

## Unitary Matrix Ensembles

### 1.1. Unitary ensemble with real analytic interaction

Recall that a matrix  $M$  is Hermitian if  $M = M^*$  (where  $M^* = \overline{M^T}$ ), so that  $M_{kj} = \overline{M_{jk}}$ . Of course, any Hermitian matrix must have real entries along the diagonal, whereas the entries below the diagonal are completely determined by the entries above the diagonal. It follows that, in order to count the real dimension of the space of  $N \times N$  Hermitian matrices, we should count the number of entries along the diagonal,  $N$ , and twice the number of entries above the diagonal to account for real and imaginary parts. Thus, if  $\mathcal{H}_N$  is the space of  $N \times N$  Hermitian matrices, then its real dimension is equal to

$$(1.1.1) \quad \dim \mathcal{H}_N = N + 2[1 + 2 + \cdots + N - 1] = N + N(N - 1) = N^2.$$

The space  $\mathcal{H}_N$  is a real Hilbert space with respect to the scalar product

$$(1.1.2) \quad (L, M) = \operatorname{Re} \operatorname{Tr}(LM^*) = \sum_{j,k=1}^N \operatorname{Re}(L_{jk}\overline{M_{jk}}) \\ = \sum_{j=1}^N L_{jj}M_{jj} + 2 \sum_{j>k}^N [(\operatorname{Re} L_{jk})(\operatorname{Re} M_{jk}) + (\operatorname{Im} L_{jk})(\operatorname{Im} M_{jk})].$$

Being a subspace of the space of  $N \times N$  matrices with complex entries,  $\mathcal{H}_N$  embeds naturally into  $\mathbb{C}^{N^2}$ . The Euclidean distance inherited from this embedding is given as

$$(1.1.3) \quad \operatorname{dist}(L, M) = \|L - M\| = \left( \sum_{j,k=1}^N |L_{jk} - M_{jk}|^2 \right)^{1/2} \\ = \left( \sum_{j=1}^N |L_{jj} - M_{jj}|^2 + 2 \sum_{j>k}^N |L_{jk} - M_{jk}|^2 \right)^{1/2}.$$

The scalar product  $(L, M)$  and the distance  $\operatorname{dist}(L, M)$  are invariant with respect to the conjugation by any unitary matrix  $U \in \operatorname{U}(N)$ ,

$$(1.1.4) \quad M \rightarrow U^{-1}MU, \quad U \in \operatorname{U}(N).$$

Let  $dM$  be the  $N^2$ -dimensional Lebesgue measure,

$$(1.1.5) \quad dM = \prod_{j=1}^N dM_{jj} \prod_{j \neq k}^N d \operatorname{Re} M_{jk} d \operatorname{Im} M_{jk}.$$

We will consider the probability distribution on  $\mathcal{H}_N$  given by

$$(1.1.6) \quad d\mu_N(M) = \frac{1}{Z_N} e^{-N \operatorname{Tr} V(M)} dM,$$

where  $V(x)$  is a real analytic function satisfying the growth condition that

$$(1.1.7) \quad \frac{V(x)}{\log(|x|^2 + 1)} \rightarrow +\infty \quad \text{as } |x| \rightarrow \infty.$$

This growth condition is a technical condition for subsequent analysis. Indeed, the reader may simply think of  $V$  as a polynomial of even degree and with positive leading coefficient, in which case it is clear what is meant by the matrix  $V(M)$  in (1.1.6). If  $V$  is not a polynomial, then the matrix  $V(M)$  can be understood by applying  $V$  to the spectrum of  $M$ . That is,  $M$  can be diagonalized as  $M = U\Lambda U^*$ , where  $U$  is a unitary matrix and  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$  is the matrix of eigenvalues, and we can then define

$$(1.1.8) \quad V(M) = U \text{diag}(V(\lambda_1), V(\lambda_2), \dots, V(\lambda_N))U^*.$$

Notice then that  $\text{Tr} V(M)$  is invariant with respect to unitary conjugations given in (1.1.4). Since the distance  $\text{dist}(L, M)$  induces the measure  $2^{N(N-1)/2} dM$  on  $\mathcal{H}_N$ , the Lebesgue measure  $dM$  is invariant with respect to unitary conjugations (1.1.4) as well. It follows that the distribution  $d\mu_N(M)$  is invariant with respect to any unitary conjugation (1.1.4), hence the name of the ensemble. The normalizing constant  $Z_N$ , called the partition function, is defined such that  $\mu_N$  is a probability measure. That is, it is the matrix integral

$$(1.1.9) \quad Z_N = \int_{\mathcal{H}_N} e^{-N \text{Tr} V(M)} dM.$$

**Example (Gaussian unitary ensemble).** For  $V(M) = M^2$ , the measure  $\mu_N$  is the probability distribution of the Gaussian unitary ensemble (GUE). This is the oldest and most well known of the invariant matrix ensembles. In this case,

$$(1.1.10) \quad \text{Tr} V(M) = \text{Tr} M^2 = \sum_{j,k=1}^N M_{kj} M_{jk} = \sum_{j=1}^N M_{jj}^2 + 2 \sum_{j>k} |M_{jk}|^2,$$

hence

$$(1.1.11) \quad d\mu_N^{\text{GUE}}(M) = \frac{1}{Z_N^{\text{GUE}}} \prod_{j=1}^N (e^{-NM_{jj}^2}) \prod_{j>k} (e^{-2N|M_{jk}|^2}) dM,$$

so that the matrix elements in GUE are independent Gaussian random variables. The partition function of GUE is evaluated as

$$(1.1.12) \quad \begin{aligned} Z_N^{\text{GUE}} &= \int_{\mathcal{H}_N} \prod_{j=1}^N (e^{-NM_{jj}^2}) \prod_{j>k} (e^{-2N|M_{jk}|^2}) dM \\ &= \left(\frac{\pi}{N}\right)^{N/2} \left(\frac{\pi}{2N}\right)^{N(N-1)/2} = \left(\frac{\pi}{N}\right)^{N^2/2} \left(\frac{1}{2}\right)^{N(N-1)/2}. \end{aligned}$$

The GUE is somewhat special in that it lies at the intersection of the invariant ensembles, which are invariant with respect to some sort of matrix conjugation (in this case unitary conjugation), and the Wigner ensembles, for which the matrix entries are independent. If the function  $V(x)$  is not quadratic, then the matrix entries become dependent.

## 1.2. Ensemble of eigenvalues

The central topic in random matrix theory is the distribution of the eigenvalues of a random matrix. We can write a formula for the distribution of eigenvalues of an Hermitian matrix  $M$  from distribution (1.1.6) by writing  $M$  in terms of its eigenvalues and eigenvectors, so that  $M = U\Lambda U^*$ , where  $U$  is a unitary matrix (of eigenvectors) and  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$  is the matrix of eigenvalues. In order to make this map one-to-one, let us consider  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$  to be ordered, so that  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ . In fact, dropping a set of measure zero, we can assume  $\lambda_j \neq \lambda_k$  for  $j \neq k$ , and thus consider  $\lambda$  in the Weyl chamber

$$(1.2.1) \quad \lambda_1 < \lambda_2 < \dots < \lambda_N.$$

Also, instead of  $U \in \text{U}(N)$  we may consider matrix  $UD$ , where

$$D = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_N}) \in \text{D}(N)$$

is any diagonal unitary matrix. Consider therefore the equivalence class

$$(1.2.2) \quad \tilde{U} = \{UD, D \in \text{D}(N)\},$$

and the homogeneous space  $\tilde{\text{U}}(N)$  of the equivalence classes  $\tilde{U}$ . Then the map

$$(1.2.3) \quad (\tilde{U}, \Lambda) \mapsto M = U\Lambda U^*, \quad U \in \tilde{U},$$

is one-to-one, and we may consider its Jacobian

$$(1.2.4) \quad J = \frac{dM}{d\tilde{U} d\lambda},$$

where  $d\tilde{U}$  is the projection of the Haar measure on  $\text{U}(N)$  onto  $\tilde{\text{U}}(N)$  and  $d\lambda = d\lambda_1 \cdots d\lambda_N$ . Since  $dM$  is invariant with respect to the unitary conjugations, and  $d\tilde{U}$  is invariant with respect to the unitary left shifts, the Jacobian  $J$  does not depend on  $\tilde{U}$ . Its dependence on  $\lambda = (\lambda_1, \dots, \lambda_N)$  is described as follows.

**Proposition 1.2.1 (Weyl's formula).** *For some constant  $C_N > 0$ ,*

$$(1.2.5) \quad J = C_N \prod_{j < k} |\lambda_k - \lambda_j|^2.$$

PROOF. Since  $J$  does not depend on  $\tilde{U}$ , it suffices to evaluate  $J$  at  $U = I$ , i.e., at  $\tilde{U} = \tilde{I} = \text{D}(N)$ . In a small neighborhood of  $\tilde{I}$ , the elements  $\tilde{U}$  are uniquely represented by unitary matrices  $U = e^A$ , where  $A^* = -A$ . As  $A \rightarrow 0$ ,

$$(1.2.6) \quad U = e^A = I + A + \mathcal{O}(A^2), \quad U^* = e^{-A} = I - A + \mathcal{O}(A^2),$$

and

$$(1.2.7) \quad M = U\Lambda U^{-1} = \Lambda + [A, \Lambda] + \mathcal{O}(A^2),$$

so that

$$(1.2.8) \quad M_{ii} = \lambda_i + \mathcal{O}(A^2), \quad M_{ij} = (\lambda_j - \lambda_i)A_{ij} + \mathcal{O}(A^2), \quad i < j.$$

This implies that at  $A = 0$ ,

$$(1.2.9) \quad \begin{aligned} \frac{\partial M_{ii}}{\partial \lambda_k} &= \delta_{ik}, & \frac{\partial M_{ii}}{\partial A_{kl}} &= 0, \\ \frac{\partial M_{ij}}{\partial \lambda_k} &= 0, & \frac{\partial M_{ij}}{\partial A_{kl}} &= (\lambda_j - \lambda_i)\delta_{ik}\delta_{jl}, \quad i < j, \end{aligned}$$

hence the Jacobian matrix  $\nabla_{\lambda,A}M$  is diagonal and its determinant is equal to

$$(1.2.10) \quad J = \prod_{j < k} |\lambda_k - \lambda_j|^2.$$

Since at  $A = 0$ ,

$$(1.2.11) \quad d\tilde{U} = C_N dA, \quad C_N > 0,$$

formula (1.2.5) follows.  $\square$

It follows that, if we are interested only in eigenvalues, the eigenvectors may be integrated out, and we find that the distribution of eigenvalues of  $M$  with respect to the ensemble  $\mu_N$  is given as

$$(1.2.12) \quad d\mu_N(\lambda) = \frac{1}{\tilde{Z}_N} \prod_{j > k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^N e^{-NV(\lambda_j)} d\lambda,$$

where

$$(1.2.13) \quad \tilde{Z}_N = \int \prod_{j > k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^N e^{-NV(\lambda_j)} d\lambda, \quad d\lambda = d\lambda_1 \cdots d\lambda_N.$$

Since we have considered  $\lambda$  in the Weyl chamber (1.2.1), this gives a measure on ordered eigenvalues, and the integral (1.2.13) is over the Weyl chamber. However, clearly (1.2.12) is symmetric in  $\lambda$ , and therefore the measure can be lifted to an ensemble of unordered eigenvalues, where the integral in (1.2.13) is then understood to be over  $\mathbb{R}^N$ .

Notice that

$$(1.2.14) \quad Z_N = \tilde{Z}_N \frac{C_N \text{Vol}(\tilde{U}(N))}{N!},$$

and thus the ratio  $\tilde{Z}_N/Z_N$  does not depend on the potential  $V$ . We can calculate this ratio in the case of the GUE, and the result will hold for any unitary ensemble given by (1.1.6). Indeed, for GUE,

$$(1.2.15) \quad d\mu_N^{\text{GUE}}(\lambda) = \frac{1}{\tilde{Z}_N^{\text{GUE}}} \prod_{j > k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^N e^{-N\lambda_j^2} d\lambda,$$

where

$$(1.2.16) \quad \tilde{Z}_N^{\text{GUE}} = \int \prod_{j > k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^N e^{-N\lambda_j^2} d\lambda.$$

The constant  $\tilde{Z}_N^{\text{GUE}}$  is a Selberg integral, and its exact value is

$$(1.2.17) \quad \tilde{Z}_N^{\text{GUE}} = \frac{(2\pi)^{N/2}}{(2N)^{N^2/2}} \prod_{k=1}^N k!,$$

see, e.g., [61]. A proof of formula (1.2.17) from the discrete string equations for orthogonal polynomials is given subsequently in Section 1.3. We therefore have that the partition functions  $Z_N$  and  $\tilde{Z}_N$  are related as

$$(1.2.18) \quad \frac{\tilde{Z}_N}{Z_N} = \frac{\tilde{Z}_N^{\text{GUE}}}{Z_N^{\text{GUE}}} = \frac{1}{\pi^{N(N-1)/2}} \prod_{k=1}^N k!.$$

One of the main problems in random matrix theory is to evaluate the large  $N$  asymptotics of the partition function  $\tilde{Z}_N$  and of the correlations between eigenvalues.

From (1.2.12), the joint probability density function for the eigenvalues is given by

$$(1.2.19) \quad p_N(x_1, \dots, x_N) = \tilde{Z}_N^{-1} \prod_{j>k} (x_j - x_k)^2 \prod_{j=1}^N e^{-NV(x_j)}.$$

Integrating out  $(N - m)$  variables, we obtain the marginal probability density function for  $m$  eigenvalues,

$$(1.2.20) \quad p_{mN}(x_1, \dots, x_m) = \int_{\mathbb{R}^{N-m}} p_N(x_1, \dots, x_N) dx_{m+1} \cdots dx_N.$$

The  $m$ -point correlation function is then defined as

$$(1.2.21) \quad R_{mN}(x_1, \dots, x_m) := \frac{N!}{(N - m)!} p_{mN}(x_1, \dots, x_m),$$

see, e.g., [2, 3, 36, 61]. Remarkably, these correlation functions can all be expressed in terms of a system of orthogonal polynomials. Let  $\{P_k(x)\}_{k=0}^\infty$  be the system of monic orthogonal polynomials defined from the orthogonality condition

$$(1.2.22) \quad \int_{-\infty}^{\infty} P_j(x) P_k(x) e^{-NV(x)} dx = h_k \delta_{jk},$$

for some system of normalizing constants  $\{h_k\}_{k=0}^\infty$ . Existence and uniqueness of these polynomials is guaranteed by condition (1.1.7). Define also the functions

$$(1.2.23) \quad \psi_k(x) = \frac{1}{h_k^{1/2}} P_k(x) e^{-NV(x)/2},$$

which form an orthonormal basis in  $L^2(\mathbb{R}^1)$ . We have the following proposition.

**Proposition 1.2.2.** *The correlation function (1.2.21) has the determinantal form*

$$(1.2.24) \quad R_{mN}(x_1, \dots, x_m) = \det(K_N(x_k, x_l))_{k,l=1}^m,$$

where

$$(1.2.25) \quad K_N(x, y) = \sum_{n=0}^{N-1} \psi_n(x) \psi_n(y).$$

Furthermore, the partition function  $\tilde{Z}_N$  can be written in terms of the orthogonal polynomials (1.2.22) as

$$(1.2.26) \quad \tilde{Z}_N = N! \prod_{j=0}^{N-1} h_j.$$

An ensemble whose correlations can be expressed by such a determinantal formula is called a *determinantal point process*, see [18, 20, 44, 45]. In particular notice that the one point correlation function enables us to write the density of eigenvalues on the real line, which we notate  $\rho_N$  in the simple form

$$(1.2.27) \quad \rho_N(x) := \frac{R_{1N}(x)}{N} = \frac{K_N(x, x)}{N}.$$

Before proving Proposition 1.2.2, let us point out some unique properties of the function  $K_N$ . Observe that  $K_N$  is the kernel of the projection operator onto the  $N$ -dimensional space generated by the first  $N$  functions  $\psi_n$ ,  $n = 0, \dots, N-1$ . The function  $K_N(x, y)$  is called the *reproducing kernel* and it has the following properties:

$$(1.2.28) \quad \begin{aligned} \int_{\mathbb{R}} K_N(x, x) dx &= N, \\ \int_{\mathbb{R}} K_N(x, y) K_N(y, z) dy &= K(x, z). \end{aligned}$$

Indeed, by (1.2.25),

$$(1.2.29) \quad \int_{\mathbb{R}} K_N(x, x) dx = \sum_{j=0}^{N-1} \int_{\mathbb{R}} \psi_j(x)^2 dx = \sum_{j=0}^{N-1} 1 = N,$$

and

$$(1.2.30) \quad \begin{aligned} \int_{\mathbb{R}} K_N(x, y) K_N(y, z) dy &= \sum_{j, k=0}^{N-1} \int_{\mathbb{R}} \psi_j(x) \psi_j(y) \psi_k(y) \psi_k(z) dy \\ &= \sum_{j=0}^{N-1} \psi_j(x) \psi_j(z) = K(x, z). \end{aligned}$$

Let us now prove formula (1.2.26) for the partition function. Recall the formula for the Vandermonde determinant,

$$(1.2.31) \quad \det [x_j^{k-1}]_{j, k=1}^N = \prod_{\substack{j, k=1 \\ j < k}}^N (x_k - x_j).$$

The main point in the proof of (1.2.26) is that the function  $p_N(x_1, \dots, x_N)$  in the integrand of (1.2.16) is the product of the square of the Vandermonde determinant and factors which are independent and identical on each of the coordinates  $x_j$ . The form of the Vandermonde matrix and multilinearity of the determinant function allow us to replace the  $j$ th row of the Vandermonde matrix with any monic polynomial of degree  $(j-1)$ . In particular, we may use the orthogonal polynomials described in (1.2.22), so that

$$(1.2.32) \quad \begin{aligned} &\tilde{Z}_N \\ &= \int_{\mathbb{R}^N} p_N(x_1, \dots, x_N) dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^N} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \\ x_1^2 & x_2^2 & \cdots & x_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \cdots & x_N^{N-1} \end{pmatrix}^2 \prod_{j=1}^N e^{-NV(x_j)} dx_j \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^N} \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ P_1(x_1) & P_1(x_2) & \dots & P_1(x_N) \\ P_2(x_1) & P_2(x_2) & \dots & P_2(x_N) \\ \vdots & \vdots & \dots & \vdots \\ P_{N-1}(x_1) & P_{N-1}(x_2) & \dots & P_{N-1}(x_N) \end{pmatrix}^2 \prod_{j=1}^N e^{-NV(x_j)} dx_j \\
&= \int_{\mathbb{R}^N} \left( \sum_{\sigma \in \mathcal{S}_N} (-1)^\sigma \prod_{j=1}^N P_{\sigma(j)-1}(x_j) \right)^2 \prod_{j=1}^N e^{-NV(x_j)} dx_j \\
&= \int_{\mathbb{R}^N} \sum_{\sigma \in \mathcal{S}_N} \sum_{\sigma' \in \mathcal{S}_N} (-1)^{\sigma+\sigma'} \prod_{j=1}^N P_{\sigma(j)-1}(x_j) P_{\sigma'(j)-1}(x_j) \prod_{j=1}^N e^{-NV(x_j)} dx_j \\
&= \sum_{\sigma \in \mathcal{S}_N} \sum_{\sigma' \in \mathcal{S}_N} (-1)^{\sigma+\sigma'} \prod_{j=1}^N \int_{\mathbb{R}} P_{\sigma(j)-1}(x_j) P_{\sigma'(j)-1}(x_j) e^{-NV(x_j)} dx_j.
\end{aligned}$$

The orthogonality condition ensures that only diagonal terms,  $\sigma = \sigma'$ , are nonzero. Thus we have

$$(1.2.33) \quad \tilde{Z}_N = \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^N \int_{\mathbb{R}} P_{\sigma(j)-1}^2(x_j) e^{-NV(x_j)} dx_j = N! \prod_{j=0}^{N-1} h_j.$$

Let us now prove the formula (1.2.24) for the correlation kernel.

Consider first the case  $m = N$ , when

(1.2.34)

$$\begin{aligned}
R_{NN}(x_1, \dots, x_N) &= N! p_N(x_1, \dots, x_N) = \frac{N!}{\tilde{Z}_N} \prod_{j>k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^N e^{-NV(\lambda_j)} \\
&= \frac{1}{\prod_{j=0}^{N-1} h_j} \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \\ x_1^2 & x_2^2 & \dots & x_N^2 \\ \vdots & \vdots & \dots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \end{pmatrix}^2 \prod_{j=1}^N e^{-NV(x_j)}.
\end{aligned}$$

As in (1.2.32), we obtain that

(1.2.35)  $R_{NN}(x_1, \dots, x_N)$

$$\begin{aligned}
&= \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ P_1(x_1) & P_1(x_2) & \dots & P_1(x_N) \\ P_2(x_1) & P_2(x_2) & \dots & P_2(x_N) \\ \vdots & \vdots & \dots & \vdots \\ P_{N-1}(x_1) & P_{N-1}(x_2) & \dots & P_{N-1}(x_N) \end{pmatrix}^2 \frac{\prod_{j=1}^N e^{-NV(x_j)}}{\prod_{j=0}^{N-1} h_j} \\
&= \det \begin{pmatrix} \psi_0(x_1) & \psi_0(x_2) & \dots & \psi_0(x_N) \\ \psi_1(x_1) & \psi_1(x_2) & \dots & \psi_1(x_N) \\ \psi_2(x_1) & \psi_2(x_2) & \dots & \psi_2(x_N) \\ \vdots & \vdots & \dots & \vdots \\ \psi_{N-1}(x_1) & \psi_{N-1}(x_2) & \dots & \psi_{N-1}(x_N) \end{pmatrix}^2,
\end{aligned}$$

hence

$$\begin{aligned}
(1.2.36) \quad & R_{NN}(x_1, \dots, x_N) \\
&= \det \begin{pmatrix} \psi_0(x_1) & \psi_1(x_1) & \dots & \psi_{N-1}(x_1) \\ \psi_0(x_2) & \psi_1(x_2) & \dots & \psi_{N-1}(x_2) \\ \vdots & \vdots & & \vdots \\ \psi_0(x_N) & \psi_1(x_N) & \dots & \psi_{N-1}(x_N) \end{pmatrix} \begin{pmatrix} \psi_0(x_N) & \psi_0(x_2) & \dots & \psi_0(x_N) \\ \psi_1(x_1) & \psi_1(x_2) & \dots & \psi_1(x_N) \\ \vdots & \vdots & & \vdots \\ \psi_{N-1}(x_1) & \psi_{N-1}(x_2) & \dots & \psi_{N-1}(x_N) \end{pmatrix} \\
&= \det(K_N(x_k, x_l))_{k,l=1}^N.
\end{aligned}$$

We will prove (1.2.24) by induction in  $m$ , starting from  $m = N$  down, but we first need to prove the following inductive lemma.

**Lemma 1.2.3.** *For any  $1 < m \leq N$ ,*

$$(1.2.37) \quad \int_{\mathbb{R}} \det(K_N(x_j, x_k))_{j,k=1}^m dx_m = (N - m + 1) \det(K_N(x_j, x_k))_{j,k=1}^{m-1}.$$

PROOF. Let us expand  $\det(K_N(x_j, x_k))_{j,k=1}^m$  with respect to the last column:

$$\begin{aligned}
(1.2.38) \quad & \det(K_N(x_j, x_k))_{j,k=1}^m \\
&= \sum_{i=1}^m (-1)^{m+i} K_N(x_i, x_m) \det(K_N(x_j, x_k))_{\substack{1 \leq j \leq m, j \neq i \\ 1 \leq k \leq m-1}}.
\end{aligned}$$

When we integrate the last term,  $i = m$ , in this formula, we obtain, by (1.2.28),

$$(1.2.39) \quad \int_{\mathbb{R}} K_N(x_m, x_m) \det(K_N(x_j, x_k))_{j,k=1}^{m-1} dx_m = N \det(K_N(x_j, x_k))_{j,k=1}^{m-1}.$$

When we integrate the  $i$ th term in (1.2.38), we expand  $\det(K_N(x_j, x_k))_{\substack{1 \leq j \leq m, j \neq i \\ 1 \leq k \leq m-1}}$  with respect to the last row:

$$\begin{aligned}
(1.2.40) \quad & \det(K_N(x_j, x_k))_{\substack{1 \leq j \leq m, j \neq i \\ 1 \leq k \leq m-1}} \\
&= \sum_{l=1}^{m-1} (-1)^{m-1+l} K_N(x_m, x_l) \det(K_N(x_j, x_k))_{\substack{1 \leq j \leq m, j \neq i \\ 1 \leq k \leq m-1, k \neq l}}
\end{aligned}$$

By (1.2.28), the integration of the  $i$ th term in (1.2.38) gives

$$(1.2.41) \quad \sum_{l=1}^{m-1} (-1)^{i+l-1} K_N(x_i, x_l) \det(K_N(x_j, x_k))_{\substack{1 \leq j \leq m, j \neq i \\ 1 \leq k \leq m-1, k \neq l}},$$

which coincides with the expansion of  $-\det(K_N(x_j, x_k))_{j,k=1}^{m-1}$  with respect to the  $i$ th row. Thus,

$$\begin{aligned}
(1.2.42) \quad & \int_{\mathbb{R}} \det(K_N(x_j, x_k))_{j,k=1}^m dx_m \\
&= N \det(K_N(x_j, x_k))_{j,k=1}^{m-1} - \sum_{i=1}^{m-1} \det(K_N(x_j, x_k))_{j,k=1}^{m-1} \\
&= (N - m + 1) \det(K_N(x_j, x_k))_{j,k=1}^{m-1}.
\end{aligned}$$

Lemma 1.2.3 is proved.  $\square$



Assume now that formula (1.2.24) is already proven for some  $m \leq N$ . Let us prove it for  $m - 1$ . From (1.2.21) we have that

$$(1.2.43) \quad R_{m-1,N}(x_1, \dots, x_{m-1}) = \frac{1}{N - m + 1} \int_{\mathbb{R}} R_{mN}(x_1, \dots, x_{m-1}, x_m) dx_m,$$

hence by the inductive assumption and Lemma 1.2.3,

$$(1.2.44) \quad \begin{aligned} R_{m-1,N}(x_1, \dots, x_{m-1}) &= \frac{1}{N - m + 1} \int_{\mathbb{R}} \det(K_N(x_j, x_k))_{j,k=1}^m dx_m \\ &= \det(K_N(x_j, x_k))_{j,k=1}^{m-1}. \end{aligned}$$

This proves formula (1.2.24) by induction.

The kernel  $K_N$  can be expressed in terms of  $\psi_{N-1}$  and  $\psi_N$  only, due to the Christoffel–Darboux formula, which is presented in the next section. We also consider recurrence and differential equations for the functions  $\psi_n$ .

### 1.3. Recurrence equations and discrete string equations for orthogonal polynomials

An important feature of orthogonal polynomials on the real line is the *three-term recurrence relation*, see, e.g., [72],

$$(1.3.1) \quad \begin{aligned} xP_n(x) &= P_{n+1}(x) + \beta_n P_n(x) + \gamma_n^2 P_{n-1}(x), \\ \gamma_n &= \left( \frac{h_n}{h_{n-1}} \right)^{1/2} > 0, \quad n \geq 1; \quad \gamma_0 = 0. \end{aligned}$$

For the functions  $\psi_n$  it reads as

$$(1.3.2) \quad x\psi_n(x) = \gamma_{n+1}\psi_{n+1}(x) + \beta_n\psi_n(x) + \gamma_n\psi_{n-1}(x).$$

This allows the calculation

$$(1.3.3) \quad \begin{aligned} (x - y) \sum_{n=0}^{N-1} \psi_n(x)\psi_n(y) &= \sum_{n=0}^{N-1} [(\gamma_{n+1}\psi_{n+1}(x) + \beta_n\psi_n(x) + \gamma_n\psi_{n-1}(x))\psi_n(y) \\ &\quad - \psi_n(x)(\gamma_{n+1}\psi_{n+1}(y) + \beta_n\psi_n(y) + \gamma_n\psi_{n-1}(y))] \\ &= \gamma_N[\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)], \end{aligned}$$

as the first line of (1.3.3) gives a telescoping sum. Hence,

$$(1.3.4) \quad K_N(x, y) = \sum_{n=0}^{N-1} \psi_n(x)\psi_n(y) = \gamma_N \frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{x - y},$$

which is the *Christoffel–Darboux formula*. For the eigenvalue density function  $K_N(x, x)/N$  we take the limit as  $y \rightarrow x$ , obtaining that

$$(1.3.5) \quad \rho_N(x) = \frac{K_N(x, x)}{N} = \frac{\gamma_N}{N} [\psi'_N(x)\psi_{N-1}(x) - \psi'_{N-1}(x)\psi_N(x)].$$

We now use the recurrence (1.3.2) derive the discrete string equations for orthogonal polynomials. To do so, we will consider two operators on  $L^2(\mathbb{R})$ . The first operator is that of multiplication by  $x$ . If we think of  $L^2(\mathbb{R})$  in the basis

$\{\psi_n(x)\}$ , then, according to the three-term recurrence relation, this operator has the symmetric tridiagonal Jacobi matrix,

$$(1.3.6) \quad Q = \begin{pmatrix} \beta_0 & \gamma_1 & 0 & 0 & \dots \\ \gamma_1 & \beta_1 & \gamma_2 & 0 & \dots \\ 0 & \gamma_2 & \beta_2 & \gamma_3 & \dots \\ 0 & 0 & \gamma_3 & \beta_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The other operator we will consider is that of differentiation. Let  $P = (P_{nm})_{n,m=0}^{\infty}$  be a matrix of the operator  $f(z) \mapsto f'(z)$  in the basis  $\{\psi_n(z)\}_{n=0}^{\infty}$ , so that

$$(1.3.7) \quad \psi'_n(x) = \sum_{m=0}^{\infty} P_{nm} \psi_m(x),$$

or equivalently

$$(1.3.8) \quad P_{nm} = \int_{-\infty}^{\infty} \psi'_n(x) \psi_m(x) dx.$$

Integrating by parts gives that  $P_{mn} = -P_{nm}$ . Since  $P'_n(z)$  is a polynomial of degree strictly less than  $n$ , we have

$$(1.3.9) \quad \begin{aligned} \psi'_n(z) &= -\frac{NV'(z)}{2} \psi_n(z) + \frac{P'_n(z)}{\sqrt{h_n}} e^{-NV(z)/2} \\ &= -\frac{NV'(z)}{2} \psi_n(z) + \frac{n}{\gamma_n} \psi_{n-1}(z) + \dots, \end{aligned}$$

hence

$$(1.3.10) \quad \left[ P + \frac{NV'(Q)}{2} \right]_{nm} = 0 \quad \text{for } m \geq n.$$

Due to the antisymmetry of the matrix  $P$ , equation (1.3.10) gives a full characterization of the matrix  $P$  in terms of the matrix  $V'(Q)$ . More specifically, we have

$$(1.3.11) \quad P_{nm} = \begin{cases} \left[ -\frac{NV'(Q)}{2} \right]_{nm} & \text{for } m \geq n \\ \left[ \frac{NV'(Q)}{2} \right]_{nm} & \text{for } m \leq n. \end{cases}$$

Since  $P_{nn} = 0$ , we obtain that

$$(1.3.12) \quad [V'(Q)]_{nn} = 0.$$

In addition,

$$(1.3.13) \quad \left[ -P + \frac{NV'(Q)}{2} \right]_{n,n-1} = 0, \quad \left[ P + \frac{NV'(Q)}{2} \right]_{n,n-1} = \frac{n}{\gamma_n},$$

hence

$$(1.3.14) \quad \gamma_n [V'(Q)]_{n,n-1} = \frac{n}{N}.$$

Thus we have the *discrete string equations* for the recurrence coefficients,

$$(1.3.15) \quad \begin{cases} \gamma_n [V'(Q)]_{n,n-1} = \frac{n}{N}, \\ [V'(Q)]_{nn} = 0. \end{cases}$$

The string equations can be brought to a variational form.

**Proposition 1.3.1.** *Define the infinite Hamiltonian,*

$$(1.3.16) \quad H(\gamma, \beta) = N \operatorname{Tr} V(Q) - \sum_{n=1}^{\infty} n \log \gamma_n^2, \quad \gamma = (\gamma_0, \gamma_1, \dots), \quad \beta = (\beta_0, \beta_1, \dots).$$

Then equations (1.3.15) can be written as

$$(1.3.17) \quad \frac{\partial H}{\partial \gamma_n} = 0, \quad \frac{\partial H}{\partial \beta_n} = 0; \quad n \geq 1,$$

which are the Euler–Lagrange equations for the Hamiltonian  $H$ .

PROOF. We have that

$$(1.3.18) \quad \frac{\partial H}{\partial \gamma_n} = N \operatorname{Tr} \left( V'(Q) \frac{\partial Q}{\partial \gamma_n} \right) - \frac{2n}{\gamma_n} = 2N [V'(Q)]_{n,n-1} - \frac{2n}{\gamma_n},$$

and

$$(1.3.19) \quad \frac{\partial H}{\partial \beta_n} = N \operatorname{Tr} \left( V'(Q) \frac{\partial Q}{\partial \beta_n} \right) = N [V'(Q)]_{nn},$$

hence equations (1.3.15) are equivalent to (1.3.17).  $\square$

**Example (The even quartic model).** Let us take

$$(1.3.20) \quad V(M) = \frac{t}{2} M^2 + \frac{g}{4} M^4,$$

for  $g > 0$ ,  $t \in \mathbb{R}$ . In this case, since  $V$  is even,  $\beta_n = 0$ , and we have one string equation,

$$(1.3.21) \quad \gamma_n^2 (t + g\gamma_{n-1}^2 + g\gamma_n^2 + g\gamma_{n+1}^2) = \frac{n}{N},$$

with the initial conditions

$$(1.3.22) \quad \gamma_0 = 0, \quad \gamma_1 = \frac{\int_{-\infty}^{\infty} z^2 e^{-NV(z)} dz}{\int_{-\infty}^{\infty} e^{-NV(z)} dz}.$$

The Hamiltonian is

$$(1.3.23) \quad H(\gamma) = \sum_{n=1}^{\infty} \left[ \frac{N}{2} \gamma_n^2 (2t + g\gamma_{n-1}^2 + g\gamma_n^2 + g\gamma_{n+1}^2) - n \log \gamma_n^2 \right].$$

The minimization of the functional  $H$  gives a useful procedure for a numerical solution of the string equations, see [11, 22]. The initial value problem (1.3.21) and (1.3.22) for the string equations is difficult to solve numerically, as the recursion is very unstable. However, the minimization of  $H$  with  $\gamma_0 = 0$  and some boundary conditions at  $n = N$ , say  $\gamma_N = 0$ , works very well. In fact, the boundary condition at  $n = N$  creates a narrow boundary layer near  $n = N$ , and it does not affect significantly the main part of the graph of  $\gamma_n^2$ . Figure 1.1 presents a computer solution,  $y = \gamma_n^2$ , of the string equation for the quartic model:  $g = 1$ ,  $t = -1$ ,

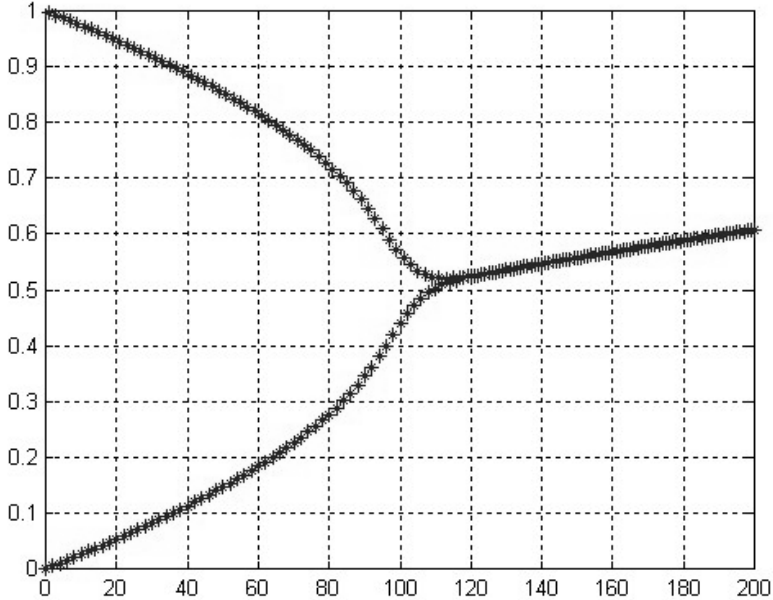


FIGURE 1.1. A computer solution,  $y = \gamma_n^2$ , of the string equation for the quartic model:  $g = 1$ ,  $t = -1$ ,  $N = 400$ .

$N = 400$ . For this solution, it is shown in [10, 11] that there is a critical value,  $\lambda_c = \frac{1}{4}$ , so that for any  $\varepsilon > 0$ , as  $N \rightarrow \infty$ ,

$$(1.3.24) \quad \gamma_n^2 = R\left(\frac{n}{N}\right) + \mathcal{O}(N^{-1}), \quad \text{if } \frac{n}{N} \geq \lambda_c + \varepsilon,$$

and

$$(1.3.25) \quad \gamma_n^2 = \begin{cases} R\left(\frac{n}{N}\right) + \mathcal{O}(N^{-1}), & n = 2k + 1, \\ L\left(\frac{n}{N}\right) + \mathcal{O}(N^{-1}), & n = 2k, \end{cases} \quad \text{if } \frac{n}{N} \leq \lambda_c - \varepsilon.$$

The functions  $R$  for  $\lambda \geq \lambda_c$  and  $R, L$  for  $\lambda \leq \lambda_c$  can be found from the string equation (1.3.21):

$$(1.3.26) \quad R(\lambda) = \frac{1 + \sqrt{1 + 12\lambda}}{6}, \quad \lambda > \lambda_c,$$

and

$$(1.3.27) \quad R(\lambda), L(\lambda) = \frac{1 \pm \sqrt{1 - 4\lambda}}{2}, \quad \lambda < \lambda_c.$$

PROOF OF FORMULA (1.2.17). We can use the the discrete string equations to obtain the recurrence coefficients for the Hermite polynomials, and therefore the formula (1.2.17). Setting  $t = 2$  and  $g = 0$  in (1.3.20) we obtain  $V(M) = M^2$ , the Gaussian unitary ensemble. Formula (1.3.21) reduces to

$$(1.3.28) \quad \gamma_n^2 = \frac{n}{2N},$$

which are the recurrence coefficients of the Hermite polynomials. We obtain from (1.3.1) that

$$(1.3.29) \quad \frac{h_n}{h_{n-1}} = \frac{n}{2N}.$$

Since

$$(1.3.30) \quad h_0 = \int_{-\infty}^{\infty} e^{-Nx^2} dx = \sqrt{\frac{\pi}{N}},$$

(1.3.29) gives that

$$(1.3.31) \quad h_n = h_0 \prod_{k=1}^n \frac{k}{2N} = \frac{\sqrt{\pi} n!}{2^n N^{(2n+1)/2}}.$$

Applying now formula (1.2.26), we obtain that

$$(1.3.32) \quad \tilde{Z}_N^{\text{GUE}} = N! \prod_{n=0}^{N-1} h_n = N! \prod_{n=0}^{N-1} \frac{\sqrt{\pi} n!}{2^n N^{(2n+1)/2}} = \frac{(2\pi)^{N/2}}{(2N)^{N^2/2}} \prod_{n=1}^N n!.$$

which gives a proof to Selberg's formula (1.2.17).  $\square$

We will discuss in the next chapter how to justify asymptotics (1.3.24), (1.3.25), and their extension for a general  $V$ .

#### 1.4. Deformation equations for the recurrence coefficients

In this section we prove some differential identities for the recurrence coefficients of a system of orthogonal polynomials. These identities are in fact quite general, and apply to systems whose measure of orthogonality may not be absolutely continuous with respect to Lebesgue measure. We will consider the setting in which  $\{P_n(x)\}_{n=0}^{\infty}$  is a system of monic polynomials satisfying the orthogonality condition

$$(1.4.1) \quad \int_{\mathbb{R}} P_n(x) P_m(x) e^{-N u_k x^k} d\mu(x) = h_n \delta_{mn},$$

where  $d\mu(x)$  is any measure on  $\mathbb{R}$  such that the system of orthogonal polynomials exists. We consider deformations of this system with respect to the parameter  $u_k$ . Let us rewrite the recurrence equation (1.3.1), explicitly noting the dependence of each recurrence coefficient on the parameter  $u_k$ :

$$(1.4.2) \quad x P_n(x) = P_{n+1}(x) + \beta_n(u_k) P_n(x) + \gamma_n^2(u_k) P_{n-1}(x), \quad \gamma_n^2(u_k) = \frac{h_n(u_k)}{h_{n-1}(u_k)}.$$

We have the following theorem (see [37, 41]).

**Theorem 1.4.1.** *Let  $Q$  be the semi-infinite matrix defined in (1.3.1). As functions of the parameter  $u_k$ , the recurrence coefficients in equation (1.4.2) satisfy the following differential identities:*

$$(1.4.3) \quad h'_n(u_k) = -N[Q^k]_{nn} h_n(u_k)$$

$$(1.4.4) \quad \gamma'_n(u_k) = \frac{N\gamma_n(u_k)}{2} ([Q^k]_{n-1, n-1} - [Q^k]_{nn})$$

$$(1.4.5) \quad \beta'_n(u_k) = N(\gamma_n(u_k)[Q^k]_{n-1, n} - \gamma_{n+1}(u_k)[Q^k]_{n, n+1}).$$

PROOF. Recall that

$$(1.4.6) \quad P_n(x) = x^n + p_{n,n-1}x^{n-1} + p_{n,n-2}x^{n-2} + \dots,$$

where each of the coefficients  $p_{n,j}$  depend on the parameter  $u_k$  as well. Differentiating (1.4.1) for  $n = m$  with respect to  $u_k$  gives

$$(1.4.7) \quad \int_{\mathbb{R}} \left[ -Nx^k P_n(x) P_n(x) + 2P_n(x) \left( \frac{d}{du_k} P_n(x) \right) \right] e^{-Nu_k x^k} d\mu(x) = h'_n(u_k).$$

According to (1.4.6),  $(d/du_k)P_n(x)$  is a polynomial of degree at most  $n-1$ , thus

$$\int_{\mathbb{R}} P_n(x) \left( \frac{d}{du_k} P_n(x) \right) e^{-Nu_k x^k} d\mu(x) = 0,$$

and we have

$$(1.4.8) \quad h'_n(u_k) = \int_{\mathbb{R}} -Nx^k P_n(x) P_n(x) e^{-Nu_k x^k} d\mu(x),$$

which is (1.4.3).

If we differentiate the identity

$$\gamma_n^2(u_k) = \frac{h_n(u_k)}{h_{n-1}(u_k)}$$

with respect to  $u_k$ , then apply the differential identity (1.4.3), we get

$$(1.4.9) \quad \begin{aligned} 2\gamma_n(u_k)\gamma'_n(u_k) &= \frac{h'_n(u_k)h_{n-1}(u_k) - h_n(u_k)h'_{n-1}(u_k)}{h_{n-1}(u_k)^2} \\ &= \frac{Nh_n(t)}{h_{n-1}(t)} ([Q^k]_{n-1,n-1} - [Q^k]_{nn}), \end{aligned}$$

which simplifies to (1.4.4).

If we differentiate (1.4.1) for  $m = n-1$  with respect to  $u_k$ , we find that

$$(1.4.10) \quad \begin{aligned} 0 &= \int_{\mathbb{R}} \left[ -Nx^k P_{n-1}(x) P_n(x) + P_{n-1}(x) \left( \frac{d}{du_k} P_n(x) \right) \right] e^{-Nu_k x^k} d\mu(x) \\ &= -\frac{N}{\gamma_n} [Q^k]_{n-1,n} h_n(u_k) + h_{n-1}(u_k) p'_{n,n-1}(u_k), \end{aligned}$$

which is

$$(1.4.11) \quad p'_{n,n-1}(u_k) = N\gamma_n [Q^k]_{n-1,n}.$$

By matching the coefficient of the  $x^n$ -term in the three term recurrence equation (1.4.2), we find that

$$(1.4.12) \quad \beta_n(u_k) = p_{n,n-1}(u_k) - p_{n+1,n}(u_k).$$

Combining (1.4.11) with (1.4.12) gives (1.4.5).  $\square$

In particular if we consider deformations with respect to the linear term in the potential, we can apply Theorem 1.4.1 with  $k = 1$ , obtaining

$$(1.4.13) \quad \frac{\partial}{\partial u_1} h_n = -N\beta_n h_n,$$

$$(1.4.14) \quad \frac{\partial}{\partial u_1} \gamma_n = \frac{N\gamma_n}{2} (\beta_{n-1} - \beta_n),$$

$$(1.4.15) \quad \frac{\partial}{\partial u_1} \beta_n = N(\gamma_n^2 - \gamma_{n+1}^2).$$

Also, if we consider deformations with respect to the quadratic term in the potential, we can apply Theorem 1.4.1 with  $k = 2$ , obtaining

$$(1.4.16) \quad \frac{\partial}{\partial u_2} h_n = -N h_n (\gamma_n^2 + \beta_n^2 + \gamma_{n+1}^2),$$

$$(1.4.17) \quad \frac{\partial}{\partial u_2} \gamma_n = \frac{N \gamma_n}{2} (\gamma_{n-1}^2 - \gamma_{n+1}^2 + \beta_{n-1}^2 - \beta_n^2),$$

$$(1.4.18) \quad \frac{\partial}{\partial u_2} \beta_n = N (\gamma_n^2 \beta_{n-1} + \gamma_n^2 \beta_n - \gamma_{n+1}^2 \beta_n - \gamma_{n+1}^2 \beta_{n+1}).$$

We can now use these deformation equations to derive some deformation equations for the partition function of the ensemble of eigenvalues in a random matrix ensemble. Recall the formula for the partition function given in (1.2.33):

$$(1.4.19) \quad \tilde{Z}_N = N! \prod_{j=0}^{N-1} h_j.$$

We have the following theorem.

**Theorem 1.4.2.** *The partition function  $\tilde{Z}_N$  satisfies the identities*

$$(1.4.20) \quad \frac{\partial^2}{\partial u_1^2} (\log \tilde{Z}_N) = N^2 \gamma_N^2$$

$$(1.4.21) \quad \frac{\partial^2}{\partial u_2^2} (\log \tilde{Z}_N) = N^2 \gamma_N^2 (\gamma_{N+1}^2 + \gamma_{N-1}^2 + \beta_{N-1}^2 + \beta_N^2 + 2\beta_{N-1} \beta_N).$$

PROOF. Differentiating (1.4.19) with respect to  $u_1$  and using the differential identity (1.4.13) gives

$$(1.4.22) \quad \frac{\partial}{\partial u_1} \tilde{Z}_N = -N \tilde{Z}_N \sum_{j=0}^{N-1} \beta_j.$$

Differentiating again with respect to  $u_1$  and applying (1.4.13) and (1.4.15) gives

$$(1.4.23) \quad \frac{\partial^2}{\partial u_1^2} \tilde{Z}_N = N^2 \left( \frac{\partial}{\partial u_1} \tilde{Z}_N \right) \left( \sum_{j=0}^{n-1} \beta_j \right)^2 - N \left( \frac{\partial}{\partial u_1} \tilde{Z}_N \right) \sum_{j=0}^{n-1} N (\gamma_j^2 - \gamma_{j+1}^2).$$

Using (1.4.22) and noting the telescoping sum in (1.4.23), we can write

$$(1.4.24) \quad \begin{aligned} \frac{\partial^2}{\partial u_1^2} \tilde{Z}_N &= \frac{(\partial \tilde{Z}_N / \partial u_1)^2}{\tilde{Z}_N} + N^2 \tilde{Z}_N \gamma_n^2 \\ &= \frac{(\partial \tilde{Z}_N / \partial u_1)^2}{\tilde{Z}_N} + N^2 \tilde{Z}_N \frac{h_n}{h_{n-1}}. \end{aligned}$$

Equivalently, we can write

$$(1.4.25) \quad \frac{\partial^2 \tilde{Z}_N / \partial u_1^2}{\tilde{Z}_N} - \left( \frac{\partial \tilde{Z}_N / \partial u_1}{\tilde{Z}_N} \right)^2 = N^2 \frac{\tilde{Z}_{N+1} \tilde{Z}_{N-1}}{\tilde{Z}_N^2}$$

or

$$\frac{\partial^2}{\partial u_1^2} (\log \tilde{Z}_N) = N^2 \frac{\tilde{Z}_{N+1} \tilde{Z}_{N-1}}{\tilde{Z}_N^2}.$$

This equation is clearly equivalent to (1.4.20), and is called the *Toda differential equation*.

To prove (1.4.21), we can write (1.4.16) as

$$(1.4.26) \quad \frac{\partial}{\partial u_2}(\log h_j) = -N(\gamma_j^2 + \beta_j^2 + \gamma_{j+1}^2).$$

Differentiating this expression once again with respect to  $u_2$  and applying (1.4.17) and (1.4.18), it is straightforward to see that

$$(1.4.27) \quad \frac{\partial^2}{\partial u_2^2}(\log h_j) = -N^2(I_j - I_{j+1}),$$

where

$$(1.4.28) \quad I_j = \gamma_j^2(\gamma_{j+1}^2 + \gamma_{j-1}^2 + \beta_{j-1}^2 + \beta_j^2 + 2\beta_{j-1}\beta_j).$$

It follows that

$$(1.4.29) \quad \begin{aligned} \frac{\partial^2}{\partial u_2^2}(\log \tilde{Z}_N) &= \sum_{j=0}^{N-1} \frac{\partial^2}{\partial u_2^2}(\log h_j) \\ &= -N^2 \sum_{j=0}^{N-1} I_j - I_{j+1} \\ &= -N^2(I_0 - I_N) \\ &= N^2 I_N, \end{aligned}$$

since  $I_0 = \gamma_0 = 0$ . This proves (1.4.21). □

### 1.5. Differential equations and Lax pair for the $\psi$ -functions

Define

$$(1.5.1) \quad \vec{\Psi}_n(z) = \begin{pmatrix} \psi_n(z) \\ \psi_{n-1}(z) \end{pmatrix}.$$

We have the following theorem (see [37]).

**Theorem 1.5.1.** *The vector  $\vec{\Psi}_n(z)$  satisfies the ODE*

$$(1.5.2) \quad \vec{\Psi}'_n(z) = N A_n(z) \vec{\Psi}_n(z),$$

where

$$(1.5.3) \quad A_n(z) = \begin{pmatrix} -V'(z)/2 - \gamma_n u_n(z) & \gamma_n v_n(z) \\ -\gamma_n v_{n-1}(z) & V'(z)/2 + \gamma_n u_n(z) \end{pmatrix}$$

and

$$(1.5.4) \quad u_n(z) = [W(Q, z)]_{n, n-1}, \quad v_n(z) = [W(Q, z)]_{nn},$$

where

$$(1.5.5) \quad W(Q, z) = \frac{V'(Q) - V'(z)}{Q - z}.$$

Observe that  $\text{Tr } A_n(z) = 0$ .



The three-term recurrence relation (1.3.1) can be written as

$$(1.5.6) \quad \vec{\Psi}_{n+1}(z) = U_n(z)\vec{\Psi}_n(z),$$

where

$$(1.5.7) \quad U_n(z) = \begin{pmatrix} \gamma_{n+1}^{-1}(z - \beta_n) & -\gamma_{n+1}^{-1}\gamma_n \\ 1 & 0 \end{pmatrix}$$

Differential equation 1.5.2 and recurrence equation (1.5.6) form the Lax pair for discrete string equations (1.3.15). This means that the compatibility condition of (1.5.2) and (1.5.6),

$$(1.5.8) \quad U'_n = N(A_{n+1}U_n - U_nA_n),$$

when written for the matrix elements, implies (1.3.15).

**Example (Even quartic model).** For  $V(M) = (t/2)M^2 + (g/4)M^4$ , the matrix  $A_n(z)$  is

$$(1.5.9) \quad A_n(z) = \begin{pmatrix} -\frac{1}{2}(tz + gz^3) - g\gamma_n^2 z & \gamma_n(gz^2 + \theta_n) \\ -\gamma_n(gz^2 + \theta_{n-1}) & \frac{1}{2}(tz + gz^3) + g\gamma_n^2 z \end{pmatrix}$$

where

$$(1.5.10) \quad \theta_n = t + g\gamma_n^2 + g\gamma_{n+1}^2.$$