathcal{A}_r(\Gamma\backslash\mathbb{H})$ satisfies the growth condition $f(\tau) = o(e^{2\pi v})$ as $v \to \infty$. Then there exist $\kappa_1, \kappa_2, c_f(n) \in \mathbb{C}$ such that*

$$f(\tau) = \kappa_1 v^r + \kappa_2 v^{1-r}\delta_r(v) + v^{\frac{1}{2}} \sum_{n \in \mathbb{Z}\setminus\{0\}} c_f(n) K_{r-\frac{1}{2}}(2\pi|n|v)e(nu),$$

*where $K_\nu$ is the $K$-Bessel function of order $\nu$, and $\delta_r(v)$ is equal to $\log(v)$ or 1, depending on whether $r = 1/2$ or $r \neq 1/2$, respectively.*

Sketch of proof. Applying $\Delta$ to (3.1) shows that the Fourier coefficients of $f$ must satisfy the differential equation

$$\frac{\partial^2}{\partial v^2}\, c_f(v;n) + \left(\frac{r(1-r)}{v^2} - 4\pi^2 n^2\right) c_f(v;n) = 0. \tag{3.2}$$

If $n = 0$, then (3.2) has the two linearly independent solutions $v^r$ and $v^{1-r}\delta_r(v)$. If $n \geq 1$, then (3.2) has the two linearly independent solutions

$$v^{\frac{1}{2}} K_{r-\frac{1}{2}}(2\pi|n|v), \quad v^{\frac{1}{2}} I_{r-\frac{1}{2}}(2\pi|n|v),$$

in terms of the $I$- and $K$-Bessel functions. The $I$-Bessel function is of exponential growth, while the $K$-Bessel function is of suitable decay as $v \to \infty$, leading to the claimed expansion. □

The Ramanujan-Petersson Conjecture for Maass forms famously predicts how the Fourier coefficients $c_f(n)$ of Maass cusp forms grow as $n \to \infty$.

Conjecture 3.4 (Ramanujan and Petersson). *For $f \in \mathcal{A}_r^0(\Gamma\backslash\mathbb{H})$ with Fourier expansion as in Lemma 3.3, we have that*

$$c_f(n) = O(\sigma_0(n)),$$

*where $\sigma_0(n) = \sum_{d|n} 1$ is the divisor function.*

## 3.3. General discussion

For an arbitrary Fuchsian group of the first kind $\Gamma$, it is unclear that Maass cusp forms on $\Gamma$ even exist, rendering their study particularly interesting. In the case that $\Gamma$ is a congruence subgroup of level $N \geq 1$, however, Selberg used the "trace formula" to show, non-constructively, that Maass cusp forms do indeed exist. (cf. Section 3.6 for more on the existence of Maass cusp forms.) Moreover, Selberg counted (on average) the number of eigenvalues for a given level $N$. To make this precise (cf. [**284**]), define the eigenvalue counting function

$$\mathcal{N}_N(T) := \#\{j \;:\; |r_j| \leq T\},$$

where the eigenvalue $\lambda_j = r_j(1 - r_j)$. This function satisfies *Weyl's law*

$$\mathcal{N}_N(T) = \frac{\mathrm{vol}(\Gamma_0(N)\backslash\mathbb{H})}{4\pi} T^2 + O\left(\sqrt{N}T\log(NT)\right),$$

for $T \geq 2$. Note that as $N$ grows, the first term (in terms of the volume of the fundamental domain) indeed dominates the error term. While this shows that there are on average roughly $NT$ eigenvalues between $T$ and $T+1$, it is non-constructive.

It can be shown that for an eigenvalue $\lambda = r(1-r)$, either $\mathrm{Re}(r) = 1/2$ or $1/2 < r \leq 1$. In the former case we have $\lambda \geq 1/4$. In the latter case we have $\lambda < 1/4$, and such eigenvalues are called *exceptional*. There can only be a finite number of exceptional $\lambda$. Selberg conjectured that for congruence subgroups, exceptional eigenvalues do not exist.

CONJECTURE 3.5 (Selberg). *Let $\Gamma$ be a congruence subgroup of level $N \geq 1$, and let $\lambda_1(\Gamma)$ be the smallest eigenvalue of $\Delta$ acting on the space of functions satisfying conditions* i), ii), *and* iv) *in Definition* 3.2. *We have that*

$$\lambda_1(\Gamma) \geq \frac{1}{4}.$$

Selberg already proved that $\lambda_1(\Gamma) \geq 3/16$. At present, the best known result towards this conjecture is the following theorem due to Kim and Sarnak [**309**].

THEOREM 3.6 (Kim and Sarnak). *Assuming the notation and hypotheses in Conjecture* 3.5*, we have that*

$$\lambda_1(\Gamma) \geq \frac{975}{4096}.$$

Booker and Strömbergsson [**70**] have also recently investigated the problem of numerically computing Maass cusp forms, and they have proved Selberg's conjecture in the case of $\Gamma = \Gamma_1(N)$ for square-free $N < 857$, extending older work of Huxley, who proved Selberg's conjecture for $N < 19$.

Many applications in number theory require the spectral theory of automorphic forms (cf. [**283, 284**]) which arises from the Petersson inner product. These applications ultimately rely on the classical analysis and the theory of Hilbert spaces for the subspace of automorphic cusp forms (on a Fuchsian group $\Gamma$ of the first kind)

$$\mathcal{C}(\Gamma\backslash\mathbb{H}) := \{\text{smooth, bounded functions satisfying i) and ii) of Definition 3.2}\}.$$

Note that due to ii), the functions in this space are also square-integrable (i.e., satisfy condition iv) of Definition 3.2). Such spaces have a spectral decomposition with respect to the operator $\Delta$ (cf. Theorem 4.7 of [**283**]). We have the following Spectral Theorem, where as usual, $\langle\cdot,\cdot\rangle$ denotes the Petersson inner product (defined by the same formula as in (5.8)).

THEOREM 3.7 (Spectral Theorem). *There exists an orthonomal basis of* $\mathcal{C}(\Gamma\backslash\mathbb{H})$ *consisting of Maass cusp forms* $\{u_j\}$ *such that for any* $f \in \mathcal{C}(\Gamma\backslash\mathbb{H})$ *we have a spectral expansion of the form*

$$f(\tau) = \sum_j \langle f, u_j \rangle \, u_j(\tau),$$

*which converges in the norm topology. If* $\Delta(f)$ *is also a smooth and bounded automorphic function on* $\Gamma$*, then this expansion converges absolutely and uniformly on compact sets.*

REMARK. We introduce *Eisenstein series* in Section 3.4 below, which are orthogonal to cusp forms. As in Theorem 3.7, one can obtain a spectral decomposition of a relevant space spanned by Eisenstein series. By considering this space, as well as the cuspidal space above, one may determine the spectral decomposition of the whole space of automorphic functions satisfying i) and iv) of Definition 3.2. We refer the interested reader to **[283]** for details.

## 3.4. Eisenstein series

A natural way to produce examples of nonholomorphic forms is to use the method of averaging. In light of the difficulty of constructing cusp forms in general, it is important to note that the Eisenstein series constructed in this way below are not cusp forms.

DEFINITION 3.8. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $\tau \in \mathbb{H}$. The **Eisenstein series** $E(\tau; s)$ is defined by

$$E(\tau; s) := \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\gamma\tau)^s.$$

Here, $\Gamma_\infty := \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \; : \; n \in \mathbb{Z}\}$ is the stabilizer group of $i\infty$ in $\mathrm{SL}_2(\mathbb{Z})$.

REMARK. One defines Eisenstein series with respect to other groups $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ and other cusps $\mathfrak{a}$ similarly.

Here we review some essential properties of the Eisenstein series.

THEOREM 3.9. *We have the following:*

i) *The function* $E(\tau; s)$ *satisfies properties* i) *and* iii) *of Definition* 3.2. *In particular, it has eigenvalue* $\lambda = s(1-s)$.
ii) *There is a meromorphic continuation of* $E(\tau; s)$ *(in the s-variable) to* $\mathbb{C}$.
iii) *The Fourier expansion of* $E(\tau; s)$ *is given by*

$$E(\tau; s) = v^s + \phi(s) v^{1-s} + 2v^{\frac{1}{2}} \sum_{n \in \mathbb{Z}\backslash\{0\}} \phi(n, s) K_{s-\frac{1}{2}}(2\pi |n| v) e(nu),$$

*where*

$$\phi(s) := \sqrt{\pi} \frac{\Gamma\left(s - \frac{1}{2}\right) \zeta(2s-1)}{\Gamma(s)\zeta(2s)},$$

$$\phi(n, s) := \pi^s \Gamma(s)^{-1} \zeta(2s)^{-1} \sum_{ab = |n|} \left(\frac{a}{b}\right)^{s-\frac{1}{2}}.$$

iv) *The function* $E(\tau; s)$ *has a simple pole at* $s = 1$*, with residue equal to* $1/\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$.

REMARK. This Eisenstein series is not square-integrable (i.e., does not satisfy condition iv) in Definition 3.2).

REMARK. Note that $E$ also has a functional equation relating its values at $s$ and $1-s$ (see also (3.31) of [**283**]).

SKETCH OF PROOF OF THEOREM 3.9. It is clear by definition that $E(\tau; s)$ is automorphic (i.e., satisfies i) of Definition 3.2). A short calculation shows that

$$\Delta(v^s) = s(1-s)v^s.$$

When combined with the fact that $\Delta$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, this leads to the fact that $E(\tau; s)$ satisfies iii) of Definition 3.2 with eigenvalue $\lambda = s(1-s)$.

The Fourier expansion for $E(\tau; s)$ in iii) is found after a technical but straightforward calculation by choosing a set of representatives for $\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})$, applying Poisson summation, using the (integral) definition of the Bessel functions, and simplifying. See Section 3.4 of [**283**] for details.

The Fourier expansion in part iii) of the theorem can be used to establish meromorphic continuation to $\mathbb{C}$; in the plane $\mathrm{Re}(s) \geq 1/2$ there is just one simple pole at $s = 1$ with constant residue. □

## 3.5. $L$-functions of Maass cusp forms

Any $f \in \mathcal{A}_r^0(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ has a Fourier expansion as in Lemma 3.3. The *L-function* for $f$ is defined by

$$L(f, s) := \sum_{n=1}^{\infty} \frac{c_f(n)}{n^s}. \tag{3.3}$$

By the trivial bound

$$c_f(n) = O\left(n^{\frac{1}{2}}\right),$$

the $L$-function converges for $s$ with $\mathrm{Re}(s) > 3/2$; however the *Rankin-Selberg* method leads to the non-trivial bound (cf. [**358**])

$$\sum_{n \leq X} |c_f(n)|^2 = O(X),$$

which shows that $L(f, s)$ actually converges for $\mathrm{Re}(s) > 1$.

These $L$-functions can be "completed" using the $\Gamma$-function into a function $\Lambda(f, s)$ which satisfies a nice functional equation. That is, let $f \in A_r^0(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ with eigenvalue $\lambda = r(1-r)$. Assume $f$ satisfies $f(u + iv) = (-1)^\varepsilon f(-u + iv)$, $\varepsilon \in \{0, 1\}$. We define the *completed L-function* for $f$ as

$$\Lambda(f, s) := \pi^{-s}\Gamma\left(\frac{s + \varepsilon + r - \frac{1}{2}}{2}\right)\Gamma\left(\frac{s + \varepsilon + \frac{1}{2} - r}{2}\right)L(f, s).$$

THEOREM 3.10. *Assume the notation and hypotheses above. Then the L-function $L(f, s)$ satisfies the following:*

i) *The L-function $L_f$ has the Euler product expansion*

$$L(f, s) = \prod_{p \text{ prime}} \left(1 - c_f(p)p^{-s} + p^{-2s}\right)^{-1}.$$

ii) *It has an analytic continuation to all of $\mathbb{C}$.*

iii) *The completed L-function* $\Lambda(f,s)$ *satisfies the functional equation*

$$\Lambda(f,s) = (-1)^{\varepsilon}\Lambda(f,1-s).$$

The value $\varepsilon \in \{0,1\}$ is called the *sign of the functional equation*.

REMARK. Theorem 3.10 also generalizes to Maass cusp forms on congruence subgroups.

REMARK. In the literature, the value $(-1)^{\varepsilon} \in \{1,-1\}$ is also referred to as the sign of the functional equation. In particular, although we find the current notation convenient in this chapter, in Chapter 19 (for example in Theorem 19.6 iii)) we will use this slightly different definition for the sign of the functional equation.

The "simplest" Dirichlet series, in the sense that all of its coefficients are equal to 1, is the Riemann $\zeta$-function (2.14). This seemingly simple $L$-function is still in many ways very elusive: the famous *Riemann Hypothesis* conjectures that all non-trivial zeros of $\zeta$ lie on the line $\mathrm{Re}(s) = 1/2$. The truth of this hypothesis would have many deep implications throughout mathematics.

In analytic number theory, many of these applications can be derived using the weaker *Lindelöf Hypothesis*, which predicts that

$$\zeta\left(\frac{1}{2}+it\right) = O\left(|t|^{\varepsilon}\right) \tag{3.4}$$

($\varepsilon > 0$). The Lindelöf Hypothesis is known to follow from the truth of the Riemann Hypothesis. The standard Phragmén-Lindelöf method establishes the weaker estimate

$$\zeta\left(\frac{1}{2}+it\right) = O\left(|t|^{\frac{1}{4}+\varepsilon}\right),$$

which is commonly referred to as the *convexity bound*.

In general, the Phragmén-Lindelöf method leads to similar convexity bounds for other types of $L$-functions $L\left(f,1/2+it\right)$ arising from Maass cusp forms, newforms, and so on. (In addition to the references listed in the discussion below, we refer the interested reader to [**280**].) In practice, breaking convexity (reducing the exponent one obtains by Phragmén-Lindelöf), or establishing a subconvexity bound, is typically very difficult. Nonetheless, a number of authors have broken convexity with respect to various types of $L$-functions in many different aspects, including Weyl, who first broke convexity for $\zeta\left(1/2+it\right)$, followed by improvements by others including Bombieri-Iwaniec and Huxley more recently. Duke, Friedlander and Iwaniec [**167, 168, 169**] also notably broke convexity for $L$-functions arising from cusp forms with respect to various parameters. The subconvexity problem has been solved with complete uniformity in the analytic conductor by Michel and Venkatesh [**387**] with an inexplicit savings over the trivial convexity bound; the savings was eventually made explicit by Wu [**501**].

As another example, consider the *Rankin-Selberg L-function*

$$L(f\otimes g,s) := \sum_{n=1}^{\infty} \frac{c_f(n)\overline{c_g(n)}}{n^s} \tag{3.5}$$

of two cusp forms or Maass cusp forms $f$ and $g$ with Fourier coefficients $c_f(n)$ and $c_g(n)$, respectively. (In the latter case, coefficients are as in Lemma 3.3.) Kowalski, Michel, and VanderKam [**330**] proved a subconvexity bound for the Rankin-Selberg $L$-function of a fixed holomorphic newform $g$, as $f$ varies over holomorphic new forms with level $N \to \infty$ of the same weight as $g$. Sarnak [**444**] established subconvexity for such an $L$-function as the weight of $f$ goes to $\infty$. Many other authors including those just mentioned have addressed the subconvexity problem with respect to various types of $L$-functions in various aspects, including Harcos-Michel [**251**], Liu-Ye [**359**], Liu-Masri-Young [**360**], and many more.

## 3.6. Maass cusp forms arising from real quadratic fields

Among the most well known examples of Maass cusp forms are those arising from real quadratic fields. In particular, if $F$ is a real quadratic field over $\mathbb{Q}$ with discriminant $D$, then one can define a Maass cusp form on $\Gamma_0(D)$ with *Hecke character* $\psi$. We first discuss Grössencharakters, now typically called Hecke characters. These are characters on number fields, first introduced by Hecke, which generalize Dirichlet characters. To define them, in what follows we consider totally real fields $F$ of degree $d$ over $\mathbb{Q}$, before later returning to our construction of Maass cusp forms associated to real quadratic fields.

**3.6.1. Hecke characters.** Let $\sigma_j, 1 \leq j \leq d$, denote the distinct real embeddings of $F$, and let $\mathcal{O}_F$ denote the ring of integers of $F$. For an ideal $A \subset \mathcal{O}_F$, let $\psi_A$ be a primitive character on $(\mathcal{O}_F/A)^\times$. The character $\psi_A$ extends to a function on $\mathcal{O}_F$ as

$$\psi_A(a) := \psi_A(a \pmod{A})$$

if $\gcd(a, A) = 1$; otherwise, $\psi_A(a) := 0$. Next, we seek to use $\psi_A$ to form a character that is trivial on units. Let $\omega_j \in i\mathbb{R}$, $1 \leq j \leq d$, be such that $\sum_{j=1}^d \omega_j = 0$, and let $\varepsilon_j \in \{0, 1\}$, $1 \leq j \leq d$. Using these parameters, one can define characters $\psi_j$ $(1 \leq j \leq d)$ on $\mathbb{R} \setminus \{0\}$ by

$$\psi_j(t) := \operatorname{sgn}(t)^{\varepsilon_j} |t|^{\omega_j}.$$

The characters $\psi_j$ give rise to a character $\psi_\infty$ on $\mathcal{O}_F^\times$ defined by

$$\psi_\infty(t) := \prod_{j=1}^d \psi_j(\sigma_j(t)).$$

In fact, it can be shown that the parameters defining $\psi_\infty$ can be chosen in such a way that

$$\psi(t) := \psi_\infty(t)\psi_A(t)$$

is trivial on $\mathcal{O}_F^\times$, yielding a character on principal ideals prime to $A$.

A *Hecke character of* $F$ on ideals in $\mathcal{O}_F$ is defined to be any character whose restriction to the subgroup of principal ideals arises in the manner described above.

**3.6.2. Maass cusp forms from real quadratic fields.** Theorem 3.11 below exhibits (weight 0) Maass cusp forms with Nebentypus. To describe this, let $F$ be a real quadratic field and $\psi$ be a Hecke character of $F$. If $A = \mathcal{O}_F$, then in the notation of Subsection 3.6.1, $\psi = \psi_\infty$, which is defined from parameters $\omega_1 = -\omega_2 =: \omega$ and $\varepsilon_1 = \varepsilon_2 =: \varepsilon$ by

$$\psi_\infty((t_1, t_2)) = \operatorname{sgn}(t_1)^{\varepsilon_1} \operatorname{sgn}(t_2)^{\varepsilon_2} \left| \frac{t_1}{t_2} \right|^\omega,$$

where $(t_1, t_2) \in \mathbb{R}^2$. Using such Hecke characters attached to real quadratic fields, Maass constructed cusp forms on congruence subgroups. These Maass cusp forms are defined from the $K$-Bessel function; one may wish to compare with Lemma 3.3. Below, $\boldsymbol{N}(A)$ denotes the norm of the ideal $A$.

THEOREM 3.11 (Maass). *Let $F$ be a real quadratic field $F$ of discriminant $D$, with ring of integers $\mathcal{O}_F$. Let $\psi$ be a Hecke character for $F$, with $\omega \neq 0$ and $\varepsilon$ defined as above. Then we have that the function*

$$M_\psi(\tau) := \begin{cases} v^{\frac{1}{2}} \sum\limits_{A \subseteq \mathcal{O}_F} \psi(A) K_\omega(2\pi \boldsymbol{N}(A)v)\cos(2\pi \boldsymbol{N}(A)u) & \text{if } \varepsilon = 0, \\ v^{\frac{1}{2}} \sum\limits_{A \subseteq \mathcal{O}_F} \psi(A) K_\omega(2\pi \boldsymbol{N}(A)v)\sin(2\pi \boldsymbol{N}(A)u) & \text{if } \varepsilon = 1 \end{cases}$$

*is a Maass cusp form on $\Gamma_0(D)$ with character $\chi_D$.*

REMARK. If $\omega = 0$, then $M_\psi(\tau)$ can be defined similarly, up to the addition of the factor $\frac{1}{2}v^{\frac{1}{2}}\mathrm{Log}(\sigma_1(\eta_0))$ involving the regulator of $F$, where $\eta_0$ is the fundamental unit of $F$.

## 3.7. Hecke theory on Maass cusp forms

In analogy to the classical Hecke theory for holomorphic modular forms, there is a Hecke theory for Maass cusp forms. We briefly recall some main properties here. To describe this, for $n \in \mathbb{N}$, let

$$\Gamma_n := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = n \right\}.$$

The $n$-th *Hecke operator* $T_n$ acts on $f \in \mathcal{A}_r^0(\mathbb{H}\backslash\mathrm{SL}_2(\mathbb{Z}))$ by

$$T_n(f)(\tau) := \frac{1}{\sqrt{n}} \sum_{M \in \mathrm{SL}_2(\mathbb{Z})\backslash\Gamma_n} f(M\tau).$$

We summarize standard properties of the Hecke operators in the following lemma.

LEMMA 3.12. *The Hecke operators $T_n$, $n \in \mathbb{N}$, are linear operators on the space of Maass cusp forms. Moreover, for $n, m \in \mathbb{N}$, the following are true:*

i) $T_m T_n = T_n T_m$.
ii) $T_n \Delta = \Delta T_n$.
iii) $T_n(f)$ *has Fourier coefficients*

$$c_{T_n(f)}(m) = \sum_{d \mid \gcd(m,n)} c_f\left(\frac{mn}{d^2}\right),$$

*where $f$ is a Maass cusp form with coefficients $c_f(m)$ $(m \neq 0)$ as in Lemma* 3.3.

iv) *The Eisenstein series $E(\tau; 1/2 + it)$ $(t \in \mathbb{R})$ are simultaneous eigenfunctions of all Hecke operators $T_n$, with eigenvalues*

$$\sum_{ad=n} \left(\frac{a}{d}\right)^{it}.$$

## 3.8. Period functions of Maass cusp forms

It is very difficult to construct non-trivial Maass cusp forms. Inspired by this problem, Lewis and Zagier have developed a program in which Maass cusp forms are in one-to-one correspondence with *period functions.*

The following theorem is due to Lewis and Zagier [**353**]. (The equivalence between i) and iii) below is due to Maass.)

THEOREM 3.13 (Lewis and Zagier). *Let $r \in \mathbb{C}$ satisfy $\mathrm{Re}(r) > 0$. There is a canonical correspondence between the following functions:*

i) *Maass cusp forms $f$ on $\mathrm{SL}_2(\mathbb{Z})$ with eigenvalue $\lambda = r(1-r)$,*
ii) *holomorphic functions $\psi\colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ satisfying*

$$\psi(z) = \psi(z+1) + (z+1)^{-2r}\psi\left(\frac{z}{z+1}\right),$$

*and*

$$\psi(z) \ll \begin{cases} |\mathrm{Im}(z)|^{-A}(1+|z|^{2A-2\mathrm{Re}(r)}), & \mathrm{Re}(z) \le 0, \\ 1, & \mathrm{Im}(z) \ge 0, |z| \le 1, \\ |z|^{-2\mathrm{Re}(r)}, & \mathrm{Re}(z) \ge 0, |z| \ge 1, \end{cases}$$

*for some $A > 0$,*

iii) *pairs of $L$-series $L_\varepsilon(s)$, $\varepsilon \in \{0,1\}$, convergent for $s$ in some half-plane, such that the functions*

$$L^*_\varepsilon(s) := \frac{1}{4\pi^{s+\varepsilon}}\Gamma\left(\frac{s+\varepsilon-r+\frac{1}{2}}{2}\right)\Gamma\left(\frac{s+\varepsilon+r-\frac{1}{2}}{2}\right)L_\varepsilon(s)$$

*are entire, of finite order, and satisfy*

$$L^*_\varepsilon(1-s) = (-1)^\varepsilon L^*_\varepsilon(s),$$

iv) *holomorphic functions $f(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, invariant under $z \mapsto z+1$ and bounded by $|\mathrm{Im}(z)|^{-t}$ for some $t > 0$ such that*

$$f(z) - z^{-2r} f\left(-\frac{1}{z}\right)$$

*extends holomorphically across the positive real axis and is bounded by a multiple of $\min\{1, |z|^{-2\mathrm{Re}(r)}\}$ in the right half-plane.*

For such an $f$, the *period function* $\psi$ is given as

$$\psi(z) := \kappa \int_0^\infty \frac{zt^r}{(z^2+t^2)^{r+1}} f(it)dt, \quad \mathrm{Re}(z) > 0,$$

where $\kappa$ is some constant.

EXAMPLE 3.14. The generalization of this theorem to congruence subgroups makes contact with some strange $q$-series studied by Ramanujan. Consider the two $q$-hypergeometric (see also (9.14)) functions (the first of which appeared in Ramanujan's lost notebook)

$$\sigma(q) := \sum_{n=0}^\infty \frac{q^{\frac{n(n+1)}{2}}}{(-q;q)_n} =: \sum_{n=0}^\infty T(n) q^{\frac{n-1}{24}}, \tag{3.6}$$

$$\sigma^*(q) := 2\sum_{n=0}^\infty \frac{(-1)^n q^{n^2}}{(q;q^2)_n} =: \sum_{n=-\infty}^{-1} T(n) q^{\frac{|n|-1}{24}}, \tag{3.7}$$

famously studied in **[32]** in relation to indefinite theta functions. Here $(a)_n = (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$. Note that the coefficients $T(n)$ above equal 0 if $|n| \not\equiv 1$

(mod 24). Cohen **[151]** beautifully showed that the coefficients of these two $q$-hypergeometric series appear as coefficients in a Maass cusp form with eigenvalue 1/4 on the congruence subgroup $\Gamma_0(2)$:

$$c(\tau) := v^{\frac{1}{2}} \sum_{n \in 1+24\mathbb{Z}} T(n) K_0\left(\frac{2\pi|n|v}{24}\right) e\left(\frac{nu}{24}\right). \tag{3.8}$$

Essentially, one can think of $q^{\frac{1}{24}}\sigma(q)+q^{-\frac{1}{24}}\sigma^*(q)$ as the Lewis-Zagier period function for the Maass cusp form $c$.

We note that these $q$-hypergeometric series are also, in fact, related to real quadratic fields. If we let

$$\varphi(q) := q^{\frac{1}{24}}\sigma(q) + q^{-\frac{1}{24}}\sigma^*(q),$$

Cohen proved the identity

$$\varphi(q) = \sum_{\mathfrak{a} \subset \mathbb{Z}[\sqrt{6}]} \chi_1(\mathfrak{a}) q^{\frac{\boldsymbol{N}(a)}{24}},$$

where $\chi_a(\mathfrak{a})$ is a character of conductor $4(3+\sqrt{6})$ and order 2 on ideals $\mathfrak{a} \subset \mathbb{Z}[\sqrt{6}]$. This gives rise to the *Artin L-function*

$$L(\chi_1, s) := \sum_{\mathfrak{a} \subset \mathbb{Z}[\sqrt{6}]} \frac{\chi_1(\mathfrak{a})}{\boldsymbol{N}(\mathfrak{a})^s},$$

which is completed as

$$\Lambda(\chi_1, s) := (1152)^{\frac{s}{2}} \pi^{-s} \Gamma\left(\frac{s}{2}\right)^2 L(\chi_1, s).$$

Cohen proves that $\Lambda(\chi_1, s)$ can be analytically continued to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(\chi_1, 1-s) = -\Lambda(\chi_1, s).$$

The connection to Maass cusp forms now follows via an extension of Theorem 3.11. Additional related identities and properties were given in **[151]**.