CHAPTER 3

The $S_N^{(\beta)}(x,y)$ Kernel and Christoffel – Darboux Formulas

In this chapter, we evaluate the sum $S_N^{(\beta)}(x,y)$ for $\beta = 1, 2$ and 4. $S_N^{(2)}(x,y)$ is a sum over products of orthogonal polynomials (1.18) and its solution has been known for decades. It is called the Christoffel – Darboux formula (see for example [77]). We compute $S_N^{(\beta)}(x,y)$, $\beta = 1$ and 4 [36, 38, 39], which are sums over products of skew-orthogonal functions (1.22). We call them the generalized Christoffel – Darboux formula. Apart from improving our understanding of the theory of skew-orthogonal functions, these results make calculations of OE and SE more transparent compared to the method used so far using Sener – Verbaarschot – Widom formalism [72, 78].

1. Method of orthogonal polynomials

The correlation function of the unitary ensemble of random matrices ($\beta = 2$) can be written as (see Chapter 2 for details)

$$R_n^{(2)}(x_1, \ldots, x_n) = \det[S_N^{(2)}(x_j, x_k)]_{j,k=1,\ldots,n}, \tag{3.1}$$

where $S_N^{(2)}(x,y)$ is the kernel function

$$S_N^{(2)}(x,y) = \sum_{j=0}^{N-1} \phi_j^{(2)}(x)\phi_j^{(2)}(y), \qquad \phi_j^{(2)}(x) = \sqrt{\frac{w(x)}{h_j}}P_j(x), \tag{3.2}$$

the $P_j(x)$ are orthogonal polynomials (1.1) of order j, $w(x)$ the weight function with respect to which they are defined, and h_j the normalization constant. The Christoffel – Darboux formula expresses (3.2) as a linear combination of products of only $\phi_N^{(2)}(x)$ and $\phi_{N-1}^{(2)}(x)$. This result is useful in studying the statistical properties of unitary ensembles of random matrices.

2. Christoffel – Darboux formula: $\beta = 2$

We begin this section with an alternative derivation of the Christoffel – Darboux formula (see the classical book of Szegő [77] for the conventional proof).

Proof. We define semi-infinite matrices $\Phi^{(2)}(x)$, comprising weighted orthogonal polynomials $\phi_n^{(2)}(x)$:

$$\Phi^{(2)}(x) := (\phi_0^{(2)}(x) \ldots \phi_n^{(2)}(x) \ldots)^{\mathrm{t}}, \qquad {}^{\mathrm{t}} = \text{transpose}. \tag{3.3}$$

We write $S_N^{(2)}(x,y)$ of (3.2) as

$$S_N^{(2)}(x,y) = \Phi^{(2)\mathrm{t}}(x) \prod_N \Phi^{(2)}(y), \tag{3.4}$$

where we use the diagonal matrix $\Pi_N = \mathrm{diag}(\underbrace{1,\ldots,1}_{N},0,\ldots,0)$ to make the sum finite.

Orthogonality ensures that they satisfy a three-term recursion relation. This is the same as saying

$$\boxed{x\Phi^{(2)}(x) = Q^{(2)}\Phi^{(2)}(x)} \tag{3.5}$$

where $Q^{(2)}$ is the well-known Jacobi matrix. Using the orthonormality condition, we get

$$Q^{(2)}_{j,k} = \int_{\mathbb{R}} x\phi^{(2)}_j(x)\phi^{(2)}_k(x)\ \mathrm{d}x = Q^{(2)}_{k,j} \implies Q^{(2)} = Q^{(2)\mathrm{t}}. \tag{3.6}$$

From equations (3.4), (3.5), and (3.6), we get

$$\begin{aligned}(x-y)S_N^{(2)}(x,y) &= (x-y)\Phi^{(2)\mathrm{t}}(x)\prod_N\Phi^{(2)}(y)\\ &= \Phi^{(2)\mathrm{t}}(x)Q^{(2)\mathrm{t}}\prod_N\Phi^{(2)}(y) - \Phi^{(2)\mathrm{t}}(x)\prod_N Q^{(2)}\Phi^{(2)}(y)\\ &= \Phi^{(2)\mathrm{t}}(x)\Big[Q^{(2)}\prod_N - \prod_N Q^{(2)}\Big]\Phi^{(2)}(y)\\ &= \Phi^{(2)\mathrm{t}}(x)\Big[Q^{(2)},\prod_N\Big]\Phi^{(2)}(y).\end{aligned} \tag{3.7}$$

Thus we have

$$\boxed{S_N^{(2)}(x,y) = \frac{\Phi^{(2)\mathrm{t}}(x)[Q^{(2)},\Pi_N]\Phi^{(2)}(y)}{(x-y)}.}$$

In terms of the matrix elements, this boils down to

$$\begin{aligned}S_N^{(2)}(x,y) &= Q^{(2)}_{N-1,N}\left(\frac{\phi^{(2)}_N(x)\phi^{(2)}_{N-1}(y) - \phi^{(2)}_N(y)\phi^{(2)}_{N-1}(x)}{x-y}\right),\\ Q^{(2)}_{N-1,N} &= \frac{k_{N-1}}{k_N}\sqrt{\frac{h_N}{h_{N-1}}},\end{aligned} \tag{3.8}$$

where k_N is the coefficient of x^N in $P_N(x)$ and h_N is the normalization constant. □

3. Method of skew-orthogonal polynomials

For an ensemble of $2N \times 2N$ random matrices invariant under the orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$) transformations, we rewrite (1.14) as

$$R_n^{(\beta)}(x_1,\ldots,x_n) = \frac{2N!}{(2N-n)!}\int \mathrm{d}x_{n+1}\cdots\int \mathrm{d}x_{2N}\ P_{\beta,N}(x_1,x_2\ldots,x_{2N}), \quad n = 1,2,\ldots. \tag{3.9}$$

As discussed in Chapter 2, for $\beta = 1$ and 4, the evaluation of such integrals requires the joint probability distribution $P_{\beta,N}(x_1,x_2,\ldots,x_{2N})$ to be written in terms of quaternion determinants (i.e., determinant of a matrix, each of whose element is a 2×2 quaternion) satisfying certain properties. From this, one can calculate (3.9). For example, the n-point correlation function $R_n^{(\beta)}(x_1,\ldots,x_n)$, and the level density $R_1^{(\beta)}(x)$, $(n = 1)$ is given by

$$R_n^{(\beta)}(x_1,\ldots,x_n) = \mathrm{Qdet}[Q^{(\beta)}_{2N}(x_j,x_k)]; \qquad j,k = 1,\ldots,n,\ \beta = 1,4, \tag{3.10}$$

where

$$Q_{2N}^{(\beta)}(x,y) = \begin{pmatrix} S_{2N}^{(\beta)}(x,y) & D_{2N}^{(\beta)}(x,y) \\ I_{2N}^{(\beta)}(x,y) - \delta_{1,\beta}\epsilon(x-y) & S_{2N}^{(\beta)}(y,x) \end{pmatrix}; \tag{3.11}$$

$$R_1^{(\beta)}(x) := S_{2N}^{(\beta)}(x,x).$$

Here δ is the Kronecker delta, introduced to write the results of the two invariant classes of ensembles under one banner. In terms of skew-orthogonal functions $\phi_n^{(\beta)}(x)$ and $\psi_n^{(\beta)}(x)$, we rewrite the expressions of (1.22) as

$$S_{2N}^{(\beta)}(x,y) := \sum_{j,k=0}^{2N-1} Z_{j,k}\phi_j^{(\beta)}(x)\psi_k^{(\beta)}(y) = \widehat{\Psi}^{(\beta)}(y) \prod_{2N} \Phi^{(\beta)}(x) = -\widehat{\Phi}^{(\beta)}(x) \prod_{2N} \Psi^{(\beta)}(y), \tag{3.12}$$

$$D_{2N}^{(\beta)}(x,y) := -\sum_{j,k=0}^{2N-1} Z_{j,k}\phi_j^{(\beta)}(x)\phi_k^{(\beta)}(y) = \widehat{\Phi}^{(\beta)}(x) \prod_{2N} \Phi^{(\beta)}(y), \tag{3.13}$$

$$I_{2N}^{(\beta)}(x,y) := \sum_{j,k=0}^{2N-1} Z_{j,k}\psi_j^{(\beta)}(x)\psi_k^{(\beta)}(y) = -\widehat{\Psi}^{(\beta)}(x) \prod_{2N} \Psi^{(\beta)}(y), \tag{3.14}$$

$$S_{2N}^{(\beta)}(y,x) = S_{2N}^{\dagger(\beta)}(x,y) = \widehat{\Phi}^{(\beta)}(x) \prod_{2N} \Psi^{(\beta)}(y). \tag{3.15}$$

where $\phi_n^{(\beta)}(x)$ and $\psi_n^{(\beta)}(x)$ are defined in (1.23) and (1.24) respectively. Here, $\Pi_{2N} = \mathrm{diag}(\underbrace{1,\ldots,1}_{2N},0,\ldots,0)$ is a diagonal matrix and $\Phi^{(\beta)}(x)$ and $\Psi^{(\beta)}(x)$ are semi-infinite vectors:

$$\Phi^{(\beta)}(x) = (\Phi_0^{(\beta)\mathrm{t}}(x)\ldots\Phi_n^{(\beta)\mathrm{t}}(x)\ldots)^{\mathrm{t}}, \qquad \widehat{\Phi}^{(\beta)}(x) = -\Phi^{(\beta)\mathrm{t}}(x)Z, \tag{3.16}$$

$$\Psi^{(\beta)}(x) = (\Psi_0^{(\beta)\mathrm{t}}(x)\ldots\Psi_n^{(\beta)\mathrm{t}}(x)\ldots)^{\mathrm{t}}, \qquad \widehat{\Psi}^{(\beta)}(x) = -\Psi^{(\beta)\mathrm{t}}(x)Z, \tag{3.17}$$

with each entry a 2×1 matrix:

$$\Phi_n^{(\beta)}(x) = \begin{pmatrix} \phi_{2n}^{(\beta)}(x) \\ \phi_{2n+1}^{(\beta)}(x) \end{pmatrix}, \qquad \widehat{\Phi}_n^{(\beta)}(x) = \begin{pmatrix} \phi_{2n+1}^{(\beta)}(x) \\ -\phi_{2n}^{(\beta)}(x) \end{pmatrix}^{\mathrm{t}}. \tag{3.18}$$

(similarly for $\Psi_n^{(\beta)}(x)$ and $\widehat{\Psi}_n^{(\beta)}(x)$). The antisymmetric block-diagonal matrix Z is given in (1.3), such that $Z = -Z^{\mathrm{t}}$ and $Z^2 = -1$.

Following (1.24), we write

$$\Psi_n^{(4)}(x) = \Phi_n'^{(4)}(x), \quad \Psi_n^{(1)}(x) = \int_{\mathbb{R}} \Phi_n^{(1)}(y)\epsilon(x-y)\,\mathrm{d}y, \qquad n\in\mathbb{N}. \tag{3.19}$$

These polynomials satisfy skew-orthonormal relations with respect to the weight function $w^2(x)$[1]:

$$(\Phi_n^{(\beta)}, \widehat{\Psi}_m^{(\beta)}) \equiv \int_{\mathbb{R}} \Phi_n^{(\beta)}(x)\widehat{\Psi}_m^{(\beta)}(x)\,\mathrm{d}x = \delta_{nm}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad n,m\in\mathbb{N}. \tag{3.20}$$

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[1]To observe duality between the two families of polynomials $\pi_n^{(\beta)}(x)$, $\beta = 1,4$, we skew-orthonormalize them with respect to $w^2(x)$ (as in (3.20)) to *set* both the families of skew-orthogonal functions on an equal footing. However, to study the statistical properties of symplectic ensembles only, this is not needed.

Thus we see that skew-orthonormalization (1.2) in ordinary space is the same as orthonormalization in the 2×2 quaternion space (3.20).

Finally, from (3.19) and (3.12), (3.13), and (3.14) we get

$$D_{2N}^{(1)}(x,y) = -\frac{\partial S_{2N}^{(1)}(x,y)}{\partial y}, \qquad S_{2N}^{(1)}(x,y) = \frac{\partial I_{2N}^{(1)}(x,y)}{\partial y}, \tag{3.21}$$

$$I_{2N}^{(4)}(x,y) = \frac{\partial S_{2N}^{(4)}(x,y)}{\partial x}, \qquad S_{2N}^{(4)}(x,y) = -\frac{\partial D_{2N}^{(4)}(x,y)}{\partial x}. \tag{3.22}$$

Thus a knowledge of the kernel function $S_{2N}^{(\beta)}(x,y)$ is enough to calculate the correlation function. In this chapter, we derive the sum $S_{2N}^{(\beta)}(x,y)$ for $\beta = 1, 4$. The large N behavior of $S_{2N}^{(\beta)}(x,y)$ is studied in Chapters 6 – 8 after we discover a method to compute large N asymptotics of skew-orthogonal functions and their recursion coefficients (Chapter 4).

4. The generalized Christoffel – Darboux formulas: $\beta = 1$ and 4

4.1. Recursion relations for skew-orthogonal functions with general weight. Let us consider skew-orthogonal polynomials defined with respect to the following weight functions:

- polynomials defined in the finite range $[x_1, x_{2d}]$, which we call the generalized Jacobi weight

$$w_{\alpha_i}(x) = (x_1 - x)^{\alpha_1}(x - x_2)^{\alpha_2} \cdots (x_{2d-1} - x)^{\alpha_{2d-1}}(x - x_{2d})^{\alpha_{2d}}, \qquad x_j \in \mathbb{R} \tag{3.23}$$

  where α_i's are suitably adjusted to ensure that they have finite moments (2.15).
- Polynomials defined in the range $[0, \infty]$, which we call the generalized Laguerre weight

$$w_\alpha(x) = x^\alpha \exp[-V(x)], \qquad V(x) = \sum_{k=1}^{2d-1} \frac{u_k x^k}{k}. \tag{3.24}$$

Finally

- polynomials defined in the range $[-\infty, \infty]$, which we call the generalized Gaussian weight

$$w(x) = \exp[-V(x)], \qquad V(x) = \sum_{k=1}^{2d} \frac{u_k x^k}{k}. \tag{3.25}$$

Evaluation of $\psi_n^{(4)}(x)$ and $\phi_n^{(1)}(x)$ involves terms like $w'_{\alpha_i}(x)/w_{\alpha_i}(x)$ which for (i) have singularity at $x_1, x_2, \ldots, x_{2d}$ and for (ii) have singularity at $x = 0$. Hence we define

$$f(x) := (x - x_1)(x_2 - x)\cdots(x_{2d-1} - x)(x - x_{2d}), \quad \text{for generalized Jacobi} \tag{3.26}$$

$$:= x, \quad \text{for generalized Laguerre,} \tag{3.27}$$

$$:= 1, \quad \text{for generalized Gaussian.} \tag{3.28}$$

To obtain recursion relations, we expand $[f(x)\phi_j^{(\beta)}(x)]'$ and $[xf(x)\phi_j^{(\beta)}(x)]'$ in terms of skew-orthogonal functions $\phi_j^{(\beta)}(x)$. We have

$$\begin{aligned} \big(f(x)\phi_j^{(\beta)}(x)\big)' &= \sum_{k=0}^{j+2d-1} A_{j,k}\phi_k^{(\beta)}(x), \\ \big(xf(x)\phi_j^{(\beta)}(x)\big)' &= \sum_{k=0}^{j+2d} B_{j,k}\phi_k^{(\beta)}(x). \end{aligned} \tag{3.29}$$

It is at this point that we introduce the semi-infinite matrices $P^{(\beta)}$ and $R^{(\beta)}$. The semi-infinite vectors $\Phi^{(\beta)}(x)$ and $\Psi^{(\beta)}(x)$ are given in (3.16) and (3.17) respectively. Using them, we have

$$\boxed{f(x)\Psi^{(4)}(x) = P^{(4)}\Phi^{(4)}(x), \qquad xf(x)\Psi^{(4)}(x) = R^{(4)}\Phi^{(4)}(x),} \tag{3.30}$$

$$\boxed{f(x)\Phi^{(1)}(x) = P^{(1)}\Psi^{(1)}(x), \qquad xf(x)\Phi^{(1)}(x) = R^{(1)}\Psi^{(1)}(x).} \tag{3.31}$$

Equation (3.31) is obtained by multiplying the above expansion by $\epsilon(y-x)$ and integrating by parts. We show that the semi-infinite matrices $P^{(\beta)}$ and $R^{(\beta)}$ have d quaternion bands above and below the central diagonal (hence a matrix with $2d+1$ quaternion bands).

For random matrix analysis, we consider the rather simple cases of Jacobi (1.6) weight function for (i), associated Laguerre (1.7) for (ii) and Gaussian (1.8) and the quartic weight (1.9) for (iii). We show that in the 2×2 quaternion space, the corresponding skew-orthogonal functions satisfy three-term recursion relations for the classical weight functions and $(2d+1)$-term recursion relations for the generalized weight functions.

Recursion relations for skew-orthogonal functions with classical weights. For the classical weights ($d = 1$), $P^{(\beta)}$ and $R^{(\beta)}$ are semi-infinite tridiagonal quaternion matrices. Now, $w'(x)/w(x)$ and hence $\Psi^{(4)}(x)$ and $\Phi^{(1)}(x)$ has singularity at $x \pm 1$ for Jacobi, and at $x = 0$ for associated Laguerre. To remove them, we have

$$f(x) = (1-x^2), \qquad \text{Jacobi} \tag{3.32}$$

$$= x, \qquad \text{Associated Laguerre} \tag{3.33}$$

$$= 1, \qquad \text{Gaussian.} \tag{3.34}$$

In other words, for classical weights, skew-orthogonal functions satisfy three-term recursion relations in quaternion space and is given by

$$f(x)\Psi_n^{(4)}(x) = \mathbf{P}_{n,n+1}^{(4)}\Phi_{n+1}^{(4)}(x) + \mathbf{P}_{n,n}^{(4)}\Phi_n^{(4)}(x) + \mathbf{P}_{n,n-1}^{(4)}\Phi_{n-1}^{(4)}(x), \tag{3.35}$$

$$xf(x)\Psi_n^{(4)}(x) = \mathbf{R}_{n,n+1}^{(4)}\Phi_{n+1}^{(4)}(x) + \mathbf{R}_{n,n}^{(4)}\Phi_n^{(4)}(x) + \mathbf{R}_{n,n-1}^{(4)}\Phi_{n-1}^{(4)}(x), \tag{3.36}$$

$$f(x)\Phi_n^{(1)}(x) = \mathbf{P}_{n,n+1}^{(1)}\Psi_{n+1}^{(1)}(x) + \mathbf{P}_{n,n}^{(1)}\Psi_n^{(1)}(x) + \mathbf{P}_{n,n-1}^{(1)}\Psi_{n-1}^{(1)}(x), \tag{3.37}$$

$$xf(x)\Phi_n^{(1)}(x) = \mathbf{R}_{n,n+1}^{(1)}\Psi_{n+1}^{(1)}(x) + \mathbf{R}_{n,n}^{(1)}\Psi_n^{(1)}(x) + \mathbf{R}_{n,n-1}^{(1)}\Psi_{n-1}^{(1)}(x), \tag{3.38}$$

where $\Phi_n^{(\beta)}(x)$ and $\Psi_n^{(\beta)}(x)$ are given in (3.18) and $\mathbf{P}_{j,k}^{(\beta)}$ and $\mathbf{R}_{j,k}^{(\beta)}$ are 2×2 quaternions. Equations (3.35) – (3.38) can be proved directly using the skew-orthogonal relation (3.20). We leave it as an exercise. In this section, we give an alternative proof by showing that the semi-infinite matrices $P^{(\beta)}$ and $R^{(\beta)}$ are tridiagonal (in the quaternion sense) and anti-self-dual.

In terms of the elements of the quaternion matrices, (3.35) and (3.36) can be written as:

$$f(x)\begin{pmatrix}\psi^{(4)}_{2n}(x)\\ \psi^{(4)}_{2n+1}(x)\end{pmatrix} = \begin{pmatrix}0 & 0\\ P^{(4)}_{2n+1,2n+2} & 0\end{pmatrix}\begin{pmatrix}\phi^{(4)}_{2n+2}(x)\\ \phi^{(4)}_{2n+3}(x)\end{pmatrix} + \begin{pmatrix}P^{(4)}_{2n,2n} & P^{(4)}_{2n,2n+1}\\ P^{(4)}_{2n+1,2n} & P^{(4)}_{2n+1,2n+1}\end{pmatrix}\begin{pmatrix}\phi^{(4)}_{2n}(x)\\ \phi^{(4)}_{2n+1}(x)\end{pmatrix} + \begin{pmatrix}0 & 0\\ P^{(4)}_{2n+1,2n-2} & 0\end{pmatrix}\begin{pmatrix}\phi^{(4)}_{2n-2}(x)\\ \phi^{(4)}_{2n-1}(x)\end{pmatrix} \tag{3.39}$$

and

$$xf(x)\begin{pmatrix}\psi^{(4)}_{2n}(x)\\ \psi^{(4)}_{2n+1}(x)\end{pmatrix} = \begin{pmatrix}R^{(4)}_{2n,2n+2} & 0\\ R^{(4)}_{2n+1,2n+2} & R^{(4)}_{2n+1,2n+3}\end{pmatrix}\begin{pmatrix}\phi^{(4)}_{2n+2}(x)\\ \phi^{(4)}_{2n+3}(x)\end{pmatrix} + \begin{pmatrix}R^{(4)}_{2n,2n} & R^{(4)}_{2n,2n+1}\\ R^{(4)}_{2n+1,2n} & R^{(4)}_{2n+1,2n+1}\end{pmatrix}\begin{pmatrix}\phi^{(4)}_{2n}(x)\\ \phi^{(4)}_{2n+1}(x)\end{pmatrix} + \begin{pmatrix}R^{(4)}_{2n,2n-2} & 0\\ R^{(4)}_{2n+1,2n-2} & R^{(4)}_{2n+1,2n-1}\end{pmatrix}\begin{pmatrix}\phi^{(4)}_{2n-2}(x)\\ \phi^{(4)}_{2n-1}(x)\end{pmatrix}. \tag{3.40}$$

For $\beta = 1$, we get similar relations, where $\Phi^{(4)}$ and $\Psi^{(4)}$ are replaced by $\Psi^{(1)}$ and $\Phi^{(1)}$ respectively.

For weight functions defined in (3.23), (3.24) and (3.25), the semi-infinite matrices $P^{(\beta)}$ and $R^{(\beta)}$ have d quaternion bands above and below the diagonal [36], which is the same as saying that they satisfy $(2d+1)$-term recursion relations. Thus skew-orthogonal functions, defined with respect to the classical weights, namely the $d = 1$ version of (3.23), (3.24) and (3.25)), satisfy three-term recursion relations in quaternion space.

Note. Here, we would like to mention that unlike orthogonal polynomials, the Jacobi matrix $Q^{(\beta)}$ coming from the relation $x\Phi^{(\beta)}(x) = Q^{(\beta)}\Phi^{(\beta)}(x)$, $\beta = 1, 4$, holds little importance in our derivation of the generalized Christoffel – Darboux formula since they do not have finite bands below the diagonal. This will be discussed further in Chapter 9.

Proof. To prove that the skew-orthogonal functions corresponding to weight functions (3.23), (3.24) and (3.25) satisfy $(2d + 1)$-term recursion relations in the quaternion space, we first show that the matrices $P^{(4)}$ and $R^{(4)}$ are anti-self-dual. For $\beta = 4$, we use the scalar products

$$\sum_j P^{(4)}_{n,j} Z_{j,m} = \big(f(x)\psi^{(4)}_n(x), \psi^{(4)}_m(x)\big) = \sum_j P^{(4)}_{m,j} Z_{j,n}, \tag{3.41}$$

$$\sum_j R^{(4)}_{n,j} Z_{j,m} = \big(xf(x)\psi^{(4)}_n(x), \psi^{(4)}_m(x)\big) = \sum_j R^{(4)}_{m,j} Z_{j,n}. \tag{3.42}$$

Replacing $\psi^{(4)}(x)$ by $\phi^{(1)}(x)$ for $\beta = 1$, we have

$$\sum_j P^{(1)}_{n,j} Z_{m,j} = \big(f(x)\phi^{(1)}_n(x), \phi^{(1)}_m(x)\big) = \sum_j P^{(1)}_{m,j} Z_{n,j}, \tag{3.43}$$

$$\sum_j R^{(1)}_{n,j} Z_{m,j} = \big(xf(x)\phi^{(1)}_n(x), \phi^{(1)}_m(x)\big) = \sum_j R^{(1)}_{m,j} Z_{n,j}. \tag{3.44}$$

Thus, we get

$$P^{(\beta)} Z = Z^{\mathrm{t}} P^{(\beta)\mathrm{t}} = -Z P^{(\beta)\mathrm{t}} \implies P^{(\beta)} = Z P^{(\beta)\mathrm{t}} Z. \tag{3.45}$$

Following the same steps for $R^{(\beta)}$, we finally get

$$\boxed{P^{(\beta)} = -P^{(\beta)\mathrm{D}}, \quad R^{(\beta)} = -R^{(\beta)\mathrm{D}}, \qquad \beta = 1, 4,} \tag{3.46}$$

where the dual of a matrix is defined as

$$A^{\mathrm{D}} := -Z A^{\mathrm{t}} Z. \tag{3.47}$$

It is straightforward to see that $P^{(\beta)}$ and $R^{(\beta)}$ have d quaternion bands (one in the case of skew-orthogonal functions defined with respect to the classical weight functions) above the diagonal. Equation (3.46) ensures that they have the same number of bands (where each entry is a 2×2 quaternion) below the diagonal. This completes the proof. □

Exercise. Verify the above results for classical and quartic weight.

4.2. Generalized Christoffel – Darboux sum. The tridiagonal symmetric Jacobi matrix $Q^{(2)}$ is instrumental in deriving the Christoffel – Darboux formula for $\beta = 2$. The symmetry (i.e., $Q^{(2)} = Q^{(2)\mathrm{t}}$) enables us to calculate the sum, while the tridiagonal property simplifies the result.

To calculate $S^{(\beta)}_{2N}(x, y)$, $\beta = 1$, 4, we need something similar to happen. In this section, we show that to evaluate $S^{(\beta)}_{2N}(x, y)$, the anti-self-dual matrix $\overline{R}^{(\beta)}(x) = R^{(\beta)} - xP^{(\beta)}$ plays a similar role for $\beta = 1, 4$ as the symmetric Jacobi matrix for $\beta = 2$. As these matrices are dependent on the weight function (or more specifically, on d), the generalized Christoffel – Darboux sum also reflects that property.

We obtain the generalized Christoffel – Darboux sum for weight functions defined in (3.23) – (3.25). To illustrate our derivation, we will refer to the classical weights, namely the Jacobi, the associated Laguerre and the Gaussian weights.

With $f(y)$ introduced to remove singularity in $w'_{\alpha_i}(x)/w_{\alpha_i}(x)$, (for example see (3.26) – (3.28) for general weight and (3.32) – (3.34) for classical weights), we use the definition of $S^{(4)}_{2N}(x, y)$ from (3.12) and the recursion relation from (3.30) to get

$$\begin{aligned} & f(y) S^{(4)}_{2N}(x, y) - f(x) S^{(4)}_{2N}(y, x) \\ &\quad = f(y)\Big[\Phi^{(4)\mathrm{t}}(x) \prod_{2N} Z \prod_{2N} \Psi^{(4)}(y)\Big] + f(x)\Big[\Psi^{(4)\mathrm{t}}(x) \prod_{2N} Z \prod_{2N} \Phi^{(4)}(y)\Big] \\ &\quad = \Big[-\Phi^{(4)\mathrm{t}}(x) Z Z \prod_{2N} Z \prod_{2N} P^{(4)} \Phi^{(4)}(y)\Big] \\ &\qquad\qquad + \Big[\Phi^{(4)\mathrm{t}}(x) Z Z P^{(4)\mathrm{t}} Z Z \prod_{2N} Z \prod_{2N} \Phi^{(4)}(y)\Big] \\ &\quad = \Big[-\widehat{\Phi}^{(4)}(x) \prod_{2N} P^{(4)} \Phi^{(4)}(y)\Big] + \Big[\widehat{\Phi}^{(4)}(x) P^{(4)} \prod_{2N} \Phi^{(4)}(y)\Big] \\ &\quad = \widehat{\Phi}^{(4)}(x) \Big[P^{(4)}, \prod_{2N}\Big] \Phi^{(4)}(y). \end{aligned} \tag{3.48}$$

Similarly,

$$
\begin{aligned}
(3.49)\quad & yf(y)S_{2N}^{(4)}(x,y) - xf(x)S_{2N}^{(4)}(y,x) \\
&= yf(y)\Big[\Phi^{(4)\mathrm{t}}(x)\prod_{2N} Z \prod_{2N} \Psi^{(4)}(y)\Big] + xf(x)\Big[\Psi^{(4)\mathrm{t}}(x)\prod_{2N} Z \prod_{2N} \Phi^{(4)}(y)\Big] \\
&= \Big[-\Phi^{(4)\mathrm{t}}(x)ZZ\prod_{2N} Z \prod_{2N} R^{(4)}\Phi^{(4)}(y)\Big] \\
&\qquad + \Big[\Phi^{(4)\mathrm{t}}(x)ZZR^{(4)\mathrm{t}}ZZ\prod_{2N} Z \prod_{2N} \Phi^{(4)}(y)\Big] \\
&= \Big[-\widehat{\Phi}^{(4)}(x)\prod_{2N} R^{(4)}\Phi^{(4)}(y)\Big] + \Big[\widehat{\Phi}^{(4)}(x)R^{(4)}\prod_{2N}\Phi^{(4)}(y)\Big] \\
&= \widehat{\Phi}^{(4)}(x)\Big[R^{(4)}, \prod_{2N}\Big]\Phi^{(4)}(y).
\end{aligned}
$$

Combining the two, the generalized Christoffel – Darboux formula for symplectic ensembles of random matrices with weight functions given in (3.23) – (3.25) is given by

$$
(3.50)\qquad \boxed{S_{2N}^{(4)}(x,y) = \frac{\widehat{\Phi}^{(4)}(x)[\overline{R}^{(4)}(x), \Pi_{2N}]\Phi^{(4)}(y)}{f(y)(y-x)}, \qquad N \geq d.}
$$

Here, we note that for the entire family of classical weights, the above result is true with $d = 1$.

For the corresponding orthogonal ensembles ($\beta = 1$), the generalized Christoffel – Darboux formula is derived using similar technique. Using the recursion relation (3.31) in our definition of $S_{2N}^{(1)}(x,y)$ (3.12), we get

$$
\begin{aligned}
(3.51)\quad & f(x)S_{2N}^{(1)}(x,y) - f(y)S_{2N}^{(1)}(y,x) \\
&= f(x)\Big[\Phi^{(1)\mathrm{t}}(x)\prod_{2N} Z \prod_{2N} \Psi^{(1)}(y)\Big] + f(y)\Big[\Psi^{(1)\mathrm{t}}(x)\prod_{2N} Z \prod_{2N} \Phi^{(1)}(y)\Big] \\
&= \widehat{\Psi}^{(1)}(x)\Big[P^{(1)}, \prod_{2N}\Big]\Psi^{(1)}(y)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.52)\quad & xf(x)S_{2N}^{(1)}(x,y) - yf(y)S_{2N}^{(1)}(y,x) \\
&= xf(x)\Big[\Phi^{(1)\mathrm{t}}(x)\prod_{2N} Z \prod_{2N} \Psi^{(1)}(y)\Big] + yf(y)\Big[\Psi^{(1)\mathrm{t}}(x)\prod_{2N} Z \prod_{2N} \Phi^{(1)}(y)\Big] \\
&= \widehat{\Psi}^{(1)}(x)\Big[R^{(1)}, \prod_{2N}\Big]\Psi^{(1)}(y).
\end{aligned}
$$

Combining the two, the generalized Christoffel – Darboux formula for orthogonal ensembles is given by

$$
(3.53)\qquad \boxed{S_{2N}^{(1)}(x,y) = \frac{\widehat{\Psi}^{(1)}(x)[\overline{R}^{(1)}(y), \Pi_{2N}]\Psi^{(1)}(y)}{f(x)(x-y)}, \qquad N \geq d.}
$$

Here $[\overline{R}^{(\beta)}(x)]/f(y)$, where

$$
(3.54)\qquad \overline{R}^{(\beta)}(x) = R^{(\beta)} - xP^{(\beta)}, \qquad \beta = 1, 4,
$$

is the skew-orthogonal analog of the Jacobi matrix for orthogonal polynomials.

For example, the generalized Christoffel – Darboux matrix for the Jacobi symplectic ensemble (including the associated Laguerre and Gaussian symplectic ensemble) has the following structure for $\beta = 4$:

$$\begin{aligned}(3.55)\quad &\widehat{\Phi}^{(4)}(x)\Big[\overline{R}^{(4)}(x), \underset{2N}{\Pi}\Big]\Phi^{(4)}(y) \\ &= \begin{pmatrix}\phi_1^{(4)}(x)\\ -\phi_0^{(4)}(x)\\ \vdots\end{pmatrix}^{\mathrm{t}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \vdots & \vdots & 0\\ 0 & 0 & 0 & -R^{(4)}_{2N-2,2N} & 0 & 0\\ 0 & 0 & 0 & -\overline{R}^{(4)}_{2N-1,2N}(x) & -R^{(4)}_{2N-1,2N+1} & 0\\ 0 & -R^{(4)}_{2N-1,2N+1} & 0 & 0 & 0 & 0\\ 0 & \overline{R}^{(4)}_{2N-1,2N}(x) & -R^{(4)}_{2N-2,2N} & 0 & \dots & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \vdots & \vdots & 0 \end{pmatrix} \begin{pmatrix}\phi_0^{(4)}(y)\\ \phi_1^{(4)}(y)\\ \vdots\end{pmatrix}\\ &= R^{(4)}_{2N-2,2N}\big[\phi^{(4)}_{2N}(x)\phi^{(4)}_{2N-1}(y) - \phi^{(4)}_{2N}(y)\phi^{(4)}_{2N-1}(x)\big]\\ &\qquad + R^{(4)}_{2N-1,2N+1}\big[\phi^{(4)}_{2N-2}(x)\phi^{(4)}_{2N+1}(y) - \phi^{(4)}_{2N-2}(y)\phi^{(4)}_{2N+1}(x)\big]\\ &\qquad + \big(R^{(4)}_{2N-1,2N} - xP^{(4)}_{2N-1,2N}\big)\big[\phi^{(4)}_{2N-2}(x)\phi^{(4)}_{2N}(y) - \phi^{(4)}_{2N-2}(y)\phi^{(4)}_{2N}(x)\big].\end{aligned}$$

Similarly, for the family of the Jacobi orthogonal ensemble, the generalized Christoffel – Darboux matrix has the following structure for $\beta = 1$:

$$\begin{aligned}(3.56)\quad &\widehat{\Psi}^{(1)}(x)\Big[\overline{R}^{(1)}(y), \underset{2N}{\Pi}\Big]\Psi^{(1)}(y)\\ &\quad = R^{(1)}_{2N-2,2N}\big[\psi^{(1)}_{2N}(x)\psi^{(1)}_{2N-1}(y) - \psi^{(1)}_{2N}(y)\psi^{(1)}_{2N-1}(x)\big]\\ &\qquad + R^{(1)}_{2N-1,2N+1}\big[\psi^{(1)}_{2N-2}(x)\psi^{(1)}_{2N+1}(y) - \psi^{(1)}_{2N-2}(y)\psi^{(1)}_{2N+1}(x)\big]\\ &\qquad + \big(R^{(1)}_{2N-1,2N} - yP^{(1)}_{2N-1,2N}\big)\big[\psi^{(1)}_{2N-2}(x)\psi^{(1)}_{2N}(y) - \psi^{(1)}_{2N-2}(y)\psi^{(1)}_{2N}(x)\big].\end{aligned}$$

For general d with $N \geq d$, the generalized Christoffel – Darboux matrix $\big[\overline{R}^{(\beta)}(x), \Pi_{2N}\big]$, in terms of the quaternion elements, takes the form

$$\begin{aligned}(3.57)\quad &\Big[\overline{\mathbf{R}}^{(\beta)}(x), \underset{N}{\Pi}\Big]\\ &= \begin{pmatrix} 0 & 0 & 0 & -\overline{\mathbf{R}}^{(\beta)}_{N-d,N}(x) & 0 & 0\\ 0 & 0 & 0 & \vdots & \ddots & 0\\ 0 & 0 & 0 & -\overline{\mathbf{R}}^{(\beta)}_{N-1,N}(x) & \dots & -\overline{\mathbf{R}}^{(\beta)}_{N-1,N+d-1}(x)\\ \overline{\mathbf{R}}^{(\beta)}_{N,N-d}(x) & \dots & \overline{\mathbf{R}}^{(\beta)}_{N,N-1}(x) & 0 & 0 & 0\\ 0 & \ddots & \vdots & \vdots & 0 & 0\\ 0 & 0 & \overline{\mathbf{R}}^{(\beta)}_{N+d-1,N-1}(x) & 0 & \dots & 0 \end{pmatrix},\\ &\hspace{20em} N \geq d.\end{aligned}$$

where the weight functions are defined in (3.23) – (3.25). Note that in the above matrix, every entry is a 2×2 quaternion, including those of the projection operator $\Pi_N = \mathrm{diag}(\underbrace{\mathbf{1}, \dots, \mathbf{1}}_{N}, 0, \dots, 0)$.

Exercise. (1) From (3.57), show that for quartic weight (i.e., $d = 2$), $S_{2N}^{(\beta)}(x,y)$ for $\beta = 1$ is given by

$$\begin{aligned}(x-y)S_{2N}^{(1)}(x,y) = {} & R_{2N-4,2N}^{(1)}[\psi_{2N-3}^{(1)}(x)\psi_{2N}^{(1)}(y) - (x \leftrightarrow y)] \\ & + R_{2N-2,2N+2}^{(1)}[\psi_{2N-1}^{(1)}(x)\psi_{2N+2}^{(1)}(y) - (x \leftrightarrow y)] \\ & - R_{2N-3,2N+1}^{(1)}[\psi_{2N-4}^{(1)}(x)\psi_{2N+1}^{(1)}(y) - (x \leftrightarrow y)] \\ & - R_{2N-1,2N+3}^{(1)}[\psi_{2N-2}^{(1)}(x)\psi_{2N+3}^{(1)}(y) - (x \leftrightarrow y)] \\ & + R_{2N-2,2N}^{(1)}[\psi_{2N-1}^{(1)}(x)\psi_{2N}^{(1)}(y) - (x \leftrightarrow y)] \\ & - yP_{2N-3,2N}^{(1)}[\psi_{2N-4}^{(1)}(y)\psi_{2N}^{(1)}(x) - (x \leftrightarrow y)] \\ & - \big[R_{2N-1,2N+1}^{(1)}[\psi_{2N-2}^{(1)}(x)\psi_{2N+1}^{(1)}(y) - (x \leftrightarrow y)] \\ & \quad - yP_{2N-1,2N+2}^{(1)}[\psi_{2N-2}^{(1)}(y)\psi_{2N+2}^{(1)}(x) - (x \leftrightarrow y)]\big] \\ & - yP_{2N-2,2N+1}^{(1)}[\psi_{2N-1}^{(1)}(x)\psi_{2N+1}^{(1)}(y) - (x \leftrightarrow y)] \\ & + yP_{2N-1,2N}^{(1)}[\psi_{2N-2}^{(1)}(x)\psi_{2N}^{(1)}(y) - (x \leftrightarrow y)].\end{aligned} \tag{3.58}$$

(2) For $\beta = 4$ and quartic weight, show

$$\begin{aligned}(y-x)S_{2N}^{(4)}(x,y) = {} & R_{2N-4,2N}^{(4)}[\phi_{2N-3}^{(4)}(x)\phi_{2N}^{(4)}(y) - (x \leftrightarrow y)] \\ & + R_{2N-2,2N+2}^{(4)}[\phi_{2N-1}^{(4)}(x)\phi_{2N+2}^{(4)}(y) - (x \leftrightarrow y)] \\ & - R_{2N-3,2N+1}^{(4)}[\phi_{2N-4}^{(4)}(x)\phi_{2N+1}^{(4)}(y) - (x \leftrightarrow y)] \\ & - R_{2N-1,2N+3}^{(4)}[\phi_{2N-2}^{(4)}(x)\phi_{2N+3}^{(4)}(y) - (x \leftrightarrow y)] \\ & + R_{2N-2,2N}^{(4)}[\phi_{2N-1}^{(4)}(x)\phi_{2N}^{(4)}(y) - (x \leftrightarrow y)] \\ & - xP_{2N-3,2N}^{(4)}[\phi_{2N-4}^{(4)}(y)\phi_{2N}^{(4)}(x) - (x \leftrightarrow y)] \\ & - [R_{2N-1,2N+1}^{(4)}[\phi_{2N-2}^{(4)}(x)\phi_{2N+1}^{(4)}(y) - (x \leftrightarrow y)] \\ & - xP_{2N-1,2N+2}^{(4)}[\phi_{2N-2}^{(4)}(y)\phi_{2N+2}^{(4)}(x) - (x \leftrightarrow y)]] \\ & - xP_{2N-2,2N+1}^{(4)}[\phi_{2N-1}^{(4)}(x)\phi_{2N+1}^{(4)}(y) - (x \leftrightarrow y)] \\ & + xP_{2N-1,2N}^{(4)}[\phi_{2N-2}^{(4)}(x)\phi_{2N}^{(4)}(y) - (x \leftrightarrow y)].\end{aligned} \tag{3.59}$$

Remark. For general weight functions (3.23)–(3.25)), there are $2d-1$ quaternions in the generalized Christoffel–Darboux formula. This is different from the Christoffel–Darboux formula for $\beta = 2$, where there is only one term irrespective of the weight function with respect to which the orthogonal polynomials are defined.