Preface

The purpose of this book is to serve as a tool for practitioners of Lie algebra and Lie group theory, i.e., for those who apply Lie algebras and Lie groups to solve problems arising in science and engineering. It is not intended to be a textbook on Lie theory, nor is it oriented towards one specific application, for instance the analysis of symmetries of differential equations. We restrict our attention to finite-dimensional Lie algebras over the fields of complex and real numbers.

In any application Lie algebras typically arise as sets of linear operators that commute with a given operator, say the Hamiltonian of a physical system. Alternatively, Lie groups arise as groups of (local) transformations leaving some object invariant; the corresponding Lie algebra then consists of vector fields generating 1-parameter subgroups. The object may be for instance the set of all solutions of a system of equations. The equations can be differential, difference, algebraic or integral ones, or some combination of such equations. They may be linear or nonlinear. In any case, the Lie algebra is realized by some operators in a basis that is usually not the standard one and that depends crucially on the manner in which it was obtained. The structure constants of Lie algebras can be calculated in any basis, but they in turn are basis dependent and reveal very little about the actual structure of the given Lie algebra.

After the Lie algebra $\mathfrak{g}$ associated with a studied problem is found, the next task that faces the researcher is to identify the Lie algebra as an abstract Lie algebra. In some cases $\mathfrak{g}$ may be isomorphic to a known algebra given in some accessible list. This is certainly the case for semisimple Lie algebras in view of Cartan's classification of all simple Lie algebras over the complex numbers, and subsequent classification of their real forms.

The fundamental Levi theorem, stating that every finite dimensional Lie algebra is isomorphic to a semidirect sum of a semisimple Lie algebra and the maximal solvable ideal (the radical) greatly simplifies the task of identifying a given Lie algebra. The weak link is that no complete classification of solvable Lie algebras exists, nor can one be expected to be produced in the future.

The problem addressed in this book is that of transforming a randomly obtained basis of a Lie algebra into a "canonical basis" in which all basis independent features of the Lie algebra are directly visible. For low dimensional Lie algebras (of dimension less or equal six) this makes it possible to identify the Lie algebra completely. In this book we give a representative list of all such Lie algebras. As stated above, in any dimension a complete identification can be performed for semisimple Lie algebras. We also describe some classes of nilpotent and solvable Lie algebras of arbitrary finite dimensions for which a complete classification exists and hence an exact identification is possible.

The book has four parts. The first presents some general results and concepts that are used in the subsequent chapters. In particular such invariant notions as the dimension of ideals in the characteristic series, and the invariants of the coadjoint representation are introduced.

In Part 2 we present algorithms that accomplish the following tasks:

(1) An algorithm for determining whether the algebra $\mathfrak{g}$ can be decomposed into a direct sum. If $\mathfrak{g}$ is decomposable the algorithm provides a basis in which $\mathfrak{g}$ is explicitly decomposed into a direct sum of indecomposable Lie subalgebras.

(2) A further algorithm is presented to find the radical $R(\mathfrak{g})$ and the Levi factor, i.e., the semisimple component of $\mathfrak{g}$.

(3) If the Lie algebra is solvable, for instance if it is the radical of a larger algebra, then it is necessary to identify its nilradical, i.e., the maximal nilpotent ideal. A rational (i.e., avoiding calculation of eigenvalues) algorithm for performing this is presented.

The text includes many examples illustrating various situations that may arise in such computations. All these algorithms have been implemented on computers.

Part 3 is devoted to solvable and nilpotent Lie algebras. While a complete classification of such algebras seems not to be feasible, it is possible to take a class of nilpotent Lie algebras and construct all extensions of these algebras to solvable ones. Finite-dimensional solvable Lie algebras with Abelian, Heisenberg, Borel, filiform and quasifiliform nilradicals are presented in Part 3.

Part 4 of the book consists of tables of all indecomposable Lie algebras of dimension n where $1 \leq n \leq 6$. They are ordered in such a way as to make the identification of any given low-dimensional Lie algebra written in an arbitrary basis as simple as possible. Any Lie algebra up to dimension 6 is isomorphic to precisely one entry in the tables. Essential characteristics of each algebra including its Casimir invariants are also provided.

The book is based on material that was previously dispersed in journal articles, many of them written by one or both of the authors of this book together with collaborators. The tables in Part 4 are based on older results and have been independently verified, in some cases corrected, unified and ordered by structural properties of the algebras (rather than by the way they were originally obtained).

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