CHAPTER 1

Moduli Spaces Associated to Dynamical Systems

In this chapter we give an introduction to the objects that we will study, dynamical systems, abstract moduli spaces, dynamical parameter spaces, and dynamical moduli spaces. Ultimately we will want to study both the geometry and the arithmetic of these objects. A basic reference for arithmetic dynamics is [110], and we refer the reader especially to Chapter 4 of [110], which deals with moduli questions.

1.1. Dynamical definitions

Discrete dynamics is the study of iteration of functions. Abstractly, we start with a set X (generally having some additional structure) and a self-map (satisfying appropriate properties)

$$\phi\colon X \longrightarrow X$$

from X to itself. The map ϕ may be iterated to yield new self-maps of X; we write

$$\phi^n = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ iterations}}$$

for the nth iterate of ϕ.

The iterates of ϕ applied to a point $x \in X$ give the (*forward*) *orbit of* x, which we denote by

$$\mathcal{O}_\phi(x) = \{x, \phi(x), \phi^2(x), \phi^3(x), \dots\}.$$

A fundamental problem in discrete dynamics is the study of orbits, and in particular, the classification of points in X according to the behavior of their orbits.

Example 1.1. We consider dynamical systems defined by analytic self-maps of complex manifolds. For example, taking $X = \mathbb{C}$, we could look at a polynomial map $\phi(z) \in \mathbb{C}[z]$ or at the exponential map $\phi(z) = e^z$. The dynamics of polynomial maps differ in many ways from the dynamics of the map e^z.

Suppose instead that we study analytic dynamics on the compact manifold $X = \mathbb{P}^1(\mathbb{C})$. Aside from the constant map $\phi(z) = \infty$, the only analytic self-maps of $\mathbb{P}^1(\mathbb{C})$ are rational functions $\phi(z) \in \mathbb{C}(z)$. We observe that the set of self-maps of $\mathbb{P}^1(\mathbb{C})$, which we denote by $\operatorname{Hom}\bigl(\mathbb{P}^1(\mathbb{C})\bigr)$, breaks up into a disjoint countable union consisting of the sets

$$\operatorname{Hom}_d\bigl(\mathbb{P}^1(\mathbb{C})\bigr) = \bigl\{\phi \in \operatorname{Hom}\bigl(\mathbb{P}^1(\mathbb{C})\bigr) : \deg(\phi) = d\bigr\} \quad \text{for } d = 0, 1, 2, \dots.$$

Further, as we will see, $\operatorname{Hom}_d\bigl(\mathbb{P}^1(\mathbb{C})\bigr)$ has a natural structure as a Zariski open subset of $\mathbb{P}^{2d+1}(\mathbb{C})$, so for each $d \ge 0$, there is a continuously varying $(2d+1)$-dimensional family of rational maps of degree d.

In fancy terminology, we start with a category $\mathcal{C}$ whose objects are sets. Then each object $X \in \operatorname{Obj}(\mathcal{C})$ and each self-morphism $\phi \in \operatorname{Hom}(X, X)$ yields a dynamical system (X, ϕ), since we can iterate the map ϕ and apply it to the points of X.

For convenience, we write $\operatorname{Hom}(X)$ for the self-morphisms of X. (The standard categorical notation would be $\operatorname{Hom}(X, X)$ or $\operatorname{End}(X)$.)

Definition 1.2. Two self-morphisms $\phi, \psi \in \operatorname{Hom}(X)$ are *equivalent* (or *conjugate*) if there is an automorphism, i.e., a self-isomorphism $f \in \operatorname{Aut}(X)$ that makes the following square commute:

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & X \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f} \\ X & \xrightarrow{\ \psi\ } & X. \end{array}$$

We write

$$\phi^f = f^{-1} \circ \phi \circ f$$

for the conjugation of ϕ by f, so ψ is equivalent to ϕ if $\psi = \phi^f$ for some $f \in \operatorname{Aut}(X)$. We may view conjugation as “changing coordinates” on X. This makes it the “right” equivalence relation for dynamics, since simultaneously making the same change of coordinates on the domain and range of ϕ gives a relation that commutes with iteration,

$$(\phi^f)^n = (\phi^n)^f.$$

Further, it is easy to see that conjugation transforms orbits in a natural way,

$$\mathcal{O}_{\phi^f}(x) = f\left(\mathcal{O}_\phi\big(f^{-1}(x)\big)\right).$$

Even if the space X has no further structure, we can still classify orbits according to their length.

Definition 1.3. A point $x \in X$ if *periodic for* ϕ if

$$\phi^n(x) = x \quad \text{for some } n \geq 1.$$

We write $\operatorname{Per}(\phi, X)$ for the set of ϕ-periodic points, and we let

$$\operatorname{Per}_n(\phi, X) = \big\{x \in X : \phi^n(x) = x\big\}.$$

The (*exact*) *period* of x is the smallest n such that $\phi^n(x) = x$. A point of period one is called a *fixed point*. There is also a notion of *formal period*, which we discuss in Section 4.1.[1]

A point $x \in X$ if *preperiodic for* ϕ if its orbit is finite, or equivalently if some point in its orbit is periodic. We write $\operatorname{PrePer}(\phi, X)$ for the set of ϕ-preperiodic points, and we let

$$\operatorname{PrePer}_{m,n}(\phi, X) = \big\{x \in X : \phi^{m+n}(x) = \phi^m(x)\big\}.$$

In particular, $\operatorname{PrePer}_{0,n}(\phi, X) = \operatorname{Per}_n(\phi, X)$.

When the space X is fixed, we omit it from the notation and write $\operatorname{Per}(\phi)$ and $\operatorname{PrePer}(\phi)$.

Remark 1.4. If $\phi, \psi \in \operatorname{Hom}(X)$ satisfy

$$f \circ \phi = \psi \circ f \tag{1.1}$$

[1]In classical dynamical terminology, the exact period is called the *prime period*, which we deprecate because of possible confusion with periods that are prime numbers.

for some morphism $f \in \mathrm{Hom}(X)$ that need not be an automorphism, then ϕ and ψ are said to be *semiconjugate*. Semi-conjugation is also useful in studying dynamics, since (1.1) implies that

$$f \circ \phi^n = \psi^n \circ f \quad \text{for all } n \geq 0.$$

1.2. Moduli spaces: what they are and why they're useful

We start with an abstract formulation of a *moduli problem*. Suppose that we are given:

- A collection of objects parametrized by a parameter space $\mathcal{P}$.
- An equivalence relation $\sim$ on $\mathcal{P}$.

The goal is to classify the equivalence classes in a functorial way. A *moduli space* for this problem is a space $\mathcal{M}$ that is "functorially" isomorphic to the quotient $\mathcal{P}/{\sim}$. We illustrate with a familiar example.

Example 1.5. Consider

$$\mathcal{P} = \bigl\{(A,B) : 4A^3 + 27B^2 \neq 0\bigr\}.$$

Then $\mathcal{P}$ parametrizes elliptic curves given by Weierstrass equations[2]

$$(A,B) \in \mathcal{P} \;\rightsquigarrow\; E_{A,B} : Y^2 = X^3 + AX + B.$$

Our equivalence relation is isomorphism of the associated elliptic curves, i.e.,

$$\begin{aligned}(A,B) \sim (A',B') &\iff E_{A,B} \text{ is isomorphic to } E_{A',B'}\\ &\iff A = u^4A' \text{ and } B = u^6B' \text{ for some } u \neq 0.\end{aligned}$$

The moduli space $\mathcal{M} = \mathcal{P}/{\sim}$ may be identified with the affine line $\mathbb{A}^1$ via the j-invariant map

$$j\colon \mathcal{P} \longrightarrow \mathbb{A}^1, \quad j(A,B) = \frac{4A^3}{4A^3 + 27B^2}.$$

We now describe the key functorially property that makes $\mathbb{A}^1$ the moduli space for this moduli problem. Let

$$E \longrightarrow T$$

be any algebraic family of elliptic curves parametrized by the points of a variety T. Then the natural map of sets

$$T \longrightarrow \mathbb{A}^1, \quad t \longmapsto j(E_t),$$

is automatically a *morphism* of varieties.

1.3. Fine moduli spaces and coarse moduli spaces

Suppose now that the parameter space $\mathcal{P}$ is a variety (scheme, stack,...) that parametrizes some sort of algebro-geometric object, e.g., curves of genus g or principally polarized abelian varieties of dimension d, and that $\sim$ identifies objects that are isomorphic.

A (*coarse*) *moduli space* is a variety (scheme, stack,...) $\mathcal{M}$ with the following properties:

- $\mathcal{M}(k) = \mathcal{P}(k)/{\sim}$ for all algebraically closed fields.

[2]To simplify the exposition, we are being a little imprecise. What we should say is that $\mathcal{P}$ parametrizes elliptic curves defined over an algebraically closed field of characteristic not equal to 2 or 3.

- If $X \to T$ is an algebraic family of the type of objects parametrized by $\mathcal{P}$, then the induced map of sets,

$$T \longrightarrow \mathcal{M}, \quad t \longmapsto (\text{equivalence class of } X_t), \tag{1.2}$$

  is a morphism.

We might ask for more, namely we might want a *universal family.* A *fine moduli space* is a pair of varieties (schemes, stacks,...) $(\mathcal{X}, \mathcal{M})$ and a map

$$\mathcal{X} \to \mathcal{M}$$

such that:

- $\mathcal{M}$ is a coarse moduli space.
- For each $m \in \mathcal{M}$, the fiber $\mathcal{X}_m$ is in the isomorphism class classified by m.
- For any algebraic family $X \to T$ of objects, let $f\colon T \to \mathcal{M}$ be the map (1.2) coming from the fact that $\mathcal{M}$ is a course moduli space. Then there are morphisms making a commutative square

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & \mathcal{M} \end{array}$$

  such that the restriction to every fiber $X_t \to \mathcal{X}_{f(t)}$ is an isomorphism.

Example 1.6. There is no fine moduli space for the elliptic curve problem. If there were such a space $\mathcal{E} \to \mathbb{A}^1$, then as you walk around the $j = 0$ or $j = 1728$ points, the fiber would be subject to a nontrivial automorphism. And even ignoring these special points, the fact that elliptic curves have an automorphism of order two, i.e., $P \to -P$, precludes the existence of a fine moduli space.

This is a general phenomenon. If $\mathcal{M}$ is a coarse moduli space and if some of the objects admit nontrivial automorphisms, then there is no universal family. A standard way to get rid of automorphisms is to add level structure, thereby allowing the construction of a fine moduli space for objects with level structure.

Example 1.7. The moduli problem

$$\left\{\begin{matrix}\text{isomorphism classes of elliptic curves} \\ \text{with a marked point of order } N\end{matrix}\right\}$$

admits a fine moduli space provided $N > 3$, since an automorphism of an elliptic curve never fixes any point of order $N > 3$.

1.4. Parameter spaces for dynamical systems

We begin with dynamical systems on the projective line $\mathbb{P}^1$. In the analytic or algebraic categories, the self-maps of $\mathbb{P}^1$ are rational maps, which leads to the dynamical parameter spaces

$$\mathrm{Hom}_d = \{\text{degree } d \text{ rational maps } \mathbb{P}^1 \to \mathbb{P}^1\}.$$

An element $\phi \in \mathrm{Hom}_d$ is a rational function

$$\phi(z) = \frac{a_0 z^d + a_1 z^{d-1} + \cdots + a_{d-1} z + a_d}{b_0 z^d + b_1 z^{d-1} + \cdots + b_{d-1} z + b_d} = \frac{F_a(z)}{F_b(z)}$$

with at least one of a_d and b_d not zero and with $\gcd(F_a, F_b) = 1$. The map ϕ does not change if we multiply the numerator and the denominator by a nonzero number,

$$\phi(z) = \frac{\lambda F_a(z)}{\lambda F_b(z)} \quad \text{for any } \lambda \neq 0.$$

Thus Hom_d has a natural description as a subvariety of projective space,

$$\begin{aligned}\mathrm{Hom}_d &= \bigl\{[a,b] \in \mathbb{P}^{2d+1} : a_0, b_0 \text{ not both } 0 \text{ and } \gcd(F_a, F_b) = 1\bigr\} \\ &= \bigl\{[a,b] \in \mathbb{P}^{2d+1} : \mathrm{Resultant}(F_a, F_b) \neq 0\bigr\}.\end{aligned}$$

Every rational self-map $\mathbb{P}^1 \to \mathbb{P}^1$ is automatically a morphism, but this is no longer true in higher dimension. We let

$$\begin{aligned}\mathrm{Rat}_d^n &= \{\text{degree } d \text{ rational maps } \mathbb{P}^n \to \mathbb{P}^n\}, \\ \mathrm{Hom}_d^n &= \{\text{degree } d \text{ morphisms } \mathbb{P}^n \to \mathbb{P}^n\}.\end{aligned}$$

Note that maps in Hom_d^n can be iterated, but iterates of general rational maps need not be well-defined. However, dominant rational maps can be iterated, although some points may have orbits that terminate in the indeterminacy locus of the map.

Every $\phi \in \mathrm{Rat}_d^n$ has the form

$$\phi = [\phi_0, \dots, \phi_n],$$

where each ϕ_i is a homogeneous polynomial of degree d in $X_0, \dots, X_n$, and the polynomials $\phi_0, \dots, \phi_n$ have no common factor. The map ϕ is in Hom_d^n if and only if $\phi_0, \dots, \phi_n$ have no nontrivial common zeros. Writing

$$\phi_i = \sum_j a_{ij} X^j \quad \text{using a multi-index } j = (j_0, \dots, j_n),$$

we identify ϕ with a point in projective space

$$\phi \leftrightarrow [a_{ij}]_{i,j} \in \mathbb{P}^N, \quad \text{where } N = N(n,d) = \binom{n+d}{d}(n+1) - 1. \tag{1.3}$$

Conversely, every point in $\mathbb{P}^N$ is identified with a rational map $\mathbb{P}^n \to \mathbb{P}^n$ of some degree. We observe that

$$\mathrm{Hom}_d^n \subset \mathrm{Rat}_d^n \subset \mathbb{P}^N$$

are Zariski open subsets of $\mathbb{P}^N$. For Hom_d^n we have the following precise result.

Theorem 1.8. *There exists a geometrically irreducible polynomial $\mathcal{R} \in \mathbb{Z}[a_{ij}]$ in the coefficients of ϕ such that*

$$\phi \in \mathrm{Hom}_d^n \iff \mathcal{R}(\phi) \neq 0.$$

In particular, Hom_d^n *is an affine variety. The polynomial $\mathcal{R}$ is called the* Macaulay resultant of ϕ. *It is multihomogeneous in the coefficients of the coordinate polynomials $\phi_0, \dots, \phi_n$ defining ϕ.*

PROOF. See [25, Theorem 3.8], [29, Chapter 3], or [56]. □

Remark 1.9. For a map $\phi = [\phi_0, \dots, \phi_n] \in \mathrm{Hom}_d^n$, the homogeneous polynomials $\phi_0, \dots, \phi_n$ are only determined up to multiplication by a nonzero constant c. The Macaulay resultant $\mathcal{R}$ depends on the choice of $\phi_0, \dots, \phi_n$, so is only well-defined up to a power of c; more precisely,

$$\mathcal{R}(c\phi_0, \dots, c\phi_n) = c^{(n+1)d^n}\, \mathcal{R}(\phi_0, \dots, \phi_n).$$

In any case, the question of whether $\mathcal{R}(\phi)$ vanishes is independent of the choice of coordinate functions for ϕ, so Theorem 1.8 makes sense. We will use the Macaulay resultant later to define the minimal resultant of a rational map; see Section 3.2.

1.5. Moduli spaces for dynamical systems

The projective linear group

$$\mathrm{PGL}_{n+1} = \mathrm{Aut}(\mathbb{P}^n)$$

acts on Rat_d^n and Hom_d^n in various ways. For dynamics we are interested in the conjugation action:

$$\phi^f = f^{-1} \circ \phi \circ f \quad \text{for } \phi \in \mathrm{Rat}_d^n \text{ and } f \in \mathrm{PGL}_{n+1}.$$

As noted earlier, this action satisfies

$$(\phi^n)^f = (\phi^f)^n$$

and corresponds to change of variables for iteration:

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\ \phi\ } & \mathbb{P}^n \\ \Big\downarrow f & & \Big\downarrow f \\ \mathbb{P}^n & \xrightarrow{\ \phi^f\ } & \mathbb{P}^n. \end{array}$$

The *moduli space of self-morphisms of* $\mathbb{P}^n$ is the set

$$\mathsf{M}_d^n = \mathrm{Hom}_d^n \,/\, \mathrm{PGL}_{n+1}\,.$$

Its points classify self-maps of $\mathbb{P}^n$ up to PGL_{n+1}-conjugation. For now, M_d^n is just a set. Our primary goal in Chapter 2 will be to give M_d^n an algebraic structure. Since the dimension one case comes up so frequently, to ease notation when $n = 1$ we write

$$\mathsf{M}_d = \mathsf{M}_d^1.$$

1.6. Level structure and the uniform boundedness conjecture

It is often useful to look at parameter and moduli spaces that classify a map together with one or more marked periodic points. So for example,

$$\mathrm{Hom}_d^n(m) = \bigl\{(\phi, P) : \phi \in \mathrm{Hom}_d^n \text{ and } P \in \mathrm{Per}_m^{**}(\phi)\bigr\},$$

where $\mathrm{Per}_m^{**}(\phi)$ denotes the set of periodic points of exact period m.[3] Notice that PGL_{n+1} acts on $\mathrm{Hom}_d^n(m)$ via

$$(\phi, P)^f = \bigl(\phi^f, f^{-1}(P)\bigr),$$

so we can form the quotient

$$\mathsf{M}_d^n(m) = \mathrm{Hom}_d^n(m)/\,\mathrm{PGL}_{n+1}\,.$$

Study of rational and algebraic points in $\mathsf{M}_d^n(m)$ is closely related to the following motivating conjecture in arithmetic dynamics.

[3]The set $\mathrm{Per}_m^{**}(\phi)$ of points of exact period m is not quite the right set. Instead one uses points of *formal period* m, which may include some points of lower exact period; cf. Section 4.1.

Conjecture 1.10 (Uniform Boundededness Conjecture; Morton–Silverman [84]**).** *For all n, d, D there is a constant $C(n,d,D)$ such that for all number fields $K/\mathbb{Q}$ of degree at most D and all rational maps $\phi \in \operatorname{Hom}_d^n(K)$,*

$$\#\operatorname{PrePer}\bigl(\phi, \mathbb{P}^n(K)\bigr) \le C(n,d,D).$$

The simplest nontrivial case of the conjecture is for quadratic polynomials

$$\phi_c(z) = z^2 + c \quad \text{with } K = \mathbb{Q}.$$

Let

$$\begin{aligned} \mathcal{Q}_m &= \bigl\{c \in \mathbb{Q} : \operatorname{Per}_m^{**}(\phi_c, \mathbb{Q}) \ne \varnothing\bigr\} \\ &= \{c \in \mathbb{Q} : \phi_c \text{ has a } \mathbb{Q}\text{-rational periodic point of exact period } m\}. \end{aligned}$$

Table 1.1 describes the meager results currently known about $\mathcal{Q}_m$. Poonen conjectures that $\mathcal{Q}_m = \varnothing$ for all $n \ge 4$.

To illustrate the depth of the uniform boundedness conjecture, we observe that it trivially implies Mazur's uniform boundedness theorem [72] for torsion on elliptic curves over $\mathbb{Q}$, and similarly Merel's theorem [77] for number fields.

Proposition 1.11. *Suppose that the uniform boundedness conjecture is true for $(n,d,D) = (1,4,1)$, i.e., for maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 4 defined over $\mathbb{Q}$. Then*

$$\#E(\mathbb{Q})_{\text{tors}} \le C(1,4,1) \quad \textit{for all elliptic curves } E/\mathbb{Q},$$

where $C(1,4,1)$ is the constant appearing in Conjecture 1.10.

Proof. Consider the Lattès map ϕ_E defined by the commutativity of the diagram

$$\begin{array}{ccc} E & \xrightarrow{P \mapsto 2P} & E \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{\phi_E} & \mathbb{P}^1. \end{array}$$

Then

$$E(\mathbb{Q})_{\text{tors}} = \operatorname{PrePer}\bigl(\phi_E, \mathbb{P}^1(\mathbb{Q})\bigr).$$

so $\#E(\mathbb{Q})_{\text{tors}} \le C(1,4,1)$. □

Although not so immediate, the full uniform boundedness conjecture implies an analogous result for torsion points on abelian varieties. This latter statement is not currently known unconditionally for any number field in any dimension greater than one.

Table 1.1. $\mathcal{Q}_m = \bigl\{c \in \mathbb{Q} : \operatorname{Per}_m^{**}(\phi_c, \mathbb{Q}) \ne \varnothing\bigr\}$

m	$\# \mathcal{Q}_m$	Notes
1,2,3	∞	Morton [82]
4	0	Morton [82]
5	0	Flynn, Poonen, Schaefer [42]
6	0	Stoll [115] (assuming B–SwD)
≥ 7	???	

Theorem 1.12 (Fakhruddin [39, 40]**).** *Assume that the uniform boundedness conjecture is true. Then for every number field K and abelian variety A/K we have*

$$\# A(K)_{\mathrm{tors}} < C\bigl(\dim A, [K:\mathbb{Q}]\bigr),$$

where as indicated the constant C depends only on the dimension of the abelian variety A and the degree of the number field K.

Proof Sketch. The idea of the proof is to embed A into projective space in such a way that the duplication map

$$A \xrightarrow{[2]} A \tag{1.4}$$

extends to a morphism of the projective space. More generally, Fakhruddin shows the following. Let $\phi\colon X \to X$ is a polarized self-morphism of a projective variety, i.e., there is an ample divisor $D \in \mathrm{Div}(X)$ and an integer $n \geq 2$ such that $\phi^* D \sim nD$. Then there is a projective embedding $X \hookrightarrow \mathbb{P}^N$ such that ϕ extends to a self-morphism of $\mathbb{P}^N$. Further, this can be done functorially, so for abelian varieties of a fixed dimension and for the duplication map (which has fixed degree), the associated projective space and self-morphism also have fixed dimension and degree. We also mention that for the map (1.4), one might be able to do explicitly write down Fakhruddin's map using Mumford's theory of equations defining abelian varieties [85–87]. □