# Preface

The idea of an identity in an algebraic structure is quite general, all the basic laws that we learn at school as the commutative, associative, and distributive laws are in fact special types of identities. Loosely speaking, an identity is a symbolic expression involving one or several operations and one or several *variables*, which is identically satisfied when the *variables* are substituted in a given algebraic structure.

In this generality, the topic is that of *universal algebra* for which we refer to the classical book of P. M. Cohn [**Coh65**]. In the present book we restrict to a special type of identities, satisfied by associative algebras, over some commutative ring $A$. We take as symbolic expressions the noncommutative polynomials, with coefficients in $A$, in some finite or infinite set of variables.

By definition, polynomials are formal linear combinations of *monomials* in a fixed *alphabet* $X$. In the noncommutative setting a monomial is just a *word* in the variables $X = \{x_1, x_2, \dots\}$. Thus a formal polynomial $p(x_1, x_2, \dots, x_n)$ is a *polynomial identity* for an algebra $R$ if $p(r_1, r_2, \dots, r_n) = 0$ whenever we substitute to the variables $x_i$ elements $r_i$ of $R$.

The simplest and, in a way, the strongest identity for an associative algebra is the commutative law $xy - yx$. The role of more complex identities for truly noncommutative algebras has been discovered by several people: Jacobson connected it with the Kurosh problem; Kaplansky proved that an algebra $R$ satisfying an identity of degree $d$ has no infinite-dimensional irreducible modules, and in fact all irreducible modules have dimension $\leq d/2$; and Amitsur and Levitzki discovered that the ring of $n \times n$ matrices over a commutative ring $A$ satisfies a kind of higher-order commutative law, the *standard identity*

$$\mathrm{St}_{2n}(x_1, x_2, \dots, x_{2n}) := \sum_{\sigma \in S_{2n}} \epsilon_\sigma x_{\sigma(1)} \cdots x_{\sigma(2n)},$$

where $S_{2n}$ denotes the symmetric group on $2n$ elements and $\epsilon_\sigma$ denotes the sign of a permutation $\sigma$.

These theorems suggest that the theory of polynomial identities (PI for short) is strongly related to finite-dimensional representations of algebras.

If an algebra $R = A\langle a_1, \dots, a_k \rangle$ is finitely generated, over a commutative domain $A$, it is easily seen that the set of representations of $R$ in an $n$-dimensional space $K^n$, where $K \supset A$ is an algebraically closed field, is an affine algebraic variety, a subvariety of the affine space of $k$-tuples of $n \times n$ matrices. The notion of isomorphism between such representations corresponds geometrically to the fact that such representations are in the same orbit under the simultaneous conjugation action of the group $\mathrm{GL}(n, K)$ of invertible matrices, on $M_n(K)^k$.

This suggests that the theory of PI should be close both to commutative algebra and to invariant theory. This we shall try to explain in this book. In parallel there is the mostly combinatorial theory of $T$-ideals; that is the ideals in the noncommutative free algebra describing identities of algebras.

The main difference between PI rings and commutative rings is that in general no analogue of the Hilbert basis theorem is available. That is, usually a finitely generated PI algebra is not Noetherian. This is the main technical difficulty and also a source of some pathological examples. The phenomenon responsible for this fact is essentially the fact that a given algebra may have irreducible representations of various dimensions. When all irreducible representations have the same dimension $d$ (and moreover the algebra satisfies all formal identities of $d \times d$ matrices) by a theorem of M. Artin, then $R$ is an Azumaya algebra, that is a possibly *nonsplit* form of the algebra of $d \times d$ matrices over a commutative ring. For such an algebra the theory is very close to the theory of commutative algebras.

Another basic technique of commutative algebra which is not really available is localization. In general, in order to construct a ring of fractions for some multiplicative system, one needs special conditions, the Ore conditions, which are often not satisfied. On the other hand representation theory of the symmetric and linear group appear in an essential way and give the main flavour to PI theory.

PI theory has developed in several steps. First the discovery of special identities, the various structure theorems, for primitive, nil, or prime algebras satisfying a PI. Then a more geometric theory strongly related to invariants of matrices. Another step is a combinatorial theory based on the representation theory of the symmetric group, investigating various problems related to the *growth* of the space of polynomial identities. The final step involves a deep structure theory analysing mostly the nil part of algebras and leading to the theorem of Razmyslov on the nilpotency of the radical of a PI algebra finitely generated over a field, and its generalizations and the theory of Kemer leading to a solution of Specht's problem, that is the finite generation of $T$-ideals (proposed by Specht in [**Spe50**]).

This last theory requires in a surprising way that we introduce superalgebras and their superidentities. In some way this is suggested by the hook theorem of Amitsur and Regev [**RA82**] and Kemer [**Kem91**]. The reason is that in the general theory the Grassmann algebra plays an important role, and a basic result of Kemer is that every PI algebra is PI equivalent to the Grassmann envelope of a finite-dimensional superalgebra; see Theorem 19.8.1.

We have the impression that Kemer's theory was not fully accessible to the PI community for more than three decades after Kemer presented his remarkable work in the mid-1980s. Fortunately, the situation has changed, and here we present a complete detailed proof (see also [**KBKR16**]).

## The plan of the book

After a brief discussion of classical finiteness problems as the Burnside and Kurosh problem, which serve as original motivations, we start our treatment by dividing the book into four main parts plus the appendix on the Golod–Shafarevich counterexample, which aims at showing typical pathologies that occur for general rings. The theme of such pathologies is somewhat skew with respect to our topics; it is a widely unexplored area where several *monsters* may appear.

**Part 1.** The first six chapters contain foundational material on representation theory and noncommutative algebra. This can be used for an introductory course in noncommutative algebra. We give detailed proofs, except for Chapters 3 and 6 where we give references to existing literature. In particular, Chapter 5 is an introduction to the theory of Azumaya algebras, which play a special role in PI theory. This theory is then completed in Chapter 10 where we apply PI theory to Azumaya algebras, giving some possibly new results. The expert reader may use this part only as reference and then start with the main topics in the remaining parts.

**Part 2.** Chapters 7 and 8 discuss mostly the combinatorial aspects of the theory, the growth theorem, the hook theorem, and Shirshov's bases. Here methods of representation theory of the symmetric group play a major role. Chapter 9 discusses in detail the PI theory of $2 \times 2$ matrices, one of the very few examples which can be fully treated.

**Part 3.** Chapter 11 begins the main body of our discussion of structure theorems for PI algebras, including the theorems of Kaplansky and Posner, the theory of central polynomials, the M. Artin theorem on Azumaya algebras, and the geometric part on the variety of semisimple representations including the foundations of the theory of Cayley–Hamilton algebras.

**Part 4.** This part is devoted first to the proof of the theorem of Razmyslov, Braun, and Kemer on the nilpotency of the nil radical for finitely generated PI algebras over Noetherian rings. We then move on to the theory of Kemer and the Specht problem, which we have split in four chapters, 17 to 20. Finally, in Chapters 21 and 22 we discuss the *PI-exponent* and codimension growth. This part uses some nontrivial analytic tools coming from probability theory.

**Appendix A.** This appendix is devoted to the counterexamples of Golod and Shafarevich to the Burnside problem.

### Differences with other books

There are several books on the topics of PI algebras; see [**Bah91**], [**KBR05**], [**KBKR16**], [**Dre00**], [**For91**], [**GZ05**], [**Kem91**], [**Mue76**], [**Pro73**], [**Raz94**]. The main differences between the present book and the remaining literature is that, while trying to give a comprehensive treatment, we stress the invariant theory and the geometric aspects. This approach appears partly in the book [**LB08**] of Lieven Le Bruyn.

**What we do not treat.** As mentioned at the beginning, we do not treat Lie identities or identities on special nonassociative algebras, as in [**Dre74**], [**Dre87**], [**Dre95**].

We do not treat various special identities ([**Dre95**] or [**Raz73**]) nor special geometric properties of the algebra of generic matrices ([**LBP87**] or [**LBVdB88**]). We treat the deep Kemer theory only in characteristic 0 and leave out the deep results of Belov and Kemer in positive characteristic; see [**Bel97**], [**Bel00**], [**Bel10**], [**Kem95**], [**Kem03**].

We treat superalgebras in a limited way, only what is needed for Kemer's theory. In particular, the reader may look at two interesting papers connected with polynomial or trace identities, [**BR87**] by A. Berele, A. Regev, and [**Raz85**] by Razmyslov.

Finally, we do not treat the deep results of Berele for nonfinitely generated PI algebras; see [**Ber08a**], [**Ber08c**], [**Ber13b**].

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