

Introduction

A prototype for the type of convexity theorem that we will be discussing in this monograph is a theorem about Hermitian matrices which was proved by Horn [36] in the mid 1950's: let \mathcal{H}_λ^n be the set of $n \times n$ matrices whose eigenvalues are the numbers, $\lambda_1, \lambda_2, \dots, \lambda_n$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, and for each $A \in \mathcal{H}_\lambda^n$, let $a_{11}, a_{22}, \dots, a_{nn}$ be the diagonal entries of A . The assignment,

$$A \longmapsto (a_{11}, a_{22}, \dots, a_{nn}),$$

defines a mapping $\Phi: \mathcal{H}_\lambda^n \rightarrow \mathbb{R}^n$ and Horn's theorem asserts that the image of this mapping is a convex polytope. More explicitly it asserts that the image of Φ is the convex hull of the vectors

$$\lambda_\sigma = (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \dots, \lambda_{\sigma(n)}),$$

where σ ranges over the set of permutations of $\{1, 2, \dots, n\}$.

In the early 1970's Kostant showed that this result was a special case of a more general result having to do with coadjoint orbits of Lie groups. Specifically, let G be a compact connected Lie group, T the Cartan subgroup of G and \mathfrak{g} and \mathfrak{t} their Lie algebras. The adjoint action of G on \mathfrak{g} dualizes to give an action of G on \mathfrak{g}^* and the coadjoint orbits are by definition the orbits of this action. Let $\pi: \mathfrak{g}^* \rightarrow \mathfrak{t}^*$ be the transpose of the inclusion map, $\mathfrak{t} \rightarrow \mathfrak{g}$. If $O \subseteq \mathfrak{g}^*$ is a coadjoint orbit, then by restricting π to O one gets a map,

$$(I.1) \quad \Phi: O \longrightarrow \mathfrak{t}^*,$$

and Kostant's theorem asserts that the image of Φ is a convex polytope. In fact, if O^T is the set of T -fixed points on O , Φ maps O^T bijectively onto an orbit of the Weyl group $N(T)/T$, and the image of Φ is the convex hull of the points on this orbit.

This result has a formulation which involves ideas from symplectic geometry. Namely by a theorem of Kirillov and Kostant, the coadjoint orbits, O , of G are symplectic manifolds, and the action of G on O preserves the symplectic form. Moreover, if G is compact and connected, these O 's are the *only* symplectic G -manifolds on which G acts transitively. In addition, the action of G on O is *Hamiltonian*: for $\xi \in \mathfrak{g}$ the action of G on O associates with ξ a vector field ξ_O on O and this vector field is a Hamiltonian vector field. In fact, if ω_O is the symplectic form on O , the interior product of ξ_O with ω_O is di^*l^ξ where $i: O \rightarrow \mathfrak{g}^*$ is the inclusion map and l^ξ the linear

functional on \mathfrak{g}^* coming from the pairing on $\xi \in \mathfrak{g}$ with elements of \mathfrak{g}^* . In other words, in the language of Section 1.1 below, *the action of G on O is a Hamiltonian action with moment map, $i: O \rightarrow \mathfrak{g}^*$* . The restriction of this action to the torus T is also a Hamiltonian action and its moment mapping is the mapping (I.1), so what Kostant's theorem asserts is that the image of this moment mapping is a convex polytope, and, more explicitly,

$$\Phi(O) = \text{conv } \Phi(O^T).$$

(Here $\text{conv } A$ denotes the convex hull of a subset A of a real vector space.) In the early 1980's it was shown by Atiyah [4] and Guillemin–Sternberg [28] that, in this symplectic version of Kostant's theorem, one can drop the assumption that O is a transitive symplectic G -space, and, in fact, get rid of the role of G entirely. Their result asserts that if M is a compact symplectic manifold, T an n -torus and $T \times M \rightarrow M$ a Hamiltonian action of T on M , and if $\Phi: M \rightarrow \mathfrak{t}^*$ is the moment map associated with this action, then

- (i) $\Phi(M^T)$ is a finite subset of \mathfrak{t}^* and
- (ii) $\Phi(M) = \text{conv } \Phi(M^T)$.

In particular the image $\Phi(M)$ is a convex polytope. We will henceforth refer to this result as the *abelian* convexity theorem (and we will sketch a proof of it in Section 1.1).

Three years after this theorem was proved, Frances Kirwan proved a much deeper *non-abelian* convexity theorem. Let G be a compact connected Lie group which is not necessarily abelian. If G acts in a Hamiltonian fashion on a compact symplectic manifold then, as above, one has a moment map

$$(I.2) \quad \Phi: M \longrightarrow \mathfrak{g}^*,$$

but in general its image is not convex. However, a much more subtle convexity result *is* true: let \mathfrak{g}^*/G be the orbit space for the coadjoint action of G on \mathfrak{g}^* . If $W := N(T)/T$ is the Weyl group of G , the action of W on \mathfrak{t} gives by duality an action of W on \mathfrak{t}^* , and the orbit spaces, \mathfrak{g}^*/G and \mathfrak{t}^*/W , are isomorphic. Let us fix a (closed) Weyl chamber, \mathfrak{t}_+^* , in \mathfrak{t}^* . This is a fundamental domain for the action of W on \mathfrak{t}^* , so one has identifications $\mathfrak{g}^*/G \cong \mathfrak{t}^*/W \cong \mathfrak{t}_+^*$ and hence from (I.2) a map

$$(I.3) \quad \Phi_+: M \longrightarrow \mathfrak{t}_+^*.$$

The Kirwan convexity theorem asserts that the image of *this* map is a convex polytope.

This theorem, which was proved by Kirwan [41] in 1984, is the main topic of this monograph. We will sketch below several proofs of it: in particular, in Chapter 2 we will describe some of the ingredients that came into Kirwan's original proof, and in Section 1.3 we will outline a short and relatively simple proof which dates from the mid 90's and is due to Lerman, Meinrenken, Tolman and Woodward.

Our main concern in this monograph will be with *constructive* versions of Kirwan's theorem. One of the defects of this theorem is that, unlike

the abelian convexity theorem, it does not come with an explicit description of the image of Φ in \mathfrak{t}_+^* . However, in a number of concrete examples such explicit descriptions *have* been found. For example, suppose M is a Hamiltonian G -manifold and X an orbit of G in M . Then from the equivariant Darboux theorem, one gets a canonical model M_{can}^X for the action of G in a G -invariant neighborhood of X , and for this canonical model there is a constructive version of the convexity theorem which we will describe in Chapter 2. Moreover, by coupling this with a Morse theory result which Kirwan uses in her proof we will obtain in Chapter 2 a “locally” constructive convexity theorem. This theorem is due to Sjamaar [60].

Chapter 3 is devoted to two special cases of the convexity theorem, both involving (like the theorem of Horn which we described above) isospectral sets of Hermitian matrices. The first of these is the Kirwan theorem for the action of $\mathbf{U}(n-1)$ on a generic coadjoint orbit of $\mathbf{U}(n)$. In its constructive form it asserts that the projection of \mathcal{H}^n onto \mathcal{H}^{n-1} which assigns to each Hermitian $n \times n$ matrix its $(n-1) \times (n-1)$ minor, maps \mathcal{H}_λ^n onto the set

$$\bigcup_{\mu} \mathcal{H}_\mu^{n-1},$$

where the union is over all $n-1$ -tuples μ such that the μ_i ’s intertwine the λ_i ’s, i.e. $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for $i = 1, 2, \dots, n-1$. We will also discuss some tie-ins of this result with Gelfand–Cetlin theory and with a topic dear to the hearts of nineteenth century geometers: the theory of confocal quadrics.

The second example of Kirwan’s theorem we will discuss in Chapter 3 concerns the diagonal action of $\mathbf{U}(n)$ on the product of two coadjoint orbits of $\mathbf{U}(n)$. The Hermitian matrix version of this theorem asserts that if μ and ν are n -tuples of real numbers, the set of λ ’s satisfying

$$(I.4) \quad \mathcal{H}_\lambda^n \subseteq \mathcal{H}_\mu^n + \mathcal{H}_\nu^n$$

is a convex polytope. The “constructive” form of this result turns out to be much harder to prove (and also much harder to formulate) than the result for $(n-1) \times (n-1)$ minors: it is, in fact, only in the last decade that a completely satisfactory description of this moment polytope was obtained, largely due to the efforts of Klyachko [42]. A set of necessary conditions for λ to satisfy (I.4) can be obtained by mini-max and Morse theoretical arguments (these we will describe in Section 3.5) and Klyachko’s great achievement was to show that these conditions are sufficient as well as necessary.¹

The “constructive” Kirwan problem for products of coadjoint orbits of $\mathbf{U}(n)$ can be reformulated as follows: if O is a coadjoint orbit of $\mathbf{U}(n) \times \mathbf{U}(n)$ and $\mathbf{U}(n) \rightarrow \mathbf{U}(n) \times \mathbf{U}(n)$ is the diagonal imbedding, what is the moment polytope associated with the action of $\mathbf{U}(n)$ on O ? This formulation admits of the following generalization: if G and H are compact Lie groups, $i: H \rightarrow G$

¹It was pointed out by Woodward that some of these conditions are redundant. Refinements of Klyachko’s results, eliminating these redundancies, have been obtained by Belkale [8], Knutson–Tao [43], and Knutson–Tao–Woodward [44].

an embedding of H in G , and O a coadjoint orbit of G , what is the moment polytope associated with the action of H on O ? Berenstein and Sjamaar answered this question in [9]. In the last section of Chapter 3 we will describe a set of inequalities which, they showed, characterize this polytope; and we will also prove the easy part of their result: the necessity of these inequalities.

In Chapter 4 we will discuss yet another version of the Kirwan theorem. Let M be a compact Kähler manifold and $T \times M \rightarrow M$ a Hamiltonian action of T on M with moment map Φ . If this action preserves the complex structure, then the action of the n -torus $T = (S^1)^n$ on M can be extended to a holomorphic (but non-Hamiltonian) action of the complex torus $T^{\mathbb{C}} = (\mathbb{C}^{\times})^n$, where \mathbb{C}^{\times} is the multiplicative group of complex numbers. Atiyah proved in [4] a “local” form of the abelian convexity theorem for this $T^{\mathbb{C}}$ -action: if $T^{\mathbb{C}}p$ is the orbit of $T^{\mathbb{C}}$ through $p \in M$, the moment image of the closure of $T^{\mathbb{C}}p$ is a convex polytope. Moreover, this polytope is the convex hull of the moment image of the set

$$\overline{T^{\mathbb{C}}p} \cap M^T.$$

A non-abelian generalization of this fact, due to Brion [16], concerns Kählerian actions of a compact connected Lie group, G , on a compact Kähler manifold, M . As above such an action extends to a holomorphic action of the complex Lie group, $G^{\mathbb{C}}$, and Brion’s result asserts that if Φ_+ is the composite moment mapping (I.3), the image with respect to Φ_+ of the $G^{\mathbb{C}}$ -orbit through an arbitrary point of M is convex. A somewhat related result concerns the Borel subgroup B of $G^{\mathbb{C}}$ associated to the *opposite* chamber $-\mathfrak{t}_+^*$. We will prove that the image with respect to $\Phi(Bp)$ of the B -orbit through a point p of M intersects the open chamber $\text{Int } \mathfrak{t}_+^*$ in a convex set and that this intersection is contained in the intersection

$$(I.5) \quad \text{Int } \mathfrak{t}_+^* \cap \bigcap_{b \in B} \Phi(T^{\mathbb{C}}bp).$$

Moreover, if M is a projective variety, $\Phi(Bp) \cap \text{Int } \mathfrak{t}_+^*$ is *equal* to the intersection (I.5) and, in particular, is a convex polytope. In future work we will show that the set (I.5) is a lower semicontinuous function of p , and from this local convexity theorem get a “constructive” version of Kirwan’s theorem for Kähler manifolds which is rather different in spirit from those we previously described.

In the pages above we have surveyed the contents of Chapters 2–4. It remains to say a few words about the material in Chapters 1 and 5. In the last two decades the convexity theorem has been generalized in a number of predictable ways and, in a few instances, in some completely unanticipated ways. There are now, for instance, convexity theorems for Hamiltonian actions of compact Lie groups on non-compact symplectic manifolds, for Hamiltonian actions of non-compact Lie groups, for Poisson actions of Lie groups on Poisson manifolds, and for quasi-Hamiltonian actions of Lie groups. For non-compact groups these results are quite complicated, and

for this reason it is rather surprising that some of the simplest and most elegant generalizations of the convexity theorem have to do with the action of *infinite-dimensional* groups, e.g. loop groups and groups of gauge transformations, on infinite-dimensional symplectic manifolds. We will attempt to give a brief account of these results, without getting too bogged down in details, in Chapter 1.

In Chapter 5 we will discuss some applications of the convexity theorem. One particularly beautiful application is the Delzant theorem [23]: let M be a compact Hamiltonian T -manifold. If T acts faithfully on M then the inequality,

$$(I.6) \quad \dim M \geq 2 \dim T,$$

holds, and if this inequality is an equality the T -action is called a “completely integrable” or “toric” action. What Delzant proved is that for such actions M is determined, up to a T -equivariant symplectomorphism, by its moment polytope. One of the most intriguing outstanding questions about Hamiltonian actions of Lie groups has to do with the non-abelian analogue of this result. If G is a compact Lie group and M a connected G -manifold, there exists a dense open subset, U , of M with the property that, for all p and q in U , the stabilizer groups, G_p and G_q , are conjugate in G , i.e. all points in U have the same orbit type. The group, G_p , (which is unique up to conjugacy) is called the *principal isotropy group* of the action; and if this group is discrete, there is an analogue of the inequality (I.6), namely

$$\dim M \geq \dim G + \text{rank } G.$$

If this inequality is an equality the action of G is called *multiplicity-free*; and for such actions Delzant conjectured that M is determined up to isomorphism by its moment polytope and its principal isotropy group. This conjecture is still unsettled, but some partial results, which we will report on in Section 5.1, indicate that it is very likely to be true.

The other application of the convexity theorem which we will discuss in Chapter 5 has to do with Kählerizability. Up until a few years ago the following seemed to be a highly plausible conjecture: let M be a compact Hamiltonian T -manifold for which the fixed point set, M^T , is finite. Then M admits a T -invariant complex structure which is compatible with its symplectic structure. What made this conjecture seem plausible is a theorem of Białynicki-Birula [10, 11], which asserts that a nonsingular complex projective variety equipped with a torus action with finitely many fixed points admits a decomposition into affine spaces. In particular, such a variety is birationally equivalent to projective space. In view of this it seemed unlikely that dropping the Kähler assumption could complicate this birational classification. In 1995 Sue Tolman found a counterexample which demolished this conjecture; and as we will describe in Section 5.2, the key ingredient in the

proof of the non-Kählerizability of her example is a corollary of Atiyah's convexity theorem for $T^{\mathbb{C}}$ -orbits, which imposes some constraints on the shape of the moment polytope when the action is a Kähler action.

One aspect of moment geometry which we have not discussed in this monograph, and whose absence we regret, is Duistermaat–Heckman theory. From this theory one sees that the moment polytope has a lot of additional structure which we have neglected to mention: in particular it decomposes into a disjoint union of open convex subpolytopes, called action chambers, and associated with each of these action chambers is a polynomial: the Duistermaat–Heckman polynomial. Fortunately there are many good expositions of this subject available. In particular, we recommend the account of Duistermaat–Heckman theory by Michèle Audin in [7] and, for an infinite-dimensional version, Atiyah's article [5].