Preface

The authors of the contributions to this volume were invited speakers at *The Coxeter Legacy—Reflections and Projections*, a conference held in Toronto in May 2004.

This collection is intended to capture the essence of the *Coxeter Legacy*—it is a mixture of survey, up-to-date information, history, storytelling, and personal memories. Each of the contributions is linked to some of Coxeter's achievements.

Donald Coxeter was able to create enthusiasm, even passion, for mathematics in people of any age, any background, any profession, any walk of life. Like Euclid, Coxeter was enchanted by Euclidean geometry—this means he was interested in the beauty, in the description, and in the exploration of the world around us, the world we live in. Although Coxeter's ideas are rooted in geometry, they often have wider implications.

In 1933, Coxeter proved that

> *every finite group of the form* $R_i^2 = (R_i R_j)^{k_{ij}} = 1$ *can be generated by reflections in the bounding primes of a spherical simplex.*

The *Coxeter group* was born, its significance established, and a fruitful link between algebra and geometry discovered.

The quoted result, combined with others previously obtained by Coxeter, yields

The complete enumeration of finite groups of the form

$$R_i^2 = (R_i R_j)^{k_{ij}} = 1,$$

which is the title of a paper by Coxeter, published in 1935 in the *Journal of the London Mathematical Society.* In it, Coxeter points out that there are only very few types of irreducible Coxeter groups. These can be described conveniently by *Coxeter diagrams*, a concept he had introduced a few years earlier.

Three decades after their appearance, Coxeter groups acquired their name. The algebraic aspect of a Coxeter group W was developed by *Tits* and *Bourbaki*, always stressing the set of generating involutions, a feature that is clearly visible already in Coxeter's approach. It is advantageous to talk about a *Coxeter system* (W, S), where W is a Coxeter group and S is a set of generating involutions. Interpreting these involutions as reflections introduces the powerful tool of *visualization* that Coxeter handles masterfully.

In general, a Coxeter group may have infinitely many elements. This happens when some of the exponents k_{ij} are infinite.

Mühlherr surveys the known results on the isomorphism problem for Coxeter groups: Let W be a Coxeter group determined by a set of generators $\{R_i\}$ and a set of exponents $\{k_{ij}\}$. Let W' be defined similarly. When are these two Coxeter groups isomorphic? Mühlherr's report exhibits vibrant activity by several researchers working together on this problem.

Borovik surprises with an unusual point of view. He reveals many hidden connections, stresses the visualization, gives guidance to further research, suggests a palindromic approach to Coxeter groups, and brings to life the magical power of Coxeter's charisma. He writes, "One of the attractive features of the Coxeter Theory is that it is saturated with beautiful examples."

Ronan sheds light on the central role that Coxeter groups play in several branches of mathematics. They are connected with the root systems in groups of Lie type. Coxeter groups are used to define apartments which in turn are the essential ingredient in Tits's approach on buildings. Ronan gives a fast-moving account of the history of group theory, touching on many important developments, and brings us up to date on the research in buildings.

Kostant deals with the question of how an arbitrary (unitary) representation of $SU(2)$ decomposes under the action of a finite subgroup Γ of $SU(2)$. Coxeter elements and Coxeter diagrams are essential tools.

Kellerhals brings together a wealth of information on activities, some of which have their roots in Coxeter's work. In 1934, Coxeter classified elliptic and parabolic Coxeter groups; however, hyperbolic Coxeter groups were only partially classified. Hyperbolic Coxeter simplexes exist only in spaces of dimension 9 or less. Some bounds exist for more general situations. Some results on the density of sphere packings that are recorded here rely on a 1954 paper of Coxeter.

Regular polytopes were Coxeter's love. He had a life-time fascination with them. He shared his encyclopaedic knowledge with us in many papers, culminating in two books, *Regular Polytopes*, which went through three editions, and *Regular Complex Polytopes*. Once, after a talk at the Fields Institute in which he discussed many details of a great number of polytopes, when I admired his knowledge of them, he explained, "You see, they are all old friends." Coxeter passed this enthusiasm on to his students and admirers.

Coxeter's devotion to classical polytopes is unsurpassed. His achievements have been praised by Grünbaum, McMullen, and Ivić Weiss in the *Notices of the AMS* of November 2003. These authors also stress the role that Coxeter groups and certain of their quotient groups play as symmetry groups of polytopes.

More recently, the purely combinatorial aspect of regular polytopes has become a centre of attention. A comprehensive record on this subject has been given by *McMullen* and *Schulte* in their book *Abstract Regular Polytopes*, published in 2002.

McMullen and *Schulte* in their contribution here discuss progress on abstract regular polytopes, in particular on their faithful realizations, since the publication of their book.

Monson and *Weiss* offer a concise history of several of Coxeter's interests, objects of study, and achievements. These include regular maps, abstract polytopes, and chiral polytopes. The latter are polytopes whose automorphism groups have precisely two flag orbits with adjacent flags in distinct orbits. In their section on graphs, they tell the charming "Coxeter Graph—My Graph" story.

Wills presents an interesting account on today's knowledge of equivelar polyhedra. He formulates a number of open problems on them. An equivelar polyhedron is a polyhedral 2-manifold with planar faces embedded in Euclidean 3-space such that all faces are p-gons and all vertices are q-valent.

Khovanskii deals with the combinatorial theory of polytopes. His main result generalizes an estimate of Nikulin by giving an upper bound on the average number of l-dimensional faces on k-dimensional faces of an n-dimensional polytope simple at the edges.

Senechal tells the story of crystals that do not satisfy the crystallographic conditions; they have axes with 5-fold symmetry. She starts gently with the 5-fold symmetry of an apple core, one of Coxeter's conversation pieces. She proceeds to tilings with an axis of 5-fold symmetry, which are of necessity nonperiodic, and subsquently moves to Penrose kites and darts. Do the Coxeter groups of the noncrystallographic types H_3 and H_4 hold the key to understanding quasicrystals?

Configurations are the essence of Euclidean geometry, so it is not surprising that Coxeter embraced topics involving configurations, and his work stimulated further research.

Grünbaum gives a comprehensive account of configurations of points and lines. He conveys the excitement of the development of the concepts, of missteps, of great progress, periods of flourishing, lulls, and of the reinvigoration of the topic by a Coxeter paper. Grünbaum is largely concerned with realization and representation, i.e., loosely speaking, with the interpretation of combinatorial configurations in the real Euclidean plane.

Richter-Gebert was inspired by a proof in the book *Geometry Revisited* by Coxeter and Greitzer, where six Menelaus configurations are combined to prove the theorem of Pappus. Richter-Gebert has elevated this method to an art. Multiplying certain expressions that occur in the statements of the theorems of Ceva and Menelaus miraculously leads to a new theorem of Euclidean geometry.

Schattschneider concentrates on the interaction of Coxeter with artists and modelmakers. Preeminent is, of course, the long and fruitful friendship between Coxeter and Escher. She also mentions that Pelletier produced the beautiful 3-D shadow of the 4-D dodecahedron that adorns the ceiling above the staircase of the atrium in the Fields Institute in Toronto.

Emmer enthusiastically describes filming Escher's art with Coxeter—related to geometry, of course! Emmer includes many illuminating and revealing quotes from the correspondence on art and mathematics between Coxeter and him and also between Coxeter and Escher.

Coxeter's involvement in art and with artists earned him admiration and adoring friends in the intellectual community all over the globe. Coxeter's devotion to polytopes lives on in his students and entranced followers. Coxeter groups can arise in various subjects in applied mathematics, and they have a permanent place in some of the most demanding and fascinating branches of abstract mathematics, such as Lie algebras, algebraic groups, Chevalley groups, and Kac-Moody groups.

Coxeter's legacy is much larger than shown here. For instance, we have not mentioned his investigations into optimally dense packings of spheres. A simple search reveals that for the year 2004 there are 110 "Coxeter" entries in *Mathematical Reviews*—giving testimony to Coxeter's everlasting popularity.

Toronto, April 2005 Erich W. Ellers

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