Introduction

This research monograph is divided into three parts. Broadly speaking, Part I belongs to the realm of category theory, while Parts II and III pertain to algebraic combinatorics, although the language of the former is present and apparent throughout. Four appendices supplement the main text.

In this introduction, we explain informally the main ideas in this monograph and provide pointers to important results in the text. We also indicate the chapter most relevant to a particular discussion. More details are given in the introduction of individual chapters.

Contents of Part I

Part I is of a general nature; it contains the material on monoidal categories on which the constructions of the later parts are laid out. Except for a few references to later parts for the purpose of examples, Part I is independent and self-contained. Our main goal here is to develop the basic theory of *bilax monoidal functors*. These are functors between braided monoidal categories which allow for transferring bimonoids in one to bimonoids in the other. These functors possess a rich theory, more so than one may perhaps anticipate. Further, a systematic study of bilax monoidal functors naturally leads us to the exciting world of higher monoidal categories. The results here constitute the beginnings of a theory which should find applications in a variety of settings and which should therefore be of wide interest.

Monoidal categories (Chapter 1)**.** Our starting point is a review of basic notions pertaining to monoidal category theory. We do not assume any previous knowledge of the subject. As our goal in Part III is the construction of certain Hopf algebras, in Chapter 1 there is a special emphasis on the notion of *Hopf monoid* (and the related notions of monoid, comonoid, and bimonoid). This notion can be defined in a braided monoidal category. When the latter is the category of (graded) vector spaces, one obtains the notion of (graded) Hopf algebra. In Part II we deal with the category of species, and Hopf monoids therein are our main concern. Chapter 1 thus lays out notions that will occupy us throughout the monograph and sets up the corresponding notation.

Graded vector spaces (Chapter 2)**.** Our next step is a review of basic notions pertaining to graded vector spaces. We discuss three monoidal structures on this category, namely, the Cauchy, Hadamard and substitution products. Our emphasis is on the Cauchy product since it is Hopf monoids with respect to this product which yield the notion of graded Hopf algebras. We discuss q-Hopf algebras, a notion obtained by deforming the braiding by a parameter q, and also Q-Hopf algebras, which are higher dimensional generalizations obtained by considering braidings parametrized by a matrix Q. Examples include the Eulerian q-Hopf algebra of Joni and Rota, and Manin's quantum linear space.

We discuss basic Hopf algebras such as the tensor algebra, the shuffle algebra, the symmetric algebra and other relatives, and explain how they relate via symmetrization and abelianization. These relations can be understood via freeness and cofreeness properties of these Hopf algebras. We also discuss comparatively less familiar objects such as the tensor algebra on a coalgebra and the quasi-shuffle bialgebra.

We also consider graded vector spaces with the added structure of boundary maps. These include chain complexes (which are used later to discuss an important example of a bilax monoidal functor) and graded vector spaces with creation-annihilation operators.

In Part II we are mainly concerned with analogues of these constructions for species. The link to graded vector spaces is made in Part III, where we study how monoidal properties of species translate to monoidal properties of graded vector spaces. Many results of Chapter 2 can then be seen as particular instances of results on species. Thus, Chapter 2 provides us in a nutshell some of the main ideas that will occupy us in later parts of this monograph.

Lax and colax monoidal functors (Chapter 3)**.** There are two types of functors between monoidal categories: lax and colax. A lax monoidal functor $\mathcal{F}$ is equipped with a transformation

$$\varphi_{A,B}\colon \mathcal{F}(A)\bullet\mathcal{F}(B)\to\mathcal{F}(A\bullet B)$$

satisfying certain associativity and unital conditions. The transformation φ neefd not be an isomorphism. Colax monoidal functors $(\mathcal{F},\psi)$ are the dual notion. These notions go back to the dawn of monoidal category theory with Bénabou [36]. Observe that if one ignores the objects A and B in the above definition, then a lax monoidal functor specializes to a monoid. This can be stated more precisely: A monoid is equivalent to a lax monoidal functor from the one-arrow category. In contrast to monoids, lax monoidal functors can be composed. The composite of lax monoidal functors is again lax (Theorem 3.21). This implies that a lax monoidal functor preserves monoids. More precisely, if (A,μ) is a monoid and $(\mathcal{F},\varphi)$ is a lax monoidal functor, then $\mathcal{F}(A)$ is a monoid whose product is given by the composite

$$\mathcal{F}(A)\bullet\mathcal{F}(A)\xrightarrow{\varphi_{A,A}}\mathcal{F}(A\bullet A)\xrightarrow{\mathcal{F}(\mu)}\mathcal{F}(A).$$

Colax monoidal functors correspond to comonoids in the same manner as lax monoidal functors correspond to monoids. A strong monoidal functor is a lax monoidal functor $(\mathcal{F},\varphi)$ for which the transformation φ is an isomorphism. This is the notion of monoidal functor most frequently used in the literature. In this situation, the distinction between lax and colax monoidal functors disappears.

We also consider adjunctions between monoidal functors. One of the main results here states that the right adjoint of a colax monoidal functor carries a canonical lax structure, and the left adjoint of a lax monoidal functor carries a canonical colax structure (Proposition 3.84). This, as well as some related results, can be obtained as special cases of general results of Kelly on adjunctions between categories with structure [195], but we provide direct proofs.

In category theory, along with categories and functors, there are natural transformations. This gives rise to the 2-category Cat. In the context of monoidal categories and (co)lax functors, there is a corresponding notion for natural transformations. We simply call them morphisms between (co)lax functors. This gives rise to the 2-categories lCat and cCat corresponding to the lax and colax cases respectively. The fact that 1-cells in a 2-category can be composed corresponds to the fact that the composite of (co)lax monoidal functors is again (co)lax.

Bilax monoidal functors (Chapter 3)**.** If the monoidal category is braided, then the notions of lax and colax monoidal functor can be combined into that of a bilax monoidal functor $(\mathcal{F}, \varphi, \psi)$, much in the same manner as the notion of bimonoid combines the notions of monoid and comonoid. Recall that bimonoids are monoids in the category of comonoids (or comonoids in the category of monoids). We provide a similar characterization of bilax monoidal functors (Proposition 3.77). Many results for bimonoids carry over to bilax monoidal functors in this manner. If the transformations φ and ψ are isomorphisms, then we say that the functor is bistrong. Just as for lax and colax monoidal functors, the composite of bilax monoidal functors is again bilax (Theorem 3.22). Further, just as for monoids and comonoids, a bimonoid is equivalent to a bilax functor from the one-arrow category. It then follows that a bilax functor preserves bimonoids.

There are two other types of functors between braided monoidal categories: braided lax and braided colax. Unlike the bilax case, these have appeared frequently in the literature. They preserve commutative monoids and cocommutative comonoids respectively. If the underlying lax structure of a braided lax functor is strong, then we say that the functor is braided strong. It is important to point out that in the strong situation, the distinction between bilax, braided lax and braided colax disappears. Thus, a bistrong monoidal functor is the same thing as a braided strong monoidal functor (Proposition 3.46). This nontrivial result may explain the lack of references in the literature to the notion of bilax monoidal functors: to encounter this notion one must look beyond the familiar setting of strong (and braided strong) monoidal functors.

A number of examples of bilax monoidal functors and morphisms between them are given in the monograph; a summary is provided in Tables 3.1, 3.2 and 3.3.

The op and cop constructions (Chapter 3)**.** Recall that to any monoid A in a braided monoidal category, one can associate the opposite monoids A^{op} and ${}^{\mathrm{op}}A$ by precomposing the product with the braiding or its inverse. Similarly, to any comonoid C, one associates the opposite comonoids ${}^{\mathrm{cop}}C$ and C^{cop} by postcomposing the coproduct with the braiding or its inverse. We refer to these as the op and cop constructions.

The same construction can be carried out for (co)lax monoidal functors. To $(\mathcal{F}, \varphi)$, we associate $(\mathcal{F}, \varphi^b)$ and $(\mathcal{F}, {}^b\varphi)$, and similarly to $(\mathcal{F}, \psi)$, we associate $(\mathcal{F}, \psi^b)$ and $(\mathcal{F}, {}^b\psi)$. These are obtained by conjugating the (co)lax structures with the braiding or its inverse. For example,

$$\varphi^b \colon \mathcal{F}(A) \bullet \mathcal{F}(B) \xrightarrow{\beta} \mathcal{F}(B) \bullet \mathcal{F}(A) \xrightarrow{\varphi_{B,A}} \mathcal{F}(B \bullet A) \xrightarrow{\mathcal{F}(\beta^{-1})} \mathcal{F}(A \bullet B).$$

These constructions can be combined to obtain the following important result. If $(\mathcal{F}, \varphi, \psi)$ is a bilax monoidal functor, then so are $(\mathcal{F}, \varphi^b, \psi^b)$ and $(\mathcal{F}, {}^b\varphi, {}^b\psi)$ (Proposition 3.16). The images of a bimonoid H under these functors yield bimonoids which are related to $\mathcal{F}(H)$ via the op and cop constructions. The precise result is given in Proposition 3.34.

If $\mathcal{F}$ is braided lax, then $\varphi = \varphi^b = {}^b\varphi$. A similar statement holds for braided colax functors. If a bilax monoidal functor is both braided lax and braided colax, then we say that it is braided bilax. In this case, conjugation does not yield anything new.

Normal bilax monoidal functors (Chapter 3)**.** A bimonoid for which the unit and counit maps are inverses is necessarily trivial (isomorphic to the unit object). In contrast, there are many interesting bilax functors for which such a property holds. We refer to them as *normal bilax monoidal functors*. It is a weakening of the notion of bistrong functors (Proposition 3.45). The Fock functors which form the focus of our attention in Part III are normal. We provide other examples as well.

Our terminology is motivated by the normalized chain complex functor discussed in Chapter 5. Needless to say, the normalized chain complex functor is an example of a normal bilax functor. The class of normal bilax functors satisfies some interesting properties (Proposition 3.41). These are related to the notion of a Frobenius monoidal functor, which has been considered in the literature. Some of these properties, for the example of the normalized chain complex functor, have also appeared in the literature.

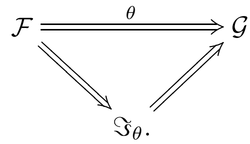
Hopf lax monoidal functors (Chapter 3)**.** At this point a natural question presents itself. If lax, colax and bilax monoidal functors correspond to monoids, comonoids and bimonoids, then what class of functors corresponds to Hopf monoids? A starting point is provided by the following result: The image of a Hopf monoid under a bistrong monoidal functor is again a Hopf monoid, in such a way that the antipode of the latter is the image of the antipode of the former (Proposition 3.50).

The answer we offer is the following. Between bilax monoidal functors and bistrong monoidal functors, there is an intermediate class of functors that preserves Hopf monoids but modifies antipodes in a predictable manner, much as the rest of the structure is modified by a bilax monoidal functor. We call them *Hopf lax monoidal functors*. We show that a normal bilax functor is Hopf lax if and only if it is bistrong (Proposition 3.60). It is worth pointing out that the analogy of Hopf lax monoidal functors with Hopf monoids is less straightforward than that of bilax monoidal functors with bimonoids, and the result on preservation of Hopf monoids (Theorem 3.70) requires a considerable amount of work. Familiar results for Hopf monoids admit generalizations to Hopf lax monoidal functors: the antipode of a Hopf lax monoidal functor is unique (Proposition 3.56), a morphism of bilax monoidal functors preserves antipodes when they exist (Proposition 3.59).

The antipode of a Hopf lax functor is related to the identity natural transformation through certain convolution formulas (Propositions 3.62–3.66). They are further developed from a more abstract point of view in Section D.4 (reviewed under *Monoids and the simplicial category*).

The results on Hopf lax functors, though interesting, are less relevant to the applications in later parts of the monograph. Indeed, the bilax full Fock functors are not Hopf lax, though they preserve Hopf monoids for other reasons. It is possible that a yet more general class of functors can be identified, so that it includes our Hopf lax functors as well as the Fock functors, and so that functors in this class preserve Hopf monoids.

The image functor (Chapter 3)**.** Recall that the image of a morphism of bialgebras is a bialgebra. More generally, the image of a morphism of bimonoids in an abelian monoidal category is itself a bimonoid. We extend this result by showing that the image of a morphism of bilax monoidal functors from a monoidal category to an abelian monoidal category is itself a bilax monoidal functor (Theorem 3.116). The diagram below shows the image $\Im_\theta$ of a transformation θ, and the factorization of the latter.



This result finds applications throughout the text. For example, the bosonic and fermionic Fock functors of Part III arise in this manner.

We obtain a concise proof of the above fact by viewing a morphism between two bilax monoidal functors from C to D as a bilax monoidal functor from C to the category of arrows in D (Proposition 3.111). There is another functor called the image functor which goes from the category of arrows in D to D. It is constructed by choosing monic-epi factorizations in D. Further, it is bistrong. Composing these two functors yields the required bilax functor.

Bilax monoidal functors in homological algebra (Chapter 5)**.** The notion of bilax monoidal functor between braided monoidal categories is of central importance to this work. Chapter 5 discusses what may be the most classical example of a bilax monoidal functor in mathematical nature. The familiar construction of a chain complex out of a simplicial module defines a functor between symmetric monoidal categories. The classical maps of Eilenberg–Zilber and Alexander–Whitney provide the lax and colax structures that turn it into a bilax monoidal functor (Theorem 5.6). While not formulated in these exact terms in the literature, this result pertains to the folklore of simplicial algebra. It was brought to our attention by Clemens Berger.

It is important to remark that we work with ordinary morphisms of chain complexes, not chain homotopy classes. If we pass to the homotopy category of chain complexes, then the chain complex functors become bistrong. In this situation the bilax axiom simplifies, and one does not need to confront it explicitly. In addition, this suffices for the applications to the construction of products in (co)homology. This may perhaps explain the lack of treatment in the literature of general bilax monoidal functors.

We state a number of well-known results which may be seen as consequences of Theorem 5.6, mainly regarding the existence of products in (co)homology. We also discuss the possibility of obtaining a one-parameter deformation of the chain complex functor. This can be done successfully provided that the boundary maps are set aside (Theorem 5.17).

2-monoidal categories (Chapter 6)**.** A careful analysis of bilax functors shows that they really belong to the world of higher monoidal categories. More precisely, they should be viewed as functors not between braided monoidal categories but rather between 2-monoidal categories, which are more general. The latter are categories with two compatible tensor products. The braiding is now replaced by an *interchange law*, which roughly speaking, specifies a way to interchange the order of the two tensor products. The familiar braiding axioms are replaced by a different set of 12 axioms.

There are many interesting examples of 2-monoidal categories and related objects ranging from directed graphs to bimodules over commutative algebras to lattices. Numerous other examples, primarily constructed out of species and graded vector spaces, along with applications are discussed in later parts of the monograph.

The definition of a 2-monoidal category raises a natural question: Despite the theorems that one may prove with such a definition, how does one know that there ought to be exactly 12 axioms? We provide an answer to this question by showing that a 2-monoidal category is an instance of a general notion in higher category theory, namely, it is a pseudomonoid in a certain monoidal 2-category (Proposition 6.73). This interpretation explains the origin of each axiom, so to speak. Further support is lent to this idea when one discovers that bilax functors are then nothing but appropriate morphisms between pseudomonoids (Proposition 6.75).

We mention that there are two other types of functors between 2-monoidal categories, namely double lax functors and double colax functors. These are generalizations of the notions of braided lax and braided colax functors. Just as bilax functors correspond to bimonoids, double (co)lax functors correspond to what we call double monoids and double comonoids. The latter, as expected, are generalizations of the familiar concepts of commutative monoids and cocommutative comonoids. Thus there are three different types of functors between 2-monoidal categories.

The Eckmann–Hilton argument (Chapter 6)**.** The classical Eckmann–Hilton argument says that if a set has two "mutually compatible" binary operations, then the two operations coincide and are commutative. This type of argument appears a number of times in the text; a summary of the results is provided below.

strong 2-monoidal category	$\longleftrightarrow$	braided monoidal category	(Proposition 6.11)
double monoid	$\longleftrightarrow$	commutative monoid	(Proposition 6.29)
double lax monoidal functor	$\longleftrightarrow$	braided lax monoidal functor	(Proposition 6.59)

A 2-monoidal category is said to be strong if the structure morphisms defining it are all isomorphisms. The first result above is due to Joyal and Street and says that a strong 2-monoidal category is equivalent to a braided monoidal category. This may be regarded as a categorical version of the Eckmann–Hilton argument: a 2-monoidal category is a category with two "mutually compatible" products and a braided monoidal category is a category with a "commutative" product. Working under this equivalence, double monoids are equivalent to commutative monoids, and double lax functors are equivalent to braided lax functors. The first of these results applied to the category of sets is essentially the classical Eckmann–Hilton argument.

Higher monoidal categories (Chapter 7)**.** The pseudomonoid interpretation for a 2-monoidal category not only sheds light on the notion of bilax monoidal functors, but also allows us to dive deeper into the world of monoidal categories. There are two simple constructions related to pseudomonoids in monoidal 2-categories; namely, the lax and colax constructions. They are discussed separately in Appendix C (reviewed under *Pseudomonoids and the looping principle*). These constructions allow us to systematically climb up the ladder of higher monoidal categories.

After 2-monoidal categories, we define 3-monoidal categories, which are monoidal categories with three compatible tensor products: A combination of any two of these products yields a 2-monoidal category, and further there is a set of 8 axioms that must be satisfied. Among these axioms there is one that stands out; we call it the *interchange axiom*. It is reminiscent of the relations in the standard presentation of the braid group.

Just as there are two types of monoidal functors between monoidal categories, and three types of functors between 2-monoidal categories, there are four types of monoidal functors between 3-monoidal categories. These are straightforward to define, using the previous definitions.

Higher monoidal categories, contrary to what one may expect, are built out of 1-, 2- and 3-monoidal categories in a rather straightforward manner, meaning that, there are no further axioms to worry about. The same is true of the monoidal functors that relate them. In general, there are $n+1$ different types of functors between two n-monoidal categories. At the level of objects, there are $n+1$ different types of monoids in a n-monoidal category, one for each type of functor.

The contragredient construction and self-duality (Chapters 3, 6 and 7)**.** We provide a framework to deal with the notion of duality for monoidal categories, monoidal functors and transformations between them. The basic idea is explained below.

Let C be a category equipped with a contravariant functor $(-)^*\colon \mathsf{C} \to \mathsf{C}$ which induces an adjoint equivalence of categories. A specific example to keep in mind is the duality functor on finite-dimensional vector spaces, which sends a space to its dual. We say that an object V in C is self-dual if $V \cong V^*$. Now let C and D be categories each equipped with such a functor, and let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a functor. Define the contragredient $\mathcal{F}^\vee$ to be the composite

$$\mathsf{C} \xrightarrow{\ *\ } \mathsf{C} \xrightarrow{\ \mathcal{F}\ } \mathsf{D} \xrightarrow{\ *\ } \mathsf{D}.$$

Further, we say that $\mathcal{F}$ is self-dual if $\mathcal{F}^\vee \cong \mathcal{F}$. Continuing with the above setup, let $\mathcal{F}$ and $\mathcal{G}$ be functors from C to D, and let $\theta\colon \mathcal{F} \Rightarrow \mathcal{G}$ be a natural transformation. Define the contragredient $\theta^\vee\colon \mathcal{G}^\vee \Rightarrow \mathcal{F}^\vee$ by

$$\mathcal{G}^\vee(A) = \mathcal{G}(A^*)^* \xrightarrow{(\theta_{A^*})^*} \mathcal{F}(A^*)^* = \mathcal{F}^\vee(A).$$

We say that $\theta\colon \mathcal{F} \Rightarrow \mathcal{F}^\vee$ is self-dual if $\theta^\vee \cong \theta$.

These notions can be extended to monoidal categories, braided monoidal categories and more generally to higher monoidal categories. This, in particular, allows us to define a self-dual monoidal category and a self-dual braided monoidal category. The contragredient of a lax functor is a colax functor and viceversa, while the contragredient of a bilax functor is a bilax functor (Proposition 3.102). This is the categorical version of the familiar statement that the dual of an algebra is a coalgebra and viceversa, and the dual of a bialgebra is a bialgebra. This setup allows us to define a self-dual bilax functor, a self-dual transformation between bilax functors, and so on. Further, one can establish results along the lines of: A self-dual bilax functor preserves self-dual bimonoids (Proposition 3.107).

We encounter plenty of examples of the contragredient construction in later parts of the monograph. The main examples of self-dual functors are summarized in Table 3.4.

Types of monoid and monoidal functors (Chapter 4)**.** The analogies between the notion of associative monoid and that of lax monoidal functor, and between the notion of commutative monoid and that of braided lax monoidal functor, which form the crux of Chapter 3, can be expanded to other types. Just as there are other types of monoid besides associative and commutative, there are other types of monoidal functors. This is the subject of Chapter 4. We first provide motivation to these ideas by explicitly introducing a number of types of monoids in monoidal categories and monoidal functors between monoidal categories. We then treat these notions in full generality by making use of the notion of operad. The necessary background on operads is given in Appendix B. For each operad $\mathbf{p}$, there is a notion of $\mathbf{p}$-monoid (in a monoidal category) and a notion of $\mathbf{p}$-lax monoidal functor (between monoidal categories). For general operads $\mathbf{p}$, the monoidal categories are required to be symmetric and linear.

The question arises as to how monoids (of a given type) transform under monoidal functors (of another type). The answer is contained in Theorem 4.28. It describes, more generally, the structure on a composite of two monoidal functors of such general types: if $\mathcal{F}$ is $\mathbf{p}$-lax and $\mathcal{G}$ is $\mathbf{q}$-lax, then the composite $\mathcal{G}\mathcal{F}$ is $(\mathbf{q} \times \mathbf{p})$-lax (under minor linearity hypotheses on the functors and categories). Here $\mathbf{q} \times \mathbf{p}$ stands for the Hadamard product of operads. Transformation of monoids is then a special case, obtained by viewing monoids as functors from the one-arrow category.

Contents of Part II

The main actors in Part II are Joyal's *species* [181]. This part is devoted to a careful study of the monoidal category of species, Hopf monoids therein, and the discussion of several examples.

Recall that a Hopf monoid in the category of graded vector spaces is the same as a graded Hopf algebra. Our ultimate goal is to provide a solid conceptual framework for the study of a large number of Hopf algebras of a combinatorial nature, which include the Hopf algebra of symmetric functions, quasi-symmetric functions, noncommutative symmetric functions, and others of prominence in the recent literature, as well as a host of new ones.

The following principle is central to our approach: A proper understanding of these objects and their interrelationships requires the consideration of a more general setting; namely, that of Hopf monoids in the monoidal category of species. This category is richer than that of graded vector spaces. The precise link between the two categories is made in Part III.

Species (Chapter 8)**.** Informally, a species is a family $\mathbf{p}$ of vector spaces, one space $\mathbf{p}[I]$ for each finite set I, which is natural in I with respect to bijections. By contrast, a graded vector space is simply a sequence of vector spaces, one space for each nonnegative integer. There is an alternative definition of a species as follows. A species is a graded vector space whose degree n component is equipped with an action of the symmetric group S_n for each n.

We are interested in studying algebraic notions in the category of species, such as monoids, Hopf monoids, and other related notions. These are the analogues of graded algebras, graded Hopf algebras, and other familiar objects pertaining to the category of graded vector spaces. Connected and positive species, which play a useful role in the theory, are the analogues of connected and positively graded vector spaces.

There are various monoidal structures on species. We are mainly interested in the *Cauchy* product (the monoids and Hopf monoids alluded above are with respect to this product), but the *Hadamard* and the *substitution* product play an important role as well. They are written

$$\mathbf{p} \cdot \mathbf{q}, \quad \mathbf{p} \times \mathbf{q}, \quad \text{and} \quad \mathbf{p} \circ \mathbf{q}$$

respectively. These are analogues of monoidal structures on graded vector spaces of the same name that are discussed in Chapter 2. The notions of (co)monoids, bimonoids and Hopf monoids in the monoidal category of species (with respect to the Cauchy product) can be made explicit. Roughly, a monoid is a species $\mathbf{p}$ equipped with maps

$$\mathbf{p}[S] \otimes \mathbf{p}[T] \to \mathbf{p}[I],$$

one such map for each decomposition $I = S \sqcup T$ of a finite set I into disjoint subsets S and T, and dually a comonoid is a species $\mathbf{p}$ equipped with maps

$$\mathbf{p}[I] \to \mathbf{p}[S] \otimes \mathbf{p}[T].$$

Similar descriptions hold for bimonoids and Hopf monoids.

Familiar properties of graded bialgebras and Hopf algebras hold also for bimonoids and Hopf monoids. For example, one can define the dual $\mathbf{p}^*$ of a species $\mathbf{p}$ by $\mathbf{p}^*[I] = \mathbf{p}[I]^*$. This association is natural in $\mathbf{p}$, so it gives rise to a (contravariant) functor on species. Further, this duality functor is bistrong, so the dual of a Hopf monoid is again a Hopf monoid.

However, bimonoids possess certain unique features not to be seen for bialgebras. The first instance is the compatibility axiom itself (Remark 8.8). Another instance is the interplay between the Cauchy and the Hadamard products on species. We show that the Hadamard product is a bilax monoidal functor with respect to the Cauchy product (Proposition 8.58). As a consequence, the Hadamard product of two Hopf monoids is again a Hopf monoid. In contrast, the Hadamard product of two graded Hopf algebras fails to be a graded Hopf algebra, in general. This is significant; we come back to this point at various places in this introduction, in particular under *Hopf algebras from geometry.*

Deformations of Hopf monoids (Chapter 9)**.** The standard braiding β on the monoidal category of species is defined by interchanging the tensor factors. Analogous to the situation for graded vector spaces, one can deform this braiding as follows.

$$\mathbf{p}[S] \otimes \mathbf{q}[T] \to \mathbf{q}[T] \otimes \mathbf{p}[S] \qquad\qquad x \otimes y \mapsto q^{|S||T|} y \otimes x.$$

This defines the braiding β_q where q is any scalar. Letting $q = 1$ recovers the previous case. Note that this braiding is a symmetry if and only if $q = \pm 1$.

A useful way to think about the coefficient in the braiding is as follows. We give the idea in rough terms. Let $S|T$ be a "state" in which every element of S precedes every element of T. Let q be the cost of changing the relative order between two elements. Then the coefficient is the cost of going from state $S|T$ to state $T|S$.

A Hopf monoid with respect to the braiding β_q is called a q-Hopf monoid. The preceding theory of Hopf monoids in species generalizes in a natural manner to the deformed setting. For example, duality continues to a bistrong functor. It turns out that the Hadamard product of a p-Hopf monoid and a q-Hopf monoid is a pq-Hopf monoid.

In addition to the above, one can also define a *signature functor* on species. It twists the action of the symmetric group by tensoring with its one-dimensional sign representation. This functor is bistrong provided the braidings are chosen carefully: if β_q is used in the source category, then use β_{-q} in the target category (Proposition 9.9). As a consequence, the signature functor takes a q-Hopf monoid to a $(-q)$-Hopf monoid. In particular, it switches Hopf monoids and (-1)-Hopf monoids.

The exponential species and the species of linear orders (Chapters 8, 9 and 11)**.** The exponential species $\mathbf{E}$ and the species of linear orders $\mathbf{L}$ are the simplest interesting and nontrivial examples of Hopf monoids in species. As a species, $\mathbf{E}$ is the graded vector space whose graded components consist of the trivial representations of the symmetric groups, while the graded components of $\mathbf{L}$ consist of the regular representations of the symmetric groups. These Hopf monoids accompany us through all our constructions. They are basic examples of universal objects: $\mathbf{L}$ is the free monoid on one generator, and $\mathbf{E}$ is the free commutative monoid on one generator. Their products and coproducts are simple to describe. We also describe their antipodes explicitly, and explain the different ways in which they can be derived.

Since we have given the rough idea of a monoid and comonoid in species, we indicate the product and coproduct of the species $\mathbf{L}$. The product is as follows: given linear orders on S and T, their product is their common extension to $S \sqcup T$ in which the elements of S precede the elements of T. The coproduct is as follows: given a linear order on I, its coproduct is its restriction to S tensored with its restriction to T.

The species $\mathbf{L}$ and $\mathbf{E}$ can be combined in various ways to obtain new examples of Hopf monoids. Many of these have a rich geometric flavor as we will see later. As an example, combining the Hadamard product construction with duality yields the Hopf monoid $\mathbf{L}^* \times \mathbf{L}$. This Hopf monoid is self-dual (isomorphic to its dual). This follows from the compatibility between duality and the Hadamard product.

The exponential species admits a signed version; we call it the signed exponential species and denote it by $\mathbf{E}^-$. As a species, its graded components consist of the sign representations of the symmetric groups. As a (-1)-Hopf monoid, it is the free commutative monoid on one generator, where commutative is now to be interpreted in the graded sense (with respect to the braiding β_{-1}). This object is intimately tied to the signature functor. The signature functor sends a species $\mathbf{p}$ to $\mathbf{p} \times \mathbf{E}^-$, that is, to its Hadamard product with $\mathbf{E}^-$. The bistrong structure of this functor arises from the bimonoid structure of $\mathbf{E}^-$ and the bilax structure of the Hadamard product. Since $\mathbf{E}$ is the unit object for the Hadamard product, it follows from this construction that the signature functor sends $\mathbf{E}$ to $\mathbf{E}^-$.

The situation for $\mathbf{L}$ is even more interesting. It admits a one-parameter deformation to a q-Hopf monoid which we denote by $\mathbf{L}_q$. Letting $q = 1$ recovers $\mathbf{L}$, while $\mathbf{L}_{-1}$ is the signature functor applied to $\mathbf{L}$. Note that $\mathbf{L}$ and $\mathbf{L}_{-1}$ are identical as species since tensoring the regular representation with the sign representation again yields the regular representation. However, their Hopf structures are different and in a sense cannot be compared since one is a Hopf monoid while the other is a (-1)-Hopf monoid.

Universal constructions (Chapter 11)**.** The free monoid on a species $\mathbf{q}$ is given by

$$\mathcal{T}(\mathbf{q}) = \mathbf{L} \circ \mathbf{q}.$$

This shows that the Cauchy and substitution products are intimately related. (This relation is encountered again in Section B.4, where the substitution product is studied in detail.) We refer to $\mathcal{T}$ as the free monoid functor; it is the species analogue of the tensor algebra for graded vector spaces. The space $\mathcal{T}(\mathbf{q})[I]$ is graded over compositions of the set I, much as the degree n component of the tensor algebra of a graded vector space is graded over compositions of n. Let $\mathbf{X}$ be the species whose value is the base field on singletons and 0 on all other finite sets. It is the unit object for the substitution product. The species $\mathbf{L}$ can be recovered as $\mathbf{L} = \mathcal{T}(\mathbf{X})$; a little more loosely, one may say that $\mathbf{L}$ is the free monoid on one generator, as already mentioned.

If $\mathbf{q}$ is a positive comonoid, then $\mathcal{T}(\mathbf{q})$ can be turned into a Hopf monoid, and its structure can be explicitly described. In addition, $\mathcal{T}(\mathbf{q})$ is the free Hopf monoid on the positive comonoid $\mathbf{q}$ (Theorem 11.9). This result is supplemented with a discussion of the monoidal properties of the corresponding adjunction.

There are commutative versions of these constructions. The free commutative monoid (Hopf monoid) on a positive species (comonoid) $\mathbf{q}$ is

$$\mathcal{S}(\mathbf{q}) = \mathbf{E} \circ \mathbf{q}.$$

The species $\mathbf{E}$ can be recovered as $\mathbf{E} = \mathcal{S}(\mathbf{X})$.

Since examples of both free or cofree Hopf monoids arise naturally, it is worth considering the dual constructions to those mentioned above. Thus we also discuss the cofree comonoid (Hopf monoid) on a positive species (monoid), and their cocommutative versions. The corresponding functors are denoted $\mathcal{T}^\vee$ and $\mathcal{S}^\vee$. The former is the species analogue of the tensor coalgebra functor for graded vector spaces.

Interestingly, the functor $\mathcal{T}$ admits a one-parameter deformation, which we denote by $\mathcal{T}_q$. It takes values in the category of q-Hopf monoids. Further, if $\mathbf{q}$ is a positive comonoid, then $\mathcal{T}_q(\mathbf{q})$ is the free q-Hopf monoid on $\mathbf{q}$. The deformation $\mathbf{L}_q$ can be recovered as $\mathbf{L}_q = \mathcal{T}_q(\mathbf{X})$. The commutative version of this construction for $q = 1$ is the functor $\mathcal{S}$. Similarly, there is an interesting commutative version of this construction for $q = -1$. We call the corresponding functor Λ. It takes values in the category of (-1)-Hopf monoids. Further, if $\mathbf{q}$ is a positive comonoid, then $\Lambda(\mathbf{q})$ is the free commutative (-1)-Hopf monoid on $\mathbf{q}$. The signed exponential species $\mathbf{E}^-$ can be recovered as $\mathbf{E}^- = \Lambda(\mathbf{X})$. We also discuss the dual functors $\mathcal{T}_q^\vee$ and $\Lambda^\vee$.

We briefly touch upon related functors such as the free Lie algebra functor, the universal enveloping algebra functor and the primitive element functor for species, along with the Poincaré-Birkhoff-Witt and Cartier-Milnor-Moore theorems for species which appear in the works of Joyal and Stover. We describe the coradical filtrations and primitive elements of the Hopf monoids which arise as values of the functors $\mathcal{T}^\vee$, $\mathcal{S}^\vee$ and $\Lambda^\vee$.

The Coxeter complex. The break, join, and projection maps (Chapter 10)**.** Symmetries of a set (bijections from the set to itself), or equivalently, the symmetric groups play a pivotal role in the theory of species. Recall that symmetric groups are Coxeter groups of type A. It turns out that key features of the theory of Coxeter groups when specialized to the example of type A can be formulated in the language of species.

To any Coxeter group is associated a simplicial complex which is called the Coxeter complex. We highlight three important properties of these objects.

- The join of Coxeter complexes is a Coxeter complex.
- The star of any face in a Coxeter complex is a Coxeter complex.
- The set of faces of a Coxeter complex is a monoid, whose product is given by the projection maps of Tits.

A set species is a family P of sets, one set $\mathrm{P}[I]$ for each finite set I, which is natural in I with respect to bijections. The family of Coxeter complexes of type A can be assimilated into one object which we denote by Σ. It is a set species with added structure. The component $\Sigma[I]$ is the set of faces of the Coxeter complex associated to the symmetric group on $|I|$ letters. An important observation specific to type A is as follows. The star of a vertex in a Coxeter complex of type A is isomorphic to the join of two smaller Coxeter complexes of type A. More precisely, for a decomposition $I = S \sqcup T$, there is a canonical identification

$$\mathrm{Star}(S|T) \cong \Sigma[S] \times \Sigma[T]$$

between the star of the vertex $S|T$ in $\Sigma[I]$ and the join of the complexes $\Sigma[S]$ and $\Sigma[T]$. We use

$$\mathrm{Star}(S|T) \underset{j_{S|T}}{\overset{b_{S|T}}{\rightleftarrows}} \Sigma[S] \times \Sigma[T]$$

to denote the inverse isomorphisms of simplicial complexes. We refer to $b_{S|T}$ and $j_{S|T}$ as the *break* and *join* maps, respectively. Further, for any vertex $S|T$ of $\Sigma[I]$, there is a map

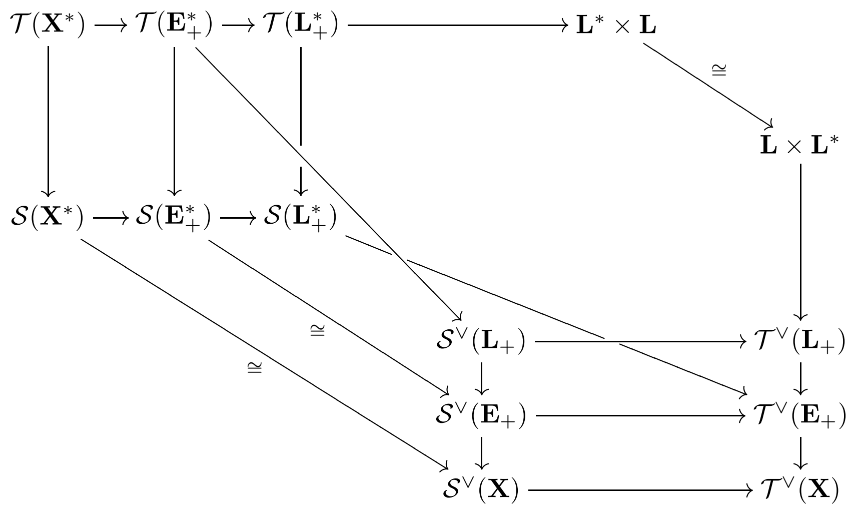
$$p_{S|T} \colon \Sigma[I] \to \mathrm{Star}(S|T)$$

which sends a face to its projection on the vertex $S|T$. It is called the *Tits projection.* These maps may be used to turn $\Sigma[I]$ into a monoid.

The break, join and projection maps are at the basis of the understanding of a number of combinatorial Hopf algebras. This geometric point of view was advocated in our previous work [12]. The compatibilities between these maps were listed as a set of coalgebra and algebra axioms [12, Sections 6.3.1 and 6.6.1]. These maps as well as their compatibilities find their most natural expression in this monograph, in the context of species.

Hopf monoids from geometry (Chapter 12)**.** In this chapter, we construct many examples of Hopf monoids in species and analyze them in considerable detail. Their nature is primarily geometric: they are associated to the Coxeter complex of type A in various ways. For this reason they admit explicit descriptions in terms of familiar combinatorial objects. The geometry of the complex, through the break, join and projection maps, is at the basis of the understanding of the algebraic structure of these objects, as evidenced by the results we present. These Hopf monoids arise by combining the species $\mathbf{X}$, $\mathbf{E}$, and $\mathbf{L}$ in various ways. They can be related by morphisms of Hopf monoids as indicated in diagram (0.1). The duality functor acts on this diagram by reflection across the diagonal.

(0.1)



Each of these Hopf monoids admits an explicit description in terms of familiar combinatorial objects. For instance, according to the definition of substitution, $\mathcal{T}(\mathbf{E}_+^*)$ is the species of set compositions. The product of this Hopf monoid is concatenation and the coproduct is dual to the quasi-shuffle of set compositions. Similarly, $\mathcal{S}(\mathbf{L}_+^*)$ is the species of *linear set partitions*. The other species and their products and coproducts can be described in similar combinatorial terms.

Some of the morphisms in (0.1) arise simply from functoriality; this is the case of the map $\mathcal{T}(\mathbf{E}_+^*) \to \mathcal{T}(\mathbf{L}_+^*)$, for instance. Some, like the map $\mathbf{L} \times \mathbf{L}^* \to \mathcal{T}^\vee(\mathbf{L}_+)$ (which relates pairs of linear orders to linear set compositions) arise from universal constructions (cofreeness of $\mathcal{T}^\vee(\mathbf{L}_+)$ in this case). Others, like the map $\mathcal{T}(\mathbf{E}_+^*) \to \mathcal{S}^\vee(\mathbf{L}_+)$ (which relates set compositions to linear set partitions), are specific to the species under consideration. In these cases, a combinatorial description can be of limited use.

Our main point in this chapter is that, in spite of their combinatorial appearance, proper understanding of these Hopf monoids and the morphisms among them demands the consideration of their geometric nature. Each of these species arises from the Coxeter complex of type A, and the products and coproducts that turn them into Hopf monoids can always be expressed in terms of the break, join and projection maps. For instance, set compositions correspond to *faces* of the complex, and linear set partitions to *directed flats*. Table 12.1 summarizes the combinatorial and geometric description of these species.

A bilinear form on chambers. Varchenko's result (Chapter 10)**.** Consider a hyperplane arrangement in which each hyperplane is assigned a weight. For any pair of chambers (top-dimensional faces) C and D in the arrangement, define $\langle C, D\rangle$ to be the product of the weights of the hyperplanes which separate C and D. This defines a bilinear form on the space spanned by chambers. Varchenko obtained a factorization of the determinant of this bilinear form, see equation (10.129). The special case when the hyperplane arrangement is the braid arrangement and all weights are equal was treated earlier by Zagier.

It follows from Varchenko's result that for generic weights on the hyperplanes the bilinear form on chambers is nondegenerate. There is a useful generalization of this result in which one puts weights on half-spaces instead of hyperplanes (Lemma 10.27). We use this result to deduce rigidity results of a very general kind in two different contexts. The first context is that of the norm transformation between $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$ and its higher dimensional generalization. The second context is that of the norm transformation between the deformed full Fock functors of Part III. This is explained in more detail below, under *Relations among the universal objects. The norm transformation* and *Relations among the Fock functors. The norm transformation.*

Hopf monoids from combinatorics (Chapter 13)**.** In this chapter, we discuss Hopf monoids that are based on combinatorial structures such as relations, preposets, posets, graphs, rooted forests, planar rooted forests, set-graded posets, closure operators, matroids and topologies. The origin of many of these ideas can be found in the paper of Joni and Rota [179]. The emphasis of this chapter is on the construction of interesting morphisms from these objects to objects such as $\mathcal{T}^\vee(\mathbf{E}_+)$ and $\mathcal{T}^\vee(\mathbf{X})$ which occur to the bottom right of diagram (0.1). By the universal constructions of Chapter 11, the latter are cofree objects, hence morphisms of the above kind can be constructed by minimal principles.

Many of these combinatorial objects can be interpreted geometrically. For example, posets can be viewed as appropriate unions of chambers (top-dimensional cones, to be precise) in the Coxeter complex of type A. This interpretation extends to preposets. This forges a link with the ideas of Chapters 10 and 12. We then observe that the morphisms relating the Hopf monoid of posets to the Hopf monoids based on linear orders, linear set partitions and set compositions, initially constructed using purely combinatorial or algebraic motivations, have simple geometric descriptions. The distinction between combinatorics and geometry (reflected in our chapter titles) is mainly for organizational purposes. The above examples reinforce the fact that either viewpoint may be used profitably according to the situation.

Relations among the universal objects. The norm transformation (Chapter 11)**.** To understand the relations between various universal objects (for example, to relate $\mathbf{L}$ and $\mathbf{E}$), one needs to properly understand how the functors $\mathcal{T}_q$, $\mathcal{T}_q^\vee$, $\mathcal{S}$ and $\mathcal{S}^\vee$, Λ and $\Lambda^\vee$ relate to one another. We now explain this.

First of all, the functors $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$, as well as $\mathcal{S}$ and $\mathcal{S}^\vee$, and Λ and $\Lambda^\vee$ are related through duality. Second, there are natural transformations $\pi\colon \mathcal{T} \Rightarrow \mathcal{S}$ and $\pi^\vee\colon \mathcal{S}^\vee \Rightarrow \mathcal{T}^\vee$ (projection onto coinvariants and inclusion of invariants). The former is called the *abelianization* and the latter is its dual. Similarly, there are natural transformations $\pi_{-1}\colon \mathcal{T}_{-1} \Rightarrow \Lambda$ and $\pi_{-1}^\vee\colon \Lambda^\vee \Rightarrow \mathcal{T}_{-1}^\vee$ called the *signed abelianization* and its dual.

It is not possible to relate the functors $\mathcal{T}_q$ and $\mathcal{T}_q^\vee$ in general since they take values on different type of objects. The former is defined on positive comonoids while the latter is defined on positive monoids. However, one can restrict both functors to the category of positive species, and in that case, there is a natural transformation

$$\kappa_q\colon \mathcal{T}_q \Rightarrow \mathcal{T}_q^\vee.$$

This is called the *q-norm transformation* (for its relation to the norm map in group theory). As mentioned under *The image functor*, transformations can be factored. The factorization of the q-norm can be explicitly understood for $q = \pm 1$. It is as follows.

$$\begin{array}{ccc} \mathcal{T} & \overset{\kappa}{\Longrightarrow} & \mathcal{T}^\vee \\ {\scriptstyle \pi}\Big\Downarrow & & \Big\Uparrow{\scriptstyle \pi^\vee} \\ \mathcal{S} & \underset{\mathrm{id}}{\Longrightarrow} & \mathcal{S}^\vee \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{T}_{-1} & \overset{\kappa_{-1}}{\Longrightarrow} & \mathcal{T}_{-1}^\vee \\ {\scriptstyle \pi_{-1}}\Big\Downarrow & & \Big\Uparrow{\scriptstyle \pi_{-1}^\vee} \\ \Lambda & \underset{\mathrm{id}}{\Longrightarrow} & \Lambda^\vee \end{array}$$

Thus, in the restricted setting of positive species, the functors $\mathcal{S}$ and $\mathcal{S}^\vee$ are identical and equal to the image of the norm. Similarly, the functors Λ and $\Lambda^\vee$ are identical and equal to the image of the signed norm.

Applying the diagram on the left to the species $\mathbf{X}$ yields a commutative diagram of Hopf monoids relating $\mathbf{L}$, its dual $\mathbf{L}^*$, and $\mathbf{E}$. In particular, the dual of $\mathbf{E}$ is itself. This can also be checked directly. Similarly, applying the diagram on the right to the species $\mathbf{X}$ yields a commutative diagram of (-1)-Hopf monoids relating $\mathbf{L}_{-1}$, its dual $\mathbf{L}^*_{-1}$, and $\mathbf{E}^-$ (which is self-dual).

The generic case of the q-norm is quite different. In fact, we deduce from Varchenko's result that κ_q is an isomorphism if q is not a root of unity (Theorem 11.35). Under this hypothesis, for any positive species $\mathbf{q}$, the q-Hopf monoids $\mathcal{T}_q(\mathbf{p})$ and $\mathcal{T}_q^\vee(\mathbf{p})$ are isomorphic. It follows by letting $\mathbf{p} = \mathbf{X}$ that the q-Hopf monoid $\mathbf{L}_q$ is self-dual (Proposition 12.6).

An interesting situation is $q = 0$. In this case, κ_0 is the identity and $\mathcal{T}_0 = \mathcal{T}_0^\vee$. It follows that the 0-Hopf monoid $\mathbf{L}_0$ is self-dual in a canonical manner.

The Schubert cocycle (Chapters 9 and 11)**.** Let l be a linear order on I, and let $I = S \sqcup T$ be a decomposition of I into disjoint subsets S and T. Let

$$\mathrm{sch}_{S,T}(l) := |\{(i,j) \in S \times T \mid \ i > j \text{ according to } l\}|.$$

We call this the *Schubert cocycle*. If we view l as a list, then $\mathrm{sch}_{S,T}(l)$ counts the number of minimum adjacent transpositions required to bring the elements of S to the beginning of the list.

The Schubert cocycle is equivalent to a standard combinatorial statistic, which we call the Schubert statistic. Our interest in this notion stems from its relevance to deformation theory. In this regard, the following two properties of the Schubert cocycle are significant.

For any decomposition $I = R \sqcup S \sqcup T$, and for any linear order l on I,

$$\mathrm{sch}_{R,S\sqcup T}(l) + \mathrm{sch}_{S,T}(l|_{S\sqcup T}) = \mathrm{sch}_{R\sqcup S,T}(l) + \mathrm{sch}_{R,S}(l|_{R\sqcup S}),$$

where the vertical bar denotes restriction of the linear order. This is the cocycle condition. It can be understood as follows. Both sides count the number of minimum adjacent transpositions required to rearrange the list l so that elements of R are at the beginning, followed by elements of S, followed by elements of T.

Now consider a pair of decompositions $I = S \sqcup T = S' \sqcup T'$ into disjoint subsets and let A, B, C, and D be the resulting intersections:

$$A = S \cap S', \quad B = S \cap T', \quad C = T \cap S', \quad D = T \cap T'.$$

Then for any linear order l on S, and linear order m on T,

$$\mathrm{sch}_{S',T'}(l \cdot m) = \mathrm{sch}_{A,B}(l) + \mathrm{sch}_{C,D}(m) + |B||C|,$$

where $l \cdot m$ denotes the common extension of l and m to I in which the elements of l precede the elements of m. This is the multiplicative property of the cocycle. Note that the last term on the right is precisely the exponent of the coefficient of β_q (under *Deformations of Hopf monoids*).

The q-Hopf monoid $\mathbf{L}_q$ can be constructed by deforming $\mathbf{L}$ via the Schubert cocycle: Keep the product the same as before, but modify the coproduct by multiplying it by the coefficient

$$q^{\mathrm{sch}_{S,T}(l)}.$$

The fact that $\mathbf{L}_q$ is coassociative is equivalent to the cocycle condition. The product-coproduct compatibility is equivalent to the multiplicative property of the cocycle.

This idea is the driving force behind a very general construction, namely, the construction of the functor $\mathcal{T}_q$ by deforming $\mathcal{T}$. For this construction, one extends the Schubert cocycle to set compositions (which are more general combinatorial objects than linear orders), and then proceeds in the same way as above. The situation for $\mathbf{L}_q$ can be seen as a special case by specializing the general construction to the species $\mathbf{X}$.

Cohomology of comonoids in species (Chapter 9)**.** An important cohomology theory for associative algebras is Hochschild cohomology. The coefficients for this theory can be chosen to be in any bimodule over that algebra. Dually, there is a Hochschild cohomology for coalgebras with coefficients in a bicomodule.

Cohomology of a comonoid in species with coefficients in a bicomodule can be defined in the same manner. We are interested in the special case when the comonoid is the exponential species $\mathbf{E}$ and the bicomodule is any linearized comonoid (this roughly means that the coproduct is well-behaved on a basis). We develop this theory in explicit terms with emphasis on low-dimensional cocycles. If, in addition, we have a linearized bimonoid (as opposed to just a comonoid), then we can also define the notion of multiplicative cocycles. We explain how the given linearized comonoid can be deformed using a 2-cocycle on it. If we are in the setup of linearized bimonoids, then the same deformation can be carried out provided the 2-cocycle is multiplicative.

The bimonoid of linear orders $\mathbf{L}$ is linearized. As indicated by the terminology, the Schubert cocycle is a 2-cocycle on $\mathbf{L}$. Further, it is multiplicative. These facts follow from the properties of the Schubert cocycle mentioned earlier. In fact, the Schubert cocycle is the unique multiplicative cocycle (of twist 1) on $\mathbf{L}$ (Theorem 9.27). We mentioned earlier how $\mathbf{L}_q$ can be constructed from $\mathbf{L}$ using the Schubert cocycle. This can now be seen as an instance of a general construction.

Antipode formulas (Chapter 11)**.** Recall that a Hopf monoid is a bimonoid with an antipode. As for graded vector spaces, a connected bimonoid in species is automatically a Hopf monoid. The antipode can be expressed as an alternating sum, where each summand is a composite of an iterated coproduct with an iterated product. This is the species analogue of Takeuchi's antipode formula for connected Hopf algebras. Since the sum is alternating, cancellations may take place (and in concrete examples, many often do). By contrast, we would like an explicit formula for the structure constants of the antipode on a given basis. Obtaining such a formula requires understanding of these cancellations; this is often a challenging combinatorial problem.

We solve this problem for any Hopf monoid which is a universal object, that is, for any Hopf monoid which is the image under either of the functors $\mathcal{T}_q$ or $\mathcal{S}$ or Λ or their duals (Theorems 11.38–11.43). It may be surprising to note that the cancellations hinge on a simple result related to Tits projection maps (Lemma 11.37). Though our treatment of the functors $\mathcal{T}$, $\mathcal{S}$ and Λ and their duals is largely combinatorial, it is clear that projection maps do play a central role in their construction.

Instances of the antipode formulas for universal objects include antipode formulas for all objects in (0.1) except for $\mathbf{L} \times \mathbf{L}^*$ and its dual. For example, consider the six objects in the top left of this diagram. The first two are the exponential and linear order species; thus the general result recovers the antipodes of these basic objects. The antipode formulas for the remaining four objects are explicitly written down in Theorems 12.21, 12.34, 12.44 and 12.51.

We also provide cancellation-free antipode formulas for some other Hopf monoids which do not fit the above framework; see Theorems 12.17 and 12.18 for the Hopf monoid $\mathbf{L} \times \mathbf{L}^*$, and Theorems 13.4 and 13.5 for the Hopf monoids of planar rooted forests and rooted forets. The proof is very similar to the one employed for universal objects suggesting that there may be a more general framework for writing down antipode formulas which includes these examples as well.

Colored species and Q-Hopf monoids (Chapter 14). *Colored species* are higher dimensional analogues of species. They are also called *multisort species*. Roughly, an r-colored species associates a vector space to each ordered decomposition $I = S^1 \sqcup \cdots \sqcup S^r$ of a finite set I. We view the category of colored species as an analogue of the category of multigraded vector spaces.

There is a higher dimensional generalization of the preceding theory in which species are replaced by colored species. This context provides more flexibility for the definition of braidings and bilax structures. For each square matrix Q of size r, we define a braiding β_Q on r-colored species. A Hopf monoid in this braided monoidal category is called a Q-Hopf monoid.

A matrix Q is called *log-antisymmetric* if

$$q_{ij} q_{ji} = 1 \quad \text{for } 1 \le i, j \le r,$$

where q_{ij} denotes the ij-th entry of Q. The significance of these matrices is as follows. The braiding β_Q is a symmetry if and only if Q is log-antisymmetric. Note that the only log-antisymmetric matrices of size one are [1] and [−1].

One can construct a colored version of the functor $\mathcal{T}$; we denote it by $\mathcal{T}_Q$. It takes values in the category of Q-Hopf monoids. Further, if $\mathbf{q}$ is a positive colored comonoid, then $\mathcal{T}_Q(\mathbf{q})$ is the free Q-Hopf monoid on $\mathbf{q}$. Setting $Q = [q]$ recovers the functor $\mathcal{T}_q$. Similarly, one can construct a functor $\mathcal{S}_Q$ whenever Q is log-antisymmetric. This recovers $\mathcal{S}$ when $Q = [1]$ and Λ when $Q = [-1]$. If $\mathbf{q}$ is a positive colored comonoid, then $\mathcal{S}_Q(\mathbf{q})$ is the free commutative Q-Hopf monoid on $\mathbf{q}$. We provide antipode formulas for these Q-Hopf monoids in Theorems 14.18 and 14.20. The functors $\mathcal{T}_Q$, $\mathcal{S}_Q$ and their duals fit into the following commutative diagram (if one restricts them to positive colored species).

$$\begin{array}{ccc} \mathcal{T}_Q & \xrightarrow{\kappa_Q} & \mathcal{T}_Q^\vee \\ {\scriptstyle \pi_Q}\Big\Downarrow & & \Big\Uparrow{\scriptstyle \pi_Q^\vee} \\ \mathcal{S}_Q & \xrightarrow[\mathrm{id}]{} & \mathcal{S}_Q^\vee \end{array}$$

The top horizontal transformation κ_Q is the colored norm. The vertical transformations are the colored abelianization and its dual. Thus, $\mathcal{S}_Q$ is the image of the colored norm κ_Q.

Let $\mathbf{X}_{(r)}$ be the colored species which is nonzero (and equal to the base field) only if the ordered decomposition is into singletons. This is a colored version of $\mathbf{X}$. Then

$$\mathbf{L}_Q = \mathcal{T}_Q(\mathbf{X}_{(r)}) \quad \text{and} \quad \mathbf{E}_Q = \mathcal{S}_Q(\mathbf{X}_{(r)})$$

yields colored analogues of the linear order species and the exponential species (the latter for Q log-antisymmetric). These may be regarded as the simplest interesting Hopf monoids in the category of r-colored species. We also consider colored analogues of some of the other Hopf monoids occurring in diagram (0.1).

There is a colored analogue of the signature functor for any log-antisymmetric matrix Q. It sends a colored species $\mathbf{p}$ to $\mathbf{p} \times \mathbf{E}_Q$, its Hadamard product with the colored exponential species. This functor continues to be bistrong.

The generic case of the Q-norm is quite different from the log-antisymmetric case. We deduce from Varchenko's result that κ_Q is an isomorphism if no monomial in the q_{ij}'s equals one (Theorem 14.17).

Contents of Part III

In Part II, we systematically studied the monoidal category of species and Hopf monoids therein along with plenty of interesting examples constructed from combinatorial or geometric data. The goal of this part is to link the setting of Hopf monoids in species with that of graded Hopf algebras. This connection is made by means of certain bilax monoidal functors which we term *Fock functors*. The theory of bilax monoidal functors presented in Part I is extensively applied to understand how concepts involving species and graded vector spaces relate to one another via the Fock functors. This means that instead of looking at properties specific to Hopf monoids and Hopf algebras we study the Fock functors themselves. This is another principle which is central to our approach.

Sets versus numbers. A graded vector space is a family of vector spaces indexed by nonnegative integers. Recall that a species is a family of vector spaces indexed by finite sets (with further compatibilities). Thus species correspond to sets in the same way as graded vector spaces correspond to numbers. The passage from species to graded vector spaces via the Fock functors in the most naive sense amounts to replacing a set by its cardinality.

The Fock functors retain a lot of information, which is why their study is important, but at the same time, they forget or lose some information, which is why species are nicer to work with than graded vector spaces. There are a number of operations on sets for which there may or may not be any analogue for numbers. For example, disjoint union of sets corresponds to addition of numbers. This is significant because the former corresponds to the Cauchy product on species while the latter corresponds to the Cauchy product on graded vector spaces. However, note for example that intersection of sets has no analogy for numbers. Some interesting examples of well-known objects associated to numbers and the corresponding objects associated to sets can be found in Table 13.4.

Fock functors (Chapter 15)**.** The parallel between the categories of species and of graded vector spaces is reinforced by the existence of several functors between the two:

$$\mathcal{K} \quad \text{and} \quad \mathcal{K}^\vee, \qquad \text{and} \qquad \overline{\mathcal{K}} \quad \text{and} \quad \overline{\mathcal{K}}^\vee.$$

We refer to them collectively by the term *Fock functors*. For further distinction, we refer to $\mathcal{K}$ and $\mathcal{K}^\vee$ as *full Fock functors* and to $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ as *bosonic Fock functors*. The construction of these functors and the study of their properties and the relations among them constitute the main results of this chapter. The motivation for the terminology comes from Fock spaces which are studied in physics. A connection with these spaces is made later in Chapter 19 when we consider decorated versions of the Fock functors.

We show that the full Fock functors are bilax, while the bosonic Fock functors are bistrong (Theorems 15.3 and 15.6). By applying the Fock functors, one obtains four graded Hopf algebras out of each Hopf monoid in species (Theorem 15.12). These constructions of Hopf algebras were introduced by Stover [346] and further studied by Patras et al, without reference to monoidal functors. The categorical formulation is more general (for it allows the construction of other type of algebras from the corresponding type of monoids in species) and allows for a better understanding of the transfer of properties from one context to the other. Consider for instance the fact that if $\mathbf{h}$ is a commutative Hopf monoid, then of the graded Hopf algebras $\mathcal{K}(\mathbf{h})$ and $\mathcal{K}^\vee(\mathbf{h})$ only the latter is necessarily commutative. This is understood as follows: the lax monoidal functor $\mathcal{K}^\vee$ is braided, but the functor $\mathcal{K}$ is not (Propositions 15.26 and 15.28). The statement for functors is more general: it implies not only that $\mathcal{K}^\vee$ preserves commutative monoids, but also Lie monoids (and in fact any type of monoid defined from a symmetric operad, according to the results of Section 4.4). There are two other reasons why the categorical formulation is important. First, it allows us to formalize the relations between the various constructions in terms of morphisms of monoidal functors. Second, it is ripe for far-reaching generalizations, as witnessed by the results of Chapters 19 and 20.

The functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ both associate a graded vector space to a species in a very simple manner: the degree n component of $\mathcal{K}(\mathbf{p})$ is $\mathbf{p}[n]$, where $[n]$ denotes the set $\{1, \ldots, n\}$, and that of $\overline{\mathcal{K}}(\mathbf{p})$ is $\mathbf{p}[n]_{\mathrm{S}_n}$, the space of coinvariants under the action of the symmetric group S_n. Thus, roughly, the first functor forgets the action of the symmetric group while the second mods it out. The interest is in the bilax structure of $\mathcal{K}$ and $\overline{\mathcal{K}}$. This is constructed out of the functoriality of species with respect to bijections together with two basic ingredients: the unique order-preserving bijections

$$\{1, \ldots, t\} \xrightarrow{\cong} \{s+1, \ldots, s+t\} \qquad \text{and} \qquad S \xrightarrow{\cong} \{1, \ldots, s\},$$

where S is a set of integers of cardinality s. These combinatorial procedures are called shifting and standardization; they interact precisely as prescribed by the axioms of bilax monoidal functors. This provides a conceptual explanation for the repeated occurrence of shifting and standardization in the construction of Hopf algebras in combinatorics. Shifting gives rise to the lax structure of $\mathcal{K}$ and $\overline{\mathcal{K}}$ and standardization to their colax structure. Their roles can be switched; this gives rise to the other bilax functors $\mathcal{K}^\vee$ and $\overline{\mathcal{K}}^\vee$. The degree n component of $\mathcal{K}^\vee(\mathbf{p})$ is $\mathbf{p}[n]$ and that of $\overline{\mathcal{K}}^\vee(\mathbf{p})$ is $\mathbf{p}[n]^{\mathrm{S}_n}$, the space of invariants.

We study abstract properties of the Fock functors, with emphasis on the implications to the Hopf algebra constructions. We show that the full Fock functors do not preserve duality whereas the bosonic Fock functors do (at least in characteristic 0) (Corollary 15.25). We study how (co)commutativity transfers from the context of species to that of graded vector spaces. We show that $\mathcal{K}$ is braided as a colax monoidal functor, but not as a lax monoidal functor (Proposition 15.26). As a consequence, the Hopf algebra $\mathcal{K}(\mathbf{h})$ associated to a Hopf monoid $\mathbf{h}$ will be cocommutative if so is $\mathbf{h}$, but may not be commutative even when $\mathbf{h}$ is. An interesting example is that of the commutative Hopf monoid $\mathbf{L}^*$: the Hopf algebra of permutations, which is the corresponding object under the functor $\mathcal{K}$, is far from commutative (Example 15.17). Commutativity is preserved by the functor $\mathcal{K}$ in a weaker sense, however. There is an isomorphism of bilax monoidal functors

$$(\mathcal{K}, \varphi^b, \psi^b) \Rightarrow (\mathcal{K}, \varphi, \psi)$$

given by the half-twist transformation (Proposition 15.30). The lax and colax structures of the functor on the left are obtained from those of $\mathcal{K}$ by conjugation with the braidings of species and graded vector spaces (the general construction is discussed in Chapter 3). As a consequence, while the Hopf algebras $\mathcal{K}(\mathbf{h}^{\mathrm{op}})$ and $\mathcal{K}(\mathbf{h})^{\mathrm{op}}$ may not be equal, they are always canonically isomorphic. In particular, if $\mathbf{h}$ is commutative, the Hopf algebra $\mathcal{K}(\mathbf{h})$ is endowed with a canonical anti-involution. This feature has been observed for several combinatorial Hopf algebras on a case by case basis. It finds now a unified explanation.

We also study how the Fock functors interact with the primitive element functors $\mathcal{P}$ on both contexts (from Hopf monoids to Lie monoids and from graded Hopf algebras to graded Lie algebras). It is convenient to work with the functor $\mathcal{K}^\vee$ since being braided lax it preserve Lie monoids. We show that

$$\mathcal{K}^\vee\big(\mathcal{P}(\mathbf{h})\big) \subseteq \mathcal{P}\big(\mathcal{K}^\vee(\mathbf{h})\big) \qquad \text{and} \qquad \overline{\mathcal{K}}^\vee\big(\mathcal{P}(\mathbf{h})\big) = \mathcal{P}\big(\overline{\mathcal{K}}^\vee(\mathbf{h})\big)$$

as graded Lie algebras (Proposition 15.35).

The Fock functor $\mathcal{K}^\vee$ is Zinbiel-lax monoidal (Proposition 15.40). This provides a concrete example for the operad-lax monoidal functors of Chapter 4. This property is responsible for the existence of Zinbiel and dendriform structures on algebras constructed from associative or commutative monoids in species via $\mathcal{K}^\vee$. This includes some important examples of such algebras in the literature (Proposition 15.41, Examples 15.42 and 15.43).

Deformations of the Fock functors (Chapter 16)**.** The theory of Chapter 15 can be greatly generalized. The first step involves the introduction of a parameter q and the construction of q-deformations of all the objects; we explain this presently. The next step involves a further generalization to colored species and is the topic of Chapter 20.

The key idea here is to directly deform the Fock functors, rather than each object individually. We regard this as one of the main strengths of our functorial approach. This is achieved with the aid of the Schubert statistic. It is used to twist the lax or colax structure of the functors with appropriate powers of q in much the same way as the Hopf monoid of linear orders was deformed using the Schubert cocycle. The resulting bilax monoidal functors $\mathcal{K}_q$ and $\mathcal{K}_q^\vee$ map from the braided monoidal category of species to the braided monoidal category of graded vector spaces (Theorems 16.1 and 16.2). The braidings are deformed by powers of p and pq respectively. One may choose not to deform the braiding on the category of species at all; this amounts to putting p to be 1. For simplicity of exposition, we work in this setup for the present discussion.

The deformed Fock functors applied to a Hopf monoid produce q-Hopf algebras, which for $q = 1$ recover the Hopf algebras produced by the undeformed Fock functors. Thus functoriality of the construction guarantees that every Hopf algebra arising from a Hopf monoid in species can be coherently deformed. We offer this as an answer to a question raised by Thibon (in personal conversations and talks), that all "combinatorial Hopf algebras" should be the limiting case of a "quantum group".

Recall that the functor $\mathcal{K}$ is braided colax. However, its deformation $\mathcal{K}_q$ is not braided colax in general. We show that conjugating the colax structure of $\mathcal{K}_q$ with the braidings yields the functor $\mathcal{K}_{q^{-1}}$ (Proposition 16.26). This is again a result of a very general nature, and explains the nature of cocommutativity present in the q-Hopf algebras associated to $\mathcal{K}_q$.

The bosonic Fock functors admit signed analogues which we refer to as *fermionic Fock functors.* They are denoted by

$$\overline{\mathcal{K}}_{-1} \quad \text{and} \quad \overline{\mathcal{K}}^\vee_{-1}.$$

We describe them briefly. The degree n component of $\overline{\mathcal{K}}_{-1}(\mathbf{p})$ is $\mathbf{p}[n]_{\mathrm{S}_n}$, where coinvariants are now taken under a twisted action of the symmetric group S_n: tensor the usual action with the one-dimensional sign representation. The other functor $\overline{\mathcal{K}}^\vee_{-1}$ is defined similarly by using invariants instead of coinvariants. The bilax structures of the fermionic Fock functors are related to those of $\mathcal{K}_{-1}$ and $\mathcal{K}^\vee_{-1}$ and involve the Schubert statistic.

The bosonic and fermionic Fock functors are related to each other by the signature functor on species. This provides an important link between the bosonic and fermionic worlds, and results in one can be transferred to the other via properties of the signature functor. For example, since the signature functor and the bosonic Fock functor are both bistrong, it follows that so is the fermionic Fock functor.

Relations among the Fock functors. The norm transformation (Chapters 15 and 16)**.** The Fock functors in general are distinct. There are however various relations among these functors that allow us to relate the corresponding Hopf algebras in a natural manner. First of all, the functors $\mathcal{K}_q$ and $\mathcal{K}_q^\vee$, as well as $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$, and $\overline{\mathcal{K}}_{-1}$ and $\overline{\mathcal{K}}^\vee_{-1}$ are related through the contragredient construction (Propositions 15.8 and 16.3). Thus,

$$\mathcal{K}_q^\vee(-) = \mathcal{K}_q\big((-)^*\big)^*,$$

and similarly for the other pairs of functors. Second, there are natural transformations $\mathcal{K} \Rightarrow \overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee \Rightarrow \mathcal{K}^\vee$ (projection onto coinvariants and inclusion of invariants). These are morphisms of bilax monoidal functors (Theorems 15.3 and 15.6). Similarly, there are morphisms of bilax monoidal functors $\mathcal{K}_{-1} \Rightarrow \overline{\mathcal{K}}_{-1}$ and $\overline{\mathcal{K}}^\vee_{-1} \Rightarrow \mathcal{K}^\vee_{-1}$. Third, there are isomorphisms of bilax monoidal functors as follows

$$\mathcal{K}_q \cong \overline{\mathcal{K}}\big(\mathbf{L}_q \times (-)\big) \qquad \text{and} \qquad \mathcal{K}_q^\vee \cong \overline{\mathcal{K}}^\vee\big(\mathbf{L}_q^* \times (-)\big).$$

This result is given in Proposition 16.6; its signed analogue involving the fermionic Fock functor is given in Proposition 16.22. The special case $q = 1$ is discussed earlier in Propositions 15.9 and 15.10. There is thus no reason to view either one of $\mathcal{K}$, $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}_{-1}$ (or one of $\mathcal{K}^\vee$, $\overline{\mathcal{K}}^\vee$ and $\overline{\mathcal{K}}_{-1}^\vee$) as more fundamental than the others.

There is yet another relation among the functors that is particularly important in regard to later generalizations. Namely, there is a morphism of bilax monoidal functors

$$\kappa_q \colon \mathcal{K}_q \Rightarrow \mathcal{K}_q^\vee$$

(Proposition 16.15). This is called the *q-norm transformation.* For a species $\mathbf{p}$, the map $(\kappa_q)_{\mathbf{p}} \colon \mathcal{K}_q(\mathbf{p}) \to \mathcal{K}_q^\vee(\mathbf{p})$ is the action of the elements

$$\sum_{\sigma \in \mathrm{S}_n} q^{\mathrm{inv}(\sigma)} \sigma$$

on each space $\mathbf{p}[n]$, where $\mathrm{inv}(\sigma)$ denotes the number of inversions of the permutation σ. For the species of linear orders, this map has been studied by several authors in a variety of contexts (see the references in Example 16.17). The result of Zagier or of Varchenko mentioned earlier implies that it is generically invertible, and from here we deduce that over a field of characteristic 0 and if q is not a root of unity, the q-norm transformation is an isomorphism of bilax monoidal functors (Theorem 16.18). This is a rigidity result of a very general nature. The isomorphism between the tensor algebra of a vector space and the q-shuffle algebra of Duchamp, Klyachko, Krob, and Thibon, is one very special case (Example 16.31). Theorem 16.18 gives such a result for *every* Hopf algebra arising from a Hopf monoid in species. When the above hypotheses fail, the image of the transformation κ_q is truly a new bilax monoidal functor $\Im_q$, which we call the *anyonic Fock functor.* Its study appears very intriguing. A first connection with *Nichols algebras* is encountered at this point.

The parameter values ± 1 are quite interesting. The q-norm in this case is given by symmetrization or antisymmetrization. So it is indeed an instance of the norm map in group theory. Hence, the image in this situation can be understood in terms of invariants and coinvariants. Recall that the bosonic and fermionic functors are defined precisely in this manner. This leads to the following diagrams.

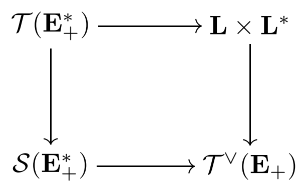
$$\begin{array}{ccc} \mathcal{K} & \overset{\kappa}{\Longrightarrow} & \mathcal{K}^\vee \\ \Downarrow & & \Uparrow \\ \overline{\mathcal{K}} & \underset{\bar\kappa}{\Longrightarrow} & \overline{\mathcal{K}}^\vee \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{K}_{-1} & \overset{\kappa_{-1}}{\Longrightarrow} & \mathcal{K}_{-1}^\vee \\ \Downarrow & & \Uparrow \\ \overline{\mathcal{K}}_{-1} & \underset{\bar\kappa_{-1}}{\Longrightarrow} & \overline{\mathcal{K}}_{-1}^\vee. \end{array}$$

In characteristic 0, the induced transformations $\bar\kappa$ and $\bar\kappa_{-1}$ are isomorphisms. Thus, in this case, one may view the isomorphic functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ as naturally associated to κ: they are the coimage and image of this morphism, respectively. The same statement holds for κ_{-1}. To summarize, in characteristic 0, for the values ± 1, the anyonic Fock functor $\Im_q$ specializes to the bosonic and fermionic Fock functors.

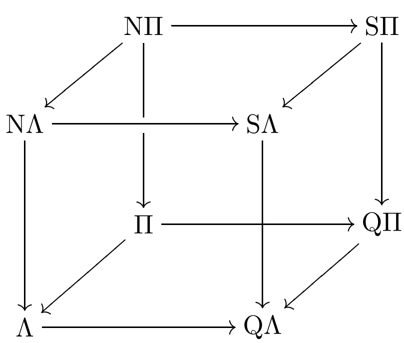
The free Fock functor and 0-bialgebras (Chapter 16)**.** As already mentioned, the Fock functors $\mathcal{K}_q$ and $\mathcal{K}_q^\vee$ are duals (contragredients) of each other. An interesting phenomenon occurs at $q = 0$, namely, these two functors coincide. In other words, $\mathcal{K}_0 = \mathcal{K}_0^\vee$; thus this resulting functor, which we call the free Fock functor, is self-dual. Further, its lax structure coincides with that of $\mathcal{K}$ and its colax structure coincides with that of $\mathcal{K}^\vee$. These results are summarized in Proposition 16.4.

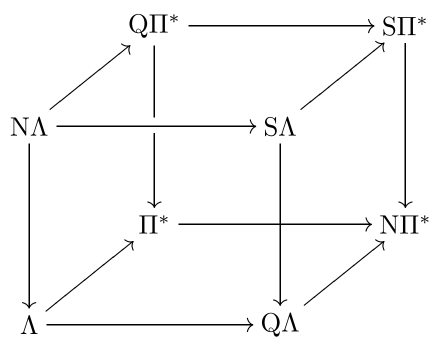
By the preceding theory, the free Fock functor sends p-Hopf monoids to 0-Hopf algebras. A rigidity result of Loday and Ronco says that any connected 0-bialgebra is free as a graded algebra and cofree as a graded coalgebra. The free Fock functor applied to any connected p-bimonoid $\mathbf{h}$ yields a connected 0-bialgebra $\mathcal{K}_0(\mathbf{h})$. As lax functors, $\mathcal{K}_0 = \mathcal{K}$, and as colax functors $\mathcal{K}_0^\vee = \mathcal{K}^\vee$. It follows that $\mathcal{K}(\mathbf{h})$ is a free graded algebra and $\mathcal{K}^\vee(\mathbf{h})$ is a cofree graded coalgebra, for any connected p-bimonoid $\mathbf{h}$ (Proposition 16.11). Results of this kind have been recently obtained by Livernet [233] and our functorial approach serves to further clarify them.

Hopf algebras from geometry (Chapter 17)**.** Many of the Hopf algebras associated to the Hopf monoids in (0.1) are familiar and have received a great deal of attention in the recent literature. For simplicity, let us work over a field of characteristic 0 and concentrate on the following small portion of diagram (0.1), which gives rise to the most familiar Hopf algebras.



Applying each of the functors $\mathcal{K}$, $\overline{\mathcal{K}}$, $\mathcal{K}^\vee$, and $\overline{\mathcal{K}}^\vee$ one obtains a commutative square of graded Hopf algebras. In view of the natural transformations $\mathcal{K} \Rightarrow \overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee \Rightarrow \mathcal{K}^\vee$, these four squares can be assembled into two commutative cubes.





The above mentioned natural transformations are responsible for the morphisms between the back and front faces. As already mentioned, in characteristic 0, the functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ are naturally isomorphic. This explains the common face of the cubes. The duality between the back faces is due to the corresponding relation between $\mathcal{K}$ and $\mathcal{K}^\vee$.

We have used the notation of [12], where all these objects are studied. Thus Λ, NΛ, QΛ, and SΛ are the Hopf algebras of symmetric functions, noncommutative symmetric functions, quasi-symmetric functions, and permutations, respectively, while Π is the Hopf algebra of symmetric functions in noncommuting variables. They provide examples of how graded Hopf algebras can be obtained through a combination of the functors $\mathcal{T}$ or $\mathcal{S}$, the Hadamard product, and duality, with the Fock functors.

Since the Hadamard product on graded vector spaces is not bilax, the Hopf algebras associated to the Hopf monoid $\mathbf{L} \times \mathbf{L}^*$ via the Fock functors cannot be described in terms of the Hopf algebras associated to $\mathbf{L}^*$ and $\mathbf{L}$. This provides evidence for our claim that proper understanding of these Hopf algebras requires the consideration of Hopf monoids in species.

The claim is further substantiated by the fact that many of the Hopf monoids we consider are universal, that is, they are special values of the functors $\mathcal{T}$, $\mathcal{S}$, or their duals, on the category of species. Depending on which Fock functors we use, these universal properties may be lost in the passage from Hopf monoids to Hopf algebras and therefore can never be fully appreciated if one is confined to the world of graded vector spaces and graded Hopf algebras. For instance, the species $\mathbf{L}$ is the free monoid on one generator, and can therefore be regarded as an analogue of the polynomial algebra in one variable. By contrast, the Hopf algebra of permutations which is its image under one of the full Fock functors has no analogous property.

Duality is another property that may be lost in the passage from Hopf monoids to Hopf algebras. For example, $\mathcal{T}(\mathbf{E}_+^*)$ are $\mathcal{T}^\vee(\mathbf{E}_+)$ are dual as Hopf monoids, but NΠ and QΠ (which are their images under $\mathcal{K}$) are not dual as graded Hopf algebras.

Two questions regarding graded Hopf algebras are often interesting and difficult: the determination of a linear basis of the space of primitive elements and the determination of the structure constants of the antipode. For most of the Hopf algebras associated to the Hopf monoids in (0.1), the first question was answered in [12]. In Part II, we consider the same questions directly for the Hopf monoids, which are more fundamental objects. In view of the results of Chapter 15, the answers to these questions for a Hopf algebra of the form $\overline{\mathcal{K}}(\mathbf{h})$ can be told from the answers for the Hopf monoid $\mathbf{h}$.

Hopf algebras from combinatorics (Chapter 17)**.** The Hopf monoids of Chapter 13 give rise to another long list of Hopf algebras. Several of these Hopf algebras have received much attention in the recent literature. We mention in particular work of Gessel and Malvenuto in connection to the Hopf algebras of posets; of Ehrenborg in connection to the Hopf algebras of set-graded posets; of Sagan, Schmitt, and Stanley in connection to the Hopf algebras of graphs; of Connes–Kreimer and Grossman–Larson in connection to the Hopf algebras of forests; of Crapo and Schmitt in connection to the Hopf algebras of matroids. Similarly, the morphisms arising from universal constructions yield well-known generating functions for the corresponding combinatorial objects. They include the enumerator of poset partitions, the enumerator of descents, the chromatic function for graphs and its variant for labeled graphs, the quasi-symmetric flag function, a generating function for matroids, and so on.

Adjoints of the Fock functors (Chapter 18)**.** This chapter is devoted to the construction of the various adjoints of the Fock functors. We view each of $\mathcal{K}$, $\overline{\mathcal{K}}$, $\mathcal{K}^\vee$, and $\overline{\mathcal{K}}^\vee$ as a functor at three levels: one from species to graded vector spaces, another from monoids to graded algebras, and another from comonoids to graded coalgebras. A complete analysis of the left and right adjoints of each of these functors (which may exist or not) is presented. This includes important notions such as the free monoid on a graded algebra (relative to each of the four functors). These notions are related through composition of adjunctions. A summary of the results is given in Tables 18.1 and 18.2. The free monoid on a graded algebra is not to be confused with the free monoid on a species discussed earlier. This chapter also serves to formalize the analogy between the tensor and symmetric algebra functors on graded vector spaces and the corresponding functors $\mathcal{T}$ and $\mathcal{S}$ on species.

Decorated Fock functors (Chapter 19)**.** The Fock functors as discussed thus far admit a decorated version: one can define a Fock functor for each vector space V, with $V = \Bbbk$ recovering the undecorated case (Theorems 19.2 and 19.33). We denote them by adding a subscript V to the previous notation. In a sense, one may view the result of applying a decorated Fock functor, say $\mathcal{K}_V$, to a species $\mathbf{p}$ as a version of the graded vector space $\mathcal{K}(\mathbf{p})$ in which the given combinatorial structure determined by the species $\mathbf{p}$ has been decorated with elements of the vector space V.

The exponential species $\mathbf{E}$ admits a decorated version which we denote by $\mathbf{E}_V$. It recovers $\mathbf{E}$ for $V = \Bbbk$ and retains all its important features. For example, it is the free commutative monoid as well as the cofree cocommutative comonoid on $\mathbf{X}_V$. The latter is the species which is V on singletons and 0 otherwise. Further, the dual of $\mathbf{E}_V$ is $\mathbf{E}_{V^*}$. In particular, if V is finite-dimensional, then $\mathbf{E}_V$ is self-dual (the self-duality depending on a choice of an isomorphism $V \cong V^*$). The significance of the decorated exponential species in the present context is brought about by the relation

$$\mathcal{K}_V(-) = \mathcal{K}\big((-) \times \mathbf{E}_V\big)$$

This allows us to quickly generalize the theory of undecorated Fock functors to the decorated setting. For instance, the bilax structure of $\mathcal{K}_V$ arises from that of the Hadamard product plus the bimonoid structure of $\mathbf{E}_V$. A similar relation holds for the other decorated functors as well.

The decorated Fock functors give a systematic procedure of decorating any graded Hopf algebra that arises from a Hopf monoid in species.

Fock spaces. Up-down and creation-annihilation operators (Chapter 19)**.** Fix a vector space V. Classical full Fock space is the underlying space of the tensor algebra on V. Similarly, bosonic Fock space is the underlying space of the symmetric algebra on V, while fermionic Fock space is the underlying space of the exterior algebra on V. In physical terms, V stands for the quantum states of a single particle, while the Fock spaces describe quantum states with a variable number of particles. The terms bosonic and fermionic are used depending on whether the particles are bosons or fermions. These spaces carry certain operators called *creation* and *annihilation*. The former increases the number of particles by 1, while the latter decreases the number of particles by 1. Further, these operators satisfy canonical commutation relations, see equations (19.4) and (19.6).

The first observation is that bosonic and fermionic Fock spaces are the values of the decorated bosonic and decorated fermionic Fock functors respectively on the exponential species. The second observation is that the exponential species is naturally equipped with what we call *up-down operators*. The third observation is that Fock functors convert up-down operators on species to creation-annihilation operators on graded vector spaces. This explains the existence of such operators on Fock spaces.

Following Guţă and Maassen [158] and Bożejko and Guţă [64], we define generalized Fock spaces to be the values of the decorated Fock functors on any species with up-down operators. In particular, we add to their constructions by paying attention to the monoidal properties of the functors. Propositions 19.16, 19.21 and 19.38 serve to illustrate this point.

We then introduce the notion of a species with balanced operators. It is a species with up-down operators in which the operators are required to satisfy further compatibilities. The motivating example is that of the exponential species. The point is that the decorated bosonic and fermionic Fock functors convert a species with balanced operators to a graded vector spaces with creation-annihilation operators which satisfy the canonical commutation relations (Propositions 19.27 and 19.39). We illustrate this on a number of examples including the species of rooted trees and the species of elements.

Colored Fock functors (Chapter 20)**.** There is a higher dimensional generalization of the preceding theory in which species are replaced by colored species, and graded vector spaces by multigraded vector spaces. Recall from *Graded vector spaces* and *Colored species and Q-Hopf monoids* that for each square matrix Q of size r, one can define a braiding on r-colored species as well as on $\mathbb{N}^r$-graded vector spaces. We construct bilax monoidal functors $\mathcal{K}_Q$ and $\mathcal{K}_Q^\vee$ from the category of r-colored species to the category of $\mathbb{N}^r$-graded vector spaces (Theorem 20.1). The Fock functors as well as their q-deformations occur as special cases when $Q = [q]$ is a matrix of size 1. The braidings are to be chosen as follows. If the square matrix P is used for r-colored species, then the matrix $P \times Q$ (the Hadamard product of P and Q) is to be used for $\mathbb{N}^r$-graded vector spaces. It follows that these functors take P-Hopf monoids to $(P \times Q)$-Hopf algebras. The constructions of these $\mathbb{N}^r$-graded Hopf algebras with respect to nontrivial braidings are now considerably more general than those of Chapter 15.

Recall that the braiding β_Q is a symmetry if and only if Q is log-antisymmetric. In this situation, the symmetric groups act via the braiding on appropriate components of a colored species (Proposition 20.3). By taking invariants and coinvariants with respect to this action, one obtains bistrong functors

$$\overline{\mathcal{K}}_Q \quad \text{and} \quad \overline{\mathcal{K}}_Q^\vee.$$

The bosonic and fermionic Fock functors correspond to the log-antisymmetric matrices $Q = [1]$ and $Q = [-1]$ respectively.

We construct a higher dimensional version of the norm transformation

$$\kappa_Q \colon \mathcal{K}_Q \Rightarrow \mathcal{K}_Q^\vee$$

and show that it is a morphism of bilax functors (Proposition 20.9). The image of κ_Q is a new bilax monoidal functor $\Im_Q$ from r-colored species to $\mathbb{N}^r$-graded vector spaces. This functor is the multivariate version of the anyonic Fock functor. If the characteristic of the field is 0 and Q is log-antisymmetric, then $\Im_Q$ coincides with $\overline{\mathcal{K}}_Q$ and $\overline{\mathcal{K}}_Q^\vee$. If Q has generic entries, then the norm κ_Q is an isomorphism (Theorem 20.11). This is again an application of Varchenko's result.

As an example, we consider a colored analog of the Hopf monoid $\mathbf{E}$ which we denote by $\mathbf{E}_{(r)}$. The functors $\mathcal{K}_Q$ and $\mathcal{K}_Q^\vee$ applied to $\mathbf{E}_{(r)}$ yield the Hopf algebra of noncommutative polynomials in r variables and the quantum shuffle algebra respectively. The Hopf algebra $\Im_Q(\mathbf{E}_{(r)})$, on the other hand, is the *quantum symmetric algebra* associated to the matrix Q. The terminology *Nichols algebra of diagonal type* is used for this object in the theory of abstract Hopf algebras. Further, special choices of Q lead to Manin's quantum linear spaces or to the nilpotent part of quantum enveloping algebras. The explicit calculation of Nichols algebras is widely regarded as a difficult problem, intimately related to the classification of pointed Hopf algebras. One may therefore expect the calculation of explicit values of the functor $\Im_Q$ to be similarly challenging and interesting.

The question of what Hopf algebras may arise when other colored species are considered is a completely open avenue. As another example, we mention that the functor $\mathcal{K}_Q$ applied to the colored linear order species yields a Q-Hopf algebra indexed by r-signed permutations.

Yang–Baxter deformation of decorated Fock functors (Chapters 19 and 20). Let R be a Yang–Baxter operator on the space of decorations V. In this setting, one can define functors $\mathcal{K}_{V,R}$ and $\mathcal{K}^\vee_{V,R}$ along with a norm transformation between them. The image of the norm yields a functor denoted $\Im_{V,R}$. These functors are not bilax in the usual sense. Just as bilax functors are the functorial analogues of bialgebras, these are the functorial analogues of braided bialgebras [356, Definition 5.1]. The Yang–Baxter operator plays a role in the lax and colax structures of these functors as well as in the braiding axiom. Applying $\Im_{V,R}$ to the exponential species $\mathbf{E}$ yields the *Nichols algebra* associated to R (also known as the *quantum symmetric algebra*).

By letting R to be the operator which switches the two tensor factors, one recovers the decorated Fock functors $\mathcal{K}_V$, $\mathcal{K}^\vee_V$ and $\Im_V$. By fixing a scalar q and letting R to be the operator

$$v \otimes w \mapsto q\, w \otimes v,$$

one obtains one-parameter deformations of the decorated Fock functors. It turns out that these are bilax in the usual sense. Thus we have $\mathcal{K}_{V,q}$ and $\mathcal{K}^\vee_{V,q}$, which deform the decorated full Fock functors, a decorated q-norm

$$\kappa_q \colon \mathcal{K}_{V,q} \Rightarrow \mathcal{K}^\vee_{V,q}$$

which is a morphism of bilax monoidal functors, and $\Im_{V,q}$, which is the image of κ_q. By letting $V = \Bbbk$, we recover the deformed Fock functors $\mathcal{K}_q$, $\mathcal{K}^\vee_q$ and $\Im_q$.

Let Q be a square matrix of size r, where r is the dimension of V. Fix a basis $x_1, x_2, \ldots, x_r$ of V, and consider the Yang–Baxter R_Q operator on V:

$$V \otimes V \to V \otimes V, \qquad x_i \otimes x_j \mapsto q_{ji}\, x_j \otimes x_i$$

where i and j vary between 1 and r, and q_{ji} denotes the entries of the matrix Q. The functors $\mathcal{K}_{V,R_Q}$, $\mathcal{K}^\vee_{V,R_Q}$ and $\Im_{V,R_Q}$ are closely related to the colored Fock functors $\mathcal{K}_Q$, $\mathcal{K}^\vee_Q$ and $\Im_Q$. The precise relation is given in Theorem 20.19.

The theory of creation-annihilation operators can also be developed in the setting of Yang–Baxter operators. They satisfy appropriately deformed commutation relations (Propositions 19.41 and 19.48).

Appendices

Four appendices supplement the text. Appendix A reviews some basic notions from category theory, including adjunctions, equivalences, and colimits. The contents of Appendices B, C and D are summarized below.

Operads (Appendix B)**.** Operads are monoids in the monoidal category of species under substitution. These objects have been at the focus of intense activity in recent times, though not often from this point of view.

We have mentioned a variety of tensor products on species, centering primarily on the Cauchy product. Of these, the substitution product is the most subtle, and a definition in full generality requires some care. When the species do not vanish on the empty set, two different versions of substitution arise. One notion of substitution (B.9) gives rise to the general notion of operad and the other (B.15) to the general notion of cooperad. We provide a complete proof of associativity for the former version of substitution (Lemma B.14) and also describe its internal Hom (Proposition B.26). Further, we explain how a proper understanding of the latter version of substitution requires a more general setup, which is that of lax monoidal categories.

The main use for operads in this monograph occurs in Chapter 4, as already mentioned. To each operad corresponds a type of monoid (in a monoidal category) and a type of monoidal functor (between monoidal categories). Types of monoids may also be understood in terms of modules over operads. The two notions are equivalent (Proposition B.27).

The substitution and Hadamard products define a 2-monoidal structure on the category of species (Propositions B.31 and B.35). This provides a context for the notion of Hopf operad.

Pseudomonoids and the looping principle (Appendix C)**.** The notion of pseudomonoid is a 2-dimensional analogue of the notion of monoid in a monoidal category. We provide a complete definition, following work of Day, McCrudden, and Street among others. The context is that of monoidal 2-categories (not to be confused with 2-monoidal categories). A pseudomonoid possesses a product that is associative up to a 2-cell. A monoidal category is an example of a pseudomonoid (in the 2-category Cat, which is monoidal under Cartesian product). Our main interest in pseudomonoids stems from a result we prove in Proposition 6.73, which states that a 2-monoidal category can be viewed as a pseudomonoid in two different monoidal 2-categories. These are the 2-categories lCat and cCat whose objects are monoidal categories and whose arrows are respectively lax and colax monoidal functors. This and other examples of pseudomonoids are summarized in Table C.1.

The passage from Cat to lCat and cCat is an instance of the *lax and colax constructions*. They are discussed in Section C.2.3. They play an important role in connection to the notion of higher monoidal categories, as already mentioned.

The set of endomorphisms of an object in a category is an ordinary monoid under composition. This is a first instance of the *looping principle* which is the subject of Section C.4. We are mainly interested in a 2-dimensional version of the principle which relates pseudomonoids (in a monoidal 2-category) to bicategories (enriched in the same monoidal 2-category). We arrive at this in Section C.4.4, after discussing simpler instances of the looping principle. We also discuss how some important examples of 2-monoidal categories arise in this manner, as loops in bicategories enriched by either lCat or cCat.

Monoids and the simplicial category (Appendix D)**.** We discuss two generalizations of the notion of monoid in a monoidal category: lax monoids and homotopy monoids. They are due to Day and Street [94] and Leinster [229], respectively. We are mainly interested in two special instances of these notions: lax monoidal categories, and a particular homotopy monoid that we construct in the context of natural transformations between monoidal functors.

The key notion on which the generalizations are based is Mac Lane's simplicial category. This category plays a universal role in connection to monoids (Proposition D.2). Relaxing the conditions in this result leads to the notions of lax monoids and homotopy monoids.

We explain the notion of a lax monoidal category in some detail (Definition D.3). This is an example of a lax monoid. It, however, plays only a minor role in this monograph. It is required for a proper understanding of the second substitution product on species, as already mentioned.

We explore a notion of convolution for natural transformations from a colax monoidal functor $\mathcal{F}$ to a lax monoidal functor $\mathcal{G}$. This may be regarded as an analogue of the convolution operation on the set $\mathrm{Hom}(C, A)$ of maps from a comonoid C to a monoid A. More precisely: The role of the set $\mathrm{Hom}(C, A)$ is played by a certain contravariant functor $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ on Mac Lane's simplicial category, that is, by an augmented simplicial set. An n-simplex in this simplicial set is a natural transformation from $\mathcal{F}_n$ to $\mathcal{G}_n$, where

$$\mathcal{F}_n(A_1, \dots, A_n) := \mathcal{F}(A_1 \bullet \cdots \bullet A_n)$$

is a functor from the n-fold Cartesian product of the source category of $\mathcal{F}$ with itself to its target category (with $\bullet$ denoting the tensor product of the source category). Convolution of natural transformations turns $\mathsf{N}_{\mathcal{F},\mathcal{G}}$ into a lax monoidal functor (Theorem D.9). This is an example of a homotopy monoid. We apply these ideas to Hopf lax functors, see Proposition D.12 and the discussion following it.

Related work

Several references to related work in the literature are given in the text. Of these, Joyal's work on braided monoidal categories and on species [181, 184] has been the most influential. We would like to view our work as a contribution to his ideas.

We would also like to highlight the work of the following authors. Schmitt and Stover, independently and at about the same time, were the first to describe constructions of Hopf algebras from Hopf monoids in species [322, 346]. The connection to combinatorial Hopf algebras was brought forth by Patras and Reutenauer [291]. Patras and Schocker [292, 293], Patras and Livernet [234] and Livernet [233] have further advanced the subject.

Our interest in Hopf monoids in species developed from a lecture by Sagan in Montréal in 2001 on the algebra Π. In trying to understand the Hopf algebras Π and $Q\Pi$ from the point of view of universal properties, as had been done for $Q\Lambda$ in [10], we were led to the consideration of species.