Introduction

The theory of random matrices has proven to have a wide reach into many areas of mathematics, physics, and statistics, and there are many excellent books on the topic. The book of Mehta [61] has become a classic for anyone interested in the subject, and several excellent books on random matrices have appeared in more recent years: [4] by Bai and Silverstein; [3] by Anderson, Guionnet, and Zeitouni; [42] by Forrester; [66] by Pastur and Shcherbina; the CRM volume of lectures [43] edited by Harnad; [73] by Tao; and the Oxford handbook on random matrix theory [1] edited by Akemann, Baik, and Di Francesco. See also the reviews [34] by Di Francesco, Ginsparg, and Zinn-Justin; the ones in the MSRI volume [12], edited by Bleher and Its; [33] by Di Francesco; and the forthcoming book [38] of Eynard. These books and reviews vary in scope and perspective, and they present different approaches to random matrices and their applications to combinatorics, statistics, and physics. In this book we outline a connection from random matrices to the six-vertex model of statistical physics. In particular, this model is related to the unitary matrix ensembles, which are among the most widely studied of the matrix ensembles. For unitary ensembles there is a direct connection to orthogonal polynomials on the real line, and the asymptotics of partition functions as well as local spectral statistics can be studied using the Riemann – Hilbert approach. The focus of this book is a description of the Riemann – Hilbert method for both continuous and discrete orthogonal polynomials, and applications of this approach to matrix models as well as to the six-vertex model.

The Riemann – Hilbert approach to ensembles of random matrices was initiated in the late 1990s in the papers [10] by Bleher and Its, and [29, 30] by Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou, and it became a powerful tool in the theory of universality and critical phenomena in random matrices. In particular, the Riemann – Hilbert method allows for an asymptotic analysis of a wide class of orthogonal polynomials, which was a vital ingredient in the proof of universality of scaling limits for correlations of eigenvalues. The main ideas of the Riemann – Hilbert approach to orthogonal polynomials and random matrices are nicely described in the the lectures [27] by Deift. Chapter 2 of this book is adapted from the paper [29].

The six-vertex model dates back to Slater [69] in the early 1940s, and is one of the integrable models of 2-d statistical physics, see [7, 67]. The domain wall boundary conditions considered in this book were introduced by Korepin [50] in 1982. In that paper certain recursions for the partition function were derived. Subsequently these recursions were used by Izergin [46] to give an explicit determinantal formula for the partition function. This formula is the basis for the asymptotic analysis described in this book, and is known as the Izergin – Korepin formula. The relation of the Izergin – Korepin formula to ensembles of random matrices and orthogonal polynomials was discovered and used by Zinn-Justin [78, 79]. For certain values of the parameters, the relevant orthogonal polynomials are classical. In these cases, the Izergin – Korepin formula was used by Colomo and Pronko [23–26] to give expressions for the 1-, 2-, and 3-enumeration of alternating sign matrices. Outside of these special cases the orthogonal polynomials are not classical, and the Riemann – Hilbert approach was employed in a series of papers by Bleher and coauthors [8, 9, 13–15].

The general outline for the book is as follows:

• In Chapter 1 we introduce the unitary matrix ensembles and describe their connections to orthogonal polynomials and integrable systems.

• In Chapter 2 we discuss the Riemann – Hilbert (RH) approach to random matrix ensembles, adapted from the original approach of the paper [29] and the book [27]. We give general formulas for asymptotics of recurrence coefficients for orthogonal polynomials, and give a proof of the universality of the sine and Airy kernels in the bulk and at the edge, respectively, of the spectrum.

• In Chapter 3 we consider an extension of the RH approach to discrete orthogonal polynomials on an infinite lattice, which was originally developed in the book [5] of Baik, Kriecherbauer, McLaughlin, and Miller for discrete orthogonal polynomials on a finite lattice, and then extended to an infinite lattice in the paper [16] by Bleher and Liechty. Again we give general formulas for asymptotics of recurrence coefficients. Universality of the local correlations in the discrete orthogonal polynomial ensemble is discussed, and we give a proof of the scaling limit of the correlation kernel at the point which separates a band from a saturated region.

• In Chapter 4 we introduce the six vertex model with with domain wall boundary conditions.

• In Chapter 5 we derive the Izergin – Korepin formula for the partition function of the six vertex model with with domain wall boundary conditions. The proof is based on the Yang – Baxter equations, and we follow the elegant approach of the papers [51, 55].

• In Chapters 6 – 8 we obtain the large n asymptotic formulas for the partition function in different phase regions on the phase diagram. These chapters follow the works [9, 13, 15]. The methods of Chapters 2 and 3 are applied, and all details of the analysis are presented.

• In Chapter 9 we discuss the asymptotics of the partition function on the critical lines between the phases, as well as the phase transitions. The results of the papers [8, 14] for the partition function on the critical lines are discussed, but we do not present detailed proofs in this book.

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