Preface to the First Edition

These notes grew out of a series of lectures I presented at the Centre de recherches mathématiques in Montreal entitled 'A survey of the Hodge conjecture', during the summer of 1990. The intent of this lecture series was to introduce non-specialists and advanced graduate students to a topic of central interest in analytic and algebraic geometry, namely the celebrated Hodge conjecture. This conjecture is a generalization of the well known Lefschetz $(1,1)$ theorem which was first proven by S. Lefschetz by analytic and topological methods, appearing in his famous 1924 monograph "L'*analysis situs* et la géométrie algébrique". One of the main ingredients in the formulation of the Hodge conjecture is the Hodge (p,q) decomposition theorem which asserts that for a compact complex Kähler manifold X [e.g. X projective algebraic], the decomposition of differential forms on X into those with $p\,dz$'s and $q\,d\overline{z}$'s descends to the cohomology level, viz $H^*(X,\mathbb{Q})\otimes\mathbb{C} = \bigoplus_{p+q=*} H^{p,q}(X)$. The statement of what is commonly understood to be the Hodge conjecture [popular version] is as follows. Let X be a projective algebraic manifold and p a positive integer. Also let $H^{2p}(X,\mathbb{Q})_{\mathrm{alg}} \subset H^{2p}(X,\mathbb{Q})$ be the subspace of algebraic cocycles, i.e. the $\mathbb{Q}$-vector space generated by the fundamental classes of algebraic subvarieties of codimension p in X. The Hodge conjecture asserts that one can 'compute' the subspace $H^{2p}(X,\mathbb{Q})_{\mathrm{alg}}$ using Hodge theory, specifically $H^{2p}(X,\mathbb{Q})_{\mathrm{alg}} = H^{p,p}(X) \cap H^{2p}(X,\mathbb{Q})$.

The original version of the Hodge conjecture appeared in Hodge's 1950 International Congress of Mathematics address[1]. After some minor amendments (due to Atiyah-Hirzebruch and Grothendieck), what is now referred to as the *general* Hodge conjecture includes the aforementioned popular version as a special case. Despite a substantial increase in example cases (since 1950) where the Hodge conjecture has been verified, it is fair to say that little progress has been made on the conjecture in general. Indeed it is not clear whether or not the conjecture should be true, and there are those who have attempted to find counterexamples.

The intent of these notes is to provide the reader with a concrete approach to the Hodge conjecture (in the general and popular form) and from the point of view of a transcendental algebraic geometer. It is expected that the reader has some familiarity with differential geometric concepts such as Stoke's theorem and de Rham cohomology, and a course in algebraic geometry including topics such as rational maps, Zariski's main theorem, the degree of a variety and Bezout's theorem. Beyond Lecture 11, it may also be helpful to have a basic understanding of the concept of a scheme, Chow varieties and Hilbert schemes. These notes are far from being polished, and may require perseverance on the part of the reader.

[1] Reference [**1**] to Lecture 7. It is worthwhile noting that his conjecture is stated there as an important open problem rather than a 'formal' conjecture; however there are indications from his published work which support the idea that he viewed the problem as a conjecture.

None the less, to cut down on prerequisites and to spare the reader an endless search for details in the literature, many of the proofs of standard (and not so standard) results have been sketched either in the lectures or in the appendices to these lectures.

An outline of these lectures is as follows. After discussing some preliminary background material in Lectures 1 through 4, the statement of the Hodge conjecture [popular version] and proof of the Lefschetz $(1,1)$ theorem (using the well known exponential short exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$) appears in Lecture 5, as well as a counterexample for certain non-algebraic complex tori. A sketch of Lefschetz's proof of his $(1,1)$ theorem is given in Lecture 6, which was included in these notes not only for its intrinsic beauty, but also as a forerunner to a discussion of Griffiths' program in Lecture 14. The Griffiths program also requires an understanding of intermediate Jacobians (vis-à-vis the discussion of normal functions), which justifies the inclusion of this topic in Lecture 12. The formulation of the general Hodge conjecture, as well as a list of examples and various techniques appear in Lectures 7, 13 and 14. A discussion of Chern class theory and some applications appears in Lectures 8 and 9. For example the Hodge numbers of complete intersections are computed (in terms of a power series). We also provide a treatment of an explicit description of the Dolbeault cohomology of hypersurfaces via residues of rational differential forms on $\mathbf{P}^n$. A proof of the Hodge theorem along the line of reasoning given by Cornalba and Griffiths appears in Lectures 10 and 11. Other topics covered in Lectures 12 through 15 include functoriality of the Abel-Jacobi mapping, the Noether-Lefschetz theorem, non-triviality of the Griffiths group, the main standard Lefschetz conjecture, and a discussion and generalization of Mumford's famous theorem on the Chow group of zero cycles on a surface. A relationship between the general Hodge conjecture and and the structure of Chow groups of a certain degree is given in Lecture 15.

I would like to express my warmest appreciation to Professor Robert Langlands for inviting and encouraging me to present this series of talks, and to my faithful audience who made my efforts all the more worthwhile. I would also like to thank Professor Francis Clarke and the staff at the C.R.M. for making my stay very agreeable. A special thanks to Professor Larry Roberts for a splendid job in reading my first draft, during a period of pressing obligations. The elimination of numerous errors and improvements in style are largely due to him. I am also grateful to Professor Chad Schoen for reading an earlier version of these notes, and for providing some useful suggestions. Finally, I wish to thank my wife, Jane for a splendid and efficient typing job in converting a rough manuscript into its present form.

James D. Lewis
Edmonton
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Preface to the Second Edition

I would like to give a thanks to the many colleagues who have expressed satisfaction with the first edition, and the need for such a book. Apart from some minor changes, additions, and improvements in style, this second edition includes the first printing in its original form. To account for some of the recent developments, I have included some notes appended to Lectures 13 and 15, an Appendix A (dealing with singular varieties), together with Appendix B, written by Professor B. Brent Gordon, which contains a detailed account of the work of the Hodge conjecture for Abelian varieties.

There are a number of topics we would have liked to have discussed, but have omitted entirely. For example, there is P. Deligne's theory of absolute Hodge cycles (which appears in the Springer Lecture Notes in Math., vol. 900). We have also excluded a discussion of Deligne cohomology, motives and regulators, and the related Hodge-$\mathcal{D}$-conjecture.

I am very grateful to Brent Gordon for generously providing this appendix, and to the **C.R.M.** for giving me the opportunity to write this second edition.

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