Preface

This is an unusual book project: While on its surface it deals with Sturm–Liouville Theory in the case where all coefficients (there are up to four such coefficients) are scalar-valued, it represents much more and includes many aspects of linear operator theory, spectral theory, some pieces of complex analysis, and special functions, such as Bessel functions. Moreover, within the topics covered in each chapter, an essentially encyclopedic approach is attempted.

More precisely, Chapter 2 provides a bit of physical motivation, Chapters 3–7 contain the classical repertoire, such as, second-order ODE fundamentals, regular and singular endpoints, Weyl–Titchmarsh theory, spectral (matrix) functions, etc., for three-coefficient Sturm–Liouville operators associated with differential expressions τ of the type

$$\tau = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right] \text{ for a.e. } x \in (a,b), \tag{0.0.1}$$

with

$$p,\ q,\ r \text{ real-valued scalar coefficients a.e. on } (a,b), \tag{0.0.2}$$

where $(a,b) \subseteq \mathbb{R}$ is an arbitrary interval. Chapter 8 develops classical oscillation theory; Chapters 9 and 10 deal with renormalized and perturbative oscillation theory, and perturbative Hardy inequalities; Chapter 11 presents boundary data maps, that is, the analogue of spectral parameter dependent Dirichlet-to-Neumann maps and their extensions to general self-adjoint boundary conditions, assuming τ is regular at a and b. Chapter 12 provides a detailed approach to the computation of traces and (modified) Fredholm determinants in connection with resolvents of three-coefficient Sturm–Liouville operators, with additional results in the special case of half-line Schrödinger operators. Chapter 13, one of the particular highlights of this monograph, revisits and extends the case of three-coefficient Sturm–Liouville operators with (strongly) singular coefficients. Chapter 14 extends most of the theory to four-coefficient Sturm–Liouville operators, a situation that can handle distributional potential coefficients in $H^{-1}_{loc}((a,b))$, and is associated with differential expressions τ of the type

$$\tau = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\left(\frac{d}{dx} + s(x)\right) + s(x)p(x)\left(\frac{d}{dx} + s(x)\right) + q(x)\right] \\ \text{for a.e. } x \in (a,b), \tag{0.0.3}$$

with

$$p,\ q,\ r,\ s \text{ real-valued scalar coefficients a.e. on } (a,b). \tag{0.0.4}$$

Our final Chapter 15, an epilogue, takes a closer look at some of the partial differential equations of Mathematical Physics which reduce to various Sturm–Liouville differential equations upon applying the time honored method of separation of variables and bringing our subject full circle with the physical motivation in Chapter 2.

While the content of Chapters 2–15 describes Sturm–Liouville Theory proper, a distinct feature of this book project consists of a large variety of appendices that develop additional material for unbounded linear operators in a Hilbert space $\mathcal{H}$, their underlying spectral theory, trace ideals $\mathcal{B}_p(\mathcal{H})$, $p \in [1,\infty)$, of compact operators $\mathcal{B}_\infty(\mathcal{H})$, and basic facts on traces and (modified) Fredholm determinants. Moreover, we discuss self-adjoint extension theory of symmetric operators with special emphasis on nonnegative extensions and their extremal cases, the celebrated Friedrichs and Krein–von Neumann extensions, a summary of boundary triples applicable to ODE situations, and sesquilinear forms and associated representation theorems. In addition, a comprehensive account of Nevanlinna–Herglotz functions and their representations (resp., exponential representations) in terms of spectral measures (resp., a Krein spectral shift function associated with absolutely continuous measures), and a crash course into the basic definitions and properties of Bessel functions, is provided.

Our quick tour through the material presented in this book would be incomplete without describing a further unique feature of this monograph, namely, the explicit presentation of a fair number of examples (ranging from elementary to highly nontrivial ones) in great, and sometimes, exhaustive detail. In this context we mention, in particular, the following situations treated in various sections:

- Constant Coefficients (Sects. 6.4, 7.4)
- The Bessel Operator on (a,∞), $a \in (0,\infty)$ (Sect. 6.4)
- The Generalized Bessel Operator on (a,∞), $a \in (0,\infty)$ (Sect. 6.4)
- Short-Range Scattering on the Half-Line $(0,\infty)$ (Sect. 6.4)
- N-Soliton Potentials (Sect. 7.4)
- Short-Range Scattering on $\mathbb{R}$ (Sect. 7.4)
- Higher-Order Trace Relations for KdV Invariants (Sect. 7.4)
- The Genus One Lamé Potential (Sect. 7.4)
- Algebro-Geometric Finite-Band Potentials (Sect. 7.4)
- Floquet Theory in the Presence of Three Periodic Coefficients (Sect. 7.5)
- The Sodin–Yuditskii Class $\mathrm{SY}(\mathcal{E})$ and some of its extensions (Sect. 7.6)
- The Bessel Operator on $(0,b)$, $b \in (0,\infty) \cup \{\infty\}$ (Sects. 13.3, 13.5, 13.7)
- The Generalized Bessel Operator on $(0,b)$, $b \in (0,\infty) \cup \{\infty\}$ (Sects. 8.4, 13.3, 13.5, 13.7)
- The Acoustic Black Hole (Sect. 13.5)
- The Jacobi Operator on $(-1,1)$ (Sect. 13.5)
- The Legendre Operator on $(-1,1)$ (Sect. 13.7)
- The Laguerre (or, Kummer, resp., confluent hypergeometric) Operator on $(0,\infty)$ (Sect. 13.7)

Here some of the particular differential expressions underlying the operators mentioned in the list of examples above are of the following type:

$$\begin{aligned}&\text{Bessel: } \tau_\gamma = -\frac{d^2}{dx^2} + \frac{\gamma^2 - (1/4)}{x^2}, \quad \gamma \in [0,\infty),\ x \in (a,b),\\ &\qquad\qquad a \in [0,\infty),\ b \in (a,\infty),\end{aligned} \tag{0.0.5}$$

$$\text{Generalized Bessel: } \tau_{\gamma,\mu,\nu} = x^{-\mu}\left[-\frac{d}{dx}x^\nu\frac{d}{dx} + \frac{(2+\mu-\nu)^2\gamma^2 - (1-\nu)^2}{4}x^{\nu-2}\right],$$

$$\gamma \in [0,\infty),\ \mu,\nu \in \mathbb{R},\ x \in (a,b),\ a \in [0,\infty),\ b \in (a,\infty), \tag{0.0.6}$$

$$\begin{aligned}&\text{Acoustic Black Hole: } \tau_{p_0,r_0,\mu,\nu} = -\frac{1}{r_0(x)x^\mu}\frac{d}{dx}p_0(x)x^\nu\frac{d}{dx}, \quad \mu,\nu \in \mathbb{R},\\ &\qquad m \le p_0(x) \le M,\ m \le r_0(x) \le M \text{ for a.e. } x \in (0,b),\ b \in (0,\infty),\end{aligned} \tag{0.0.7}$$

$$\begin{aligned}&\text{Jacobi: } \tau_{\mu,\nu,Jac} = -\frac{1}{(1-x)^\mu(1+x)^\nu}\left[\frac{d}{dx}\big((1-x)^{\mu+1}(1+x)^{\nu+1}\big)\frac{d}{dx}\right],\\ &\qquad\qquad \mu,\nu \in \mathbb{R},\ x \in (-1,1),\end{aligned} \tag{0.0.8}$$

$$\text{Legendre: } \tau_{Leg} = -\frac{d}{dx}\big(1-x^2\big)\frac{d}{dx}, \quad x \in (-1,1), \tag{0.0.9}$$

$$\text{Laguerre: } \tau_{Lag,\nu} = -x^{1-\nu}e^x\frac{d}{dx}x^\nu e^{-x}\frac{d}{dx}, \quad \nu \in (0,2),\ x \in (0,\infty). \tag{0.0.10}$$

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