Preface

The workshop *Hypergeometry, Integrability and Lie Theory* was planned to take place live at the Lorentz Center, Leiden, but due to COVID-19 measures it took place online in the week December 7-11, 2020. This volume contains the proceedings of this workshop.

1. Background

Hypergeometric functions have many spectacular appearances in mathematics and its applications. Ever since their introduction by Euler, Riemann, and Gauss, hypergeometric functions have played an important role. Initially they appear as solutions to classical differential equations that emerged from the analysis of specific applied problems. Their fundamental role in geometry and representation theory gradually became clear over many years. The workshop was focused on advances at the interface of the theory of hypergeometric type functions, quantum integrable systems and representation theory.

The study of quantum integrable systems, in particular the study of the spectrum of their Hamiltonians, is one of the most important applications of the theory of hypergeometric type functions and of the accompanying representation theory. These systems provide solvable models for quantum many-body problems, for quantum field theory and statistical mechanics. Together with topological and conformal field theories, quantum integrable systems provide an important class of models of quantum matter where the dynamics is almost completely determined by the symmetry. An important characteristic property of integrable systems is that they have sufficiently many commuting operators (quantum conservation laws) and one of the main challenges here is the description of the joint spectrum of these operators.

In all known examples the symmetries defining these models are given either by an action of a Lie group, a Lie algebra or by their deformations, such as quantum groups. This makes representation theory particularly important for the study of integrable systems. Hypergeometric functions appear in representation theory in many ways. Typically, they appear as spherical functions. For Lie groups and Lie algebras spherical functions are eigenfunctions of differential operators acting on functions on symmetric spaces and representing the radial part of Casimir elements. Generalizations of hypergeometric functions such as multivariable hypergeometric functions, basic hypergeometric functions, and elliptic hypergeometric functions appear in a similar way in representation theory of quantum groups, elliptic quantum groups, and related algebras such as Hecke algebras and double affine Hecke algebras. Some of the best known integrable systems that are related

to spherical functions and similar objects in representation theory include Calogero-Moser-Sutherland (CMS) models, Toda lattices, spin chains and elliptic Painlevé equations.

Vector-valued spherical functions also appear as eigenfunctions of radial parts of Casimir operators acting on more complicated spaces, also associated with symmetric pairs. New important developments in the theory of vector-valued spherical type functions, their asymptotic expansions in terms of Harish-Chandra series, and their applications in harmonic analysis emerged recently.

The corresponding vector-valued analogue of the (single-variable) hypergeometric differential operator was introduced in works of Tirao, Grünbaum and others. It is naturally related to matrix-valued spherical functions for symmetric spaces of rank one. More recently, the role of matrix-valued spherical functions and matrix-valued orthogonal polynomials in the integrability of the non-abelian Toda lattice has become well understood. Another application of the vector-valued spherical functions is to solutions of boundary Khnizhnik-Zamolodchikov equations and boundary fusion rules. Matrix-valued spherical functions arise as eigenfunctions of explicit realizations of the matrix-valued partial differential operators which are expected to have a close relation to quantum commuting Hamiltonians for super-integrable quantum spin CMS systems. These new connections have the potential for establishing important breakthroughs in both fields.

The quantum inverse scattering method in the study of quantum integrable systems with periodic boundary conditions led to the quantum Yang-Baxter equation and to the discovery of quantum groups such as q-deformed enveloping algebras in the 1980s by Drinfeld and Jimbo. Similarly, work by Cherednik, Sklyanin, Kulish and others in the 1980s and 1990s suggested that systems with open boundary conditions lead to the quantum reflection equation and to comodule algebras over quantum groups. However, in the early days these comodule algebras remained largely formal objects. A parallel development in the 1990s aimed to identify Macdonald polynomials as zonal spherical functions on quantum group analogues of symmetric spaces. The construction of quantum symmetric spaces is based on coideal subalgebras in quantum universal enveloping algebras. The resulting theory of quantum symmetric pairs has seen a lot of progress within the last decade, and is presently emerging as a vast generalization of the theory of Drinfeld-Jimbo quantum groups. Their representation theory is a natural extension of the representation theory of real reductive Lie groups and is a natural place where many q-hypergeometric functions appear. From the perspective of quantum integrable systems, the coideal subalgebras in the theory of quantum symmetric pairs are the symmetry objects associated to the boundary in the same way as the quantum universal enveloping algebras are associated to the bulk. Various versions of boundary Knizhnik-Zamolodchikov type equations appear in this context.

From the above it is clear that one should expect more important connections between representation theory and quantum integrable systems. The main goal of this workshop was to develop stronger interactions between these fields.

2. Structure of workshop

Each day was centred on a designated topic, which was briefly introduced in a general and accessible way by a leading expert (a moderator), after which presentations followed. Even though the workshop was online, the discussions around the talks were very lively. Going online meant that the number of participants of the workshop was almost double the originally planned number.

To give a better impression of the workshop, we summarize each day below.

Monday: Hypergeometric functions and representation theory. The moderator Hjalmar Rosengren gave an introduction to the connection of hypergeometric functions to Lie groups, Dunkl operators and non-symmetric special functions by showing how these ideas work for single-variable Jacobi polynomials.

Margit Rösler discussed the limit transition from Heckman-Opdam functions of type BC to type A. In this limit transition she paid particular attention to the interpretation on symmetric spaces, and related integral representations, and as spherical functions on Gelfand pairs of infinite-dimensional groups as introduced by G. Olshanski. These proceedings include joint work of M. Rösler and M. Voit on the construction of martingales of Heckman-Opdam diffusion processes for root systems of type A and B.

Catharina Stroppel discussed an approach to Verlinde rings using double affine Hecke algebras (DAHA) at roots of unity. A Verlinde ring is a category consisting of isomorphism classes of certain representations of the loop group associated to a Lie group, and it comes with a tensor product. She showed how properties of Verlinde rings can be obtained from representations of DAHA.

Ivan Cherednik discussed a super-version, or spinor version, of the Dunkl operators in relation to degenerate DAHA in the rank one case. Apart from several results, such as the Cherednik-Matsuo theorem, he discussed how the Bessel functions can be used to model the spread of COVID-19, especially in the first wave.

Pavel Etingof discussed quantized Kleinian singularities of type A_{n-1}, also known as generalized Weyl algebras, and twisted traces on them. These twisted traces can be used to discuss particular classes of $*$-products, and connect to 3-dimensional superconformal field theory. In the case $n = 2$ this connects to unitary spherical representations of $\mathrm{SL}(2, \mathbb{C})$, and moreover it connects to orthogonal polynomials, which for $n = 2, 3$ can be identified as polynomials from the Askey scheme, such as Meixner-Pollaczek and continuous Hahn polynomials. These proceedings also contain joint work by P. Etingof and D. Kazhdan on characteristic functions of certain p-adic integral operators, which restrict to the associated eigenspace as deformed Wronskians of solution sets of basic hypergeometric equations.

In the spirit of the theme of the first day, we also highlight a contribution to this volume by Tom Koornwinder on a natural classification of polynomials from the q-Askey scheme, developing an approach by Verde-Star.

Tuesday: Symmetric spaces and coideal subalgebras. The moderator Tom Koornwinder gave an introduction to harmonic analysis on symmetric spaces, interpolation polynomials and quantum symmetric pairs to prepare for the talks by Eric Opdam, Siddhartha Sahi, Gail Letzter and Stefan Kolb.

E. Opdam discussed some open problems in harmonic analysis related to hypergeometric functions for root systems and recent progress by T. Honda, H. Oda

and N. Shimeno. He explained how this contributes to the classification of the exceptional discrete series. Interpolation polynomials are inhomogeneous symmetric polynomials indexed by partitions whose leading terms are given by Jack polynomials. It was conjectured by F. Knop and S. Sahi in 1996 that suitable normalized interpolations polynomials have positive coefficients when expressed in terms of monomials symmetric functions. In his talk, S. Sahi presented a proof of this conjecture, see also `arXiv:2104.08598`.

The remaining two talks of the day were devoted to the theory of quantum symmetric pairs (QSP). G. Letzter focused on the interplay between the Faddeev-Reshetikhin-Takhtajan type construction of QSP (Noumi, Sugitani, Dijkhuizen) and her own Drinfeld-Jimbo type approach. She pointed out similarities between the structure of the center and the construction of Cartan subalgebras for QSP. S. Kolb presented recent joint work with M. Yakimov on the interpretation of QSP as star-product deformations of partial quantum parabolic algebras. This perspective allows a conceptual, bar-involution free description of the quasi K-matrix. This suggests constructing the bar-involution via the quasi K-matrix, as explained in S. Kolb's contribution to this volume.

Wednesday: Coideal subalgebras and integrable spin chains. On the third day of the conference the focus moved to the applications of quantum group theory to integrable quantum models and exactly solvable models from statistical mechanics (Heisenberg spin chains, vertex models, asymmetric simple exclusion processes (ASEPs), etc.). The moderator Christian Korff introduced the topic by explaining the role of representations of quantum groups and coideal subalgebras in such models.

Bernard Nienhuis, presenting joint work with O. Huygen, discussed how the affine Temperley-Lieb algebra can be used to construct new explicit expressions of quasi-local conserved quantities of the closed XXZ chain.

Robert Weston talked about ongoing work with A. Cooper and B. Vlaar on Baxter's Q-operators for the open XXZ chain, via an isomorphism of infinite-dimensional Borel subalgebra tensor product representations. Although the additional coideal subalgebras make the setting rather complex, the compatibility of the isomorphism with the boundaries can be shown.

Finally, Jan de Gier discussed joint work with Z. Chen and M. Wheeler on an application of Hecke algebra representation theory, expressed in terms of non-symmetric Macdonald polynomials, to multi-species ASEPs. Dualities between such models arise as particular cases of quantum KZ equations and allow for the exact computations of certain observables.

Thursday: Special functions and integrable systems. On the penultimate day we continued the theme of applications to integrability, with a stronger emphasis on the role of special functions. The moderator Simon Ruijsenaars presented a bird's eye view on Calogero-Moser-Sutherland type and Toda type models and important classes of hypergeometric and other special functions.

Marta Mazzocco discussed recent joint work with L. Chekhov and V. Rubtsov on the generalized Sklyanin-Painlevé algebra, a Calabi-Yau algebra with desirable properties whose rational degenerations include many known algebras with various geometric (del Pezzo surfaces) and special functions (Askey-Wilson polynomials) interpretations, which at the same time directly relate to the double affine Hecke

algebra of type $\widetilde{\mathrm{C}}_1$. This was followed by a talk by Masatoshi Noumi on collaborative work with E. Langmann and J. Shiraishi, which showcased two types of joint eigenfunctions for elliptic Ruijsenaars difference operators: symmetric deformations of Macdonald polynomials defined in a neighbourhood of the torus, and asymptotically free eigenfunctions.

Volker Schomerus talked about conformal field theories and the role of objects closely related to multivariate BC_2-symmetric Heckman-Opdam hypergeometric functions as partial waves in these theories. His contribution to these proceedings is a survey of recent works and works in progress connecting conformal field theories and integrable systems of Calogero-Moser type.

In the last talk of the day, Paul Zinn-Justin discussed the connecting role of solutions of quantum KZ equations in Brauer loop models, generalized "pipe dreams" and, through Maulik-Okounkov stable envelopes, Gröbner geometry. His joint submission with A. Garbali to this volume is on a related topic, namely the description of an isomorphism between the commutative trigonometric Feigin-Odesskii shuffle algebra and the center of the Hecke algebra, with an application in enumerative combinatorics.

Friday: Interactions. On the last day of the workshop connections between the various themes were considered. The moderator Milen Yakimov gave an introduction to the four talks of the day, which had wide-ranging topics.

In Bart Vlaar's talk on joint work with A. Appel, a formalism of cylindrical quasitriangularity for quantum symmetric pair coideal subalgebras of Kac-Moody type was discussed which generalizes results by M. Balagović and S. Kolb and produces universal solutions of twisted reflection equations. A classification of such cylindrical structures was conjectured in terms of pseudo-involutions and pseudo-fixed-point subalgebras; the Kac-Moody theory of this is featured in the joint submission of V. Regelskis and B. Vlaar.

This was followed by Yuri Berest, who talked about on algebras of quasi-invariants, which first appeared in work by Chalykh and Veselov on Calogero-Moser type systems. Y. Berest explained how his cohomological realization of this algebra generalizes a classical theorem of A. Borel on Weyl group invariants and pointed out the homotopy-theoretic nature of the construction, allowing for extensions to p-adic reflection groups.

Alexander Varchenko's presented recent joint work with R. Rimányi on a Selberg integral formula for finite fields $\mathbb{F}_p$ for an odd prime p, which is related to hypergeometric solutions of (differential) KZ equations modulo p. Two works of his have been included in this volume which are on closely related topics: one in which a determinantal formula is proven for solutions of KZ equations over $\mathbb{F}_p$ in the case the dimension of the solution space is maximal, and one which constructs polynomial solutions of these equations modulo p^s, with special attention paid to the p-adic limit $s \to \infty$.

In the workshop's last talk, given by Eric Rains, the double affine Hecke algebra featured once more, or rather its elliptic version. He explained how this new algebraic structure yields results on biorthogonal functions and the integrability of the BC_n-symmetric Hamiltonian due to J. F. van Diejen. This volume contains related work by E. Rains on families of filtered deformations of elliptic algebras (twisted homogeneous endomorphism rings of vector bundles on elliptic curves),

which computes various examples and constructs deformations from noncommutative del Pezzo surfaces.

3. A special occasion

On Wednesday December 9, 2020 we held an online celebration of the 50th birthday of Jasper Stokman, who holds these topics dear and who made fundamental contributions to them. Jasper was a PhD student at the Universiteit van Amsterdam under the supervision of Tom Koornwinder and during this time visited Kobe for 10 weeks in Spring 1997. After graduation in 1998, he was a postdoctoral researcher in Université Paris 6 and Université de Strasbourg. In 2000 he took up a position as KNAW fellow at the Universiteit van Amsterdam, which included a 3-month stay at MIT in 2002. At the UvA Jasper was appointed as an assistant professor in 2005 and an associate professor in 2007. In 2014 he was appointed professor of Lie Theory. He was also a professor by special appointment at Radboud Universiteit in the period 2012-2015.

Initially, Jasper worked on multivariable orthogonal polynomials and special functions, and their relation to quantum group structures and quantum Grassmannians. Later his attention shifted to noncompact quantum groups and related spherical functions, as well as fixed-point Lie subalgebras and the associated coideal subalgebras (reflection equation algebras). This in turn led him to topics in mathematical physics such as his work on (quantum) Knizhnik-Zamolodchikov equations and their application in the study of spin chains and loop models.

As part of this work he has also supervised many PhD students and post-docs, and has collaborated with researchers representing this wide array of topics. Jasper has also been very active in undergraduate and postgraduate teaching, and the organization of conferences, seminars and working groups in the Netherlands and beyond.

It is our pleasure to be able to include in this volume Jasper's recent work with Nicolai Reshetikhin on boundary Knizhnik-Zamolodchikov-Bernard operators and quantum Calogero-Moser spin chains, a new class of superintegrable systems. It reflects many of his interests and fits the themes of the workshop very well.

Erik Koelink
Stefan Kolb
Nicolai Reshetikhin
Bart Vlaar