Preface

The present volume reflects the contents of some of the courses and talks given at the "XX School of Mathematics *Lluís Santaló*: Research Summer School on p-adic Analysis, Arithmetic and Singularities" held at the Magdalena Palace in Santander (Spain), from June 24th to June 28th, 2019, under the support of the UIMP–Universidad Internacional Menéndez Pelayo and the RSME–Real Sociedad Matemática Española. The volume also contains some related contributions.

The School aimed to provide an introduction to a very active research area lying at the intersection of number theory, p-adic analysis, algebraic geometry and singularity theory. The main objects of study here are the *local zeta functions* and some algebraic-combinatorial counterparts called Poincaré series (sometimes Hilbert-Poincaré series), which are in some sense a kind of generating functions associated with suitable ring filtrations. The local zeta functions are ubiquitous objects in mathematics and physics.

There are several types of local zeta functions, for instance p-adic, Archimedean, topological and motivic, among others. In the Archimedean setting, the local zeta functions were introduced in the 50s by Gel'fand and Shilov. The main motivation was that the meromorphic continuation of Archimedean local zeta functions implies the existence of fundamental solutions (i.e. Green functions) for differential operators with constant coefficients. This fact was established, independently, by Atiyah and Bernstein.

In the 60s, Weil studied local zeta functions, in the Archimedean and non-Archimedean settings, in connection with the Poisson-Siegel formula. In the 70s, Igusa developed a uniform theory for local zeta functions and oscillatory integrals, with polynomial phase, in characteristic zero. In the p-adic setting, the local zeta functions are connected with the number of solutions of polynomial congruences mod p^m and with exponential sums mod p^m. Recently Denef and Loeser introduced the motivic zeta functions which constitute a vast generalization of p-adic local zeta functions. There are many intriguing conjectures relating the poles of the local zeta functions with the eigenvalues of the complex monodromy and with the roots of the Bernstein polynomials.

The p-adic strings seem to be related in some interesting ways with ordinary strings. For instance, connections through the adelic relations and through the limit when $p \to 1$. This limit gives rise to a boundary string field theory. Denef and Loeser established that the limit $p \to 1$ of Igusa's local zeta function provides an object, called topological zeta function. By using Denef-Loeser's theory of topological zeta functions, it was established recently that limits $p \to 1$ of tree-level p-adic string amplitudes give rise to certain amplitudes, which are connected with the boundary string field theory mentioned.

The construction of the motivic Igusa zeta functions requires techniques of motivic integration introduced by Kontsevich. On the other hand, Campillo, Delgado and Gusein-Zade, using integration with respect to the Euler characteristic (introduced by Viro), defined the motivic Poincaré series associated with an ideal filtration of a one-dimensional local ring. This Poincaré series is a complete topological invariant of a curve singularity: it coincides (up to a factor in the particular case of an irreducible curve) with the Alexander polynomial, and with the A'Campo zeta function of the singularity.

During the School four courses were offered, including coordinated exercise sessions, and also talks presenting current research problems. The School was aimed at doctoral and postdoctoral students working on themes broadly connected to the issues in the School. According to this spirit, we divide the volume in two parts. The first part contains a series of five lecture notes enlightening the basics of the topics of the School:

- ◇ The article "Archimedean zeta functions and oscillatory integrals" by Edwin León-Cardenal is a short survey about the relation between the Archimedean zeta functions (which were called "complex powers" originally) and oscillatory integrals. The survey provides a historical background, also it presents some central results, including a few key arguments, and it concludes with a challenging question.
- ◇ Julio J. Moyano-Fernández contributes with a survey entitled "Generalized Poincaré series for plane curve singularities", on Poincaré series of multi-index filtrations associated with plane complex curve singularities. The survey starts with an introduction to resolutions of plane curve singularities and their Poincaré series. Then, an interpretation of the Poincaré series as an integral with respect to the Euler characteristic is presented. Finally, the motivic Poincaré series of Campillo, Delgado and Gusein-Zade is introduced.
- ◇ Naud Potemans and Willem Veys give an introduction to p-adic Igusa zeta functions; it contains a careful choice of examples and exercises leading to make it easier to understand the relation of zeta functions with singularity theory, particularly with embedded resolutions. The authors consider, in full generality, the case where the polynomials are defined on fields of characteristic zero and on rings of algebraic integers.
- ◇ In the article "An introduction to p-adic and motivic integration, zeta functions and invariants of singularities", Juan Viu-Sos connects p-adic integration with motivic integration, remarking the main similarities and differences between the two theories. The survey devotes a special attention to the definition of motivic measure in the space of arcs and the formula of change of variables; after this, a relation between the Hodge invariants of birationally equivalent varieties is established.
- ◇ Wilson A. Zúñiga-Galindo contributes with a survey titled "p-adic analysis: a quick introduction", which offers an interesting introduction (with some proofs, exercises and references) to p-adic numbers, p-adic integration and harmonic analysis. This survey offers a fast way to understand the subject assuming basic knowledge in algebra and analysis.

The second part of the volume consists of four research articles dealing with current matters covered by the School:

- ◇ The article "On maximal order poles of generalized topological zeta functions" by Enrique Artal Bartolo and Manuel González Villa shows some examples of topological zeta functions associated to an isolated plane curve singular point and an allowed, in the sense of Némethi and Veys, differential form that have several poles of order two.
- ◇ José Ignacio Cogolludo-Agustín, Tamás László, Jorge Martín-Morales, and András Némethi in "Local invariants of minimal generic curves on rational surfaces" consider the non-easy problem of deciding which analytic invariants are in fact topological. They study the case of reduced curve germs $(C, 0)$ in a normal surface singularity $(X, 0)$. In fact, they recover the delta invariant of C from the topology of the embedding $(C, 0) \subseteq (X, 0)$.
- ◇ In the article "Motivic Poincaré series of cusp surface singularities" by János Nagy and András Némethi, the authors study both the equivariant multivariable analytical and the topological Poincaré series of a germ of curve, as well as their motivic versions. The authors show that they are equal and provide an explicit combinatorial expression for them. The key ingredient is a motivic multivariable series associated with the space of effective Cartier divisors of the reduced exceptional curve.
- ◇ "Non-archimedean electrostatics" by Christopher D. Sinclair discusses the p-adic analogues of 1D log-Coulomb gases and their connections with local zeta functions, which provides an interesting link between number theory and physics.

Finally, we would like to mention that the "XX School of Mathematics *Lluís Santaló*" included a parallel session under the motto *Civil dislocation and mathematical roots*. This consisted of three open talks by Antonio Campillo, Luis Español and Sebastiá Xambó-Descamps on the work and life of Lluís Santaló Sors, the Spanish mathematician who names the Research School.

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